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AN EFFICIENT COLLOCATION METHOD FOR A NON-LOCAL DIFFUSION MODEL

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Abstract. The non-local diffusion model provides an appropriate description of the deformation of a continuous body involving discontinuities or other singularities, which cannot be described properly by classical theory of solid mechanics. However, because the non-local nature of the non-local diffusion operator, the numerical methods for non-local diffusion model generate dense or even full stiffness matrices. A direct solver typically requires $O(N^3)$ of operations and $O(N^2)$ of memory where N is the number of unknowns. We develop a fast collocation method for the non-local diffusion model which has the following features: (i) It reduces the computational cost from $O(N^3)$ to $O(N \log^2 N)$ and memory requirement from $O(N^2)$ to $O(N)$. (ii) It requires only one-fold integration in the evaluation of the stiffness matrix. Numerical experiments show the utility of the method.

Key words. collocation method, dense matrices, fast methods, non-local diffusion, peridynamics.

1. Introduction

The classical theory of solid mechanics assumes that all internal forces act through zeros distance. The corresponding mathematical models are described by partial differential equations, which do not provide a proper description of problems with spontaneous formation of discontinuities or other singularities. The non-local diffusion model was proposed as a reformation of solid mechanics [8], which does not explicitly involve the notion of deformation gradients.

Galerkin finite element methods were previously developed and analyzed for the non-local diffusion model [3, 5, 7, 10]. However, these methods face two challenges:

(1) Because of the non-local nature of the non-local diffusion operator, Galerkin finite element methods generate dense or even full matrices. The direct solvers used in solving the resulting discrete systems often require $O(N^3)$ of computational work and $O(N^2)$ of memory where N is the number of degree of freedoms, which are significantly more expensive than the Galerkin finite element methods for the classical models described by differential equations.
(2) Each entry in the stiffness matrix involves two-folds of integration, which makes the evaluation of the stiffness matrix more expensive. Further, a fast solution method can be developed only for a uniform mesh.

In this paper we develop a fast collocation method for the non-local diffusion model. The method has the following features:

(1) The fast method can be developed on both a uniform mesh and a geometrically decreasing mesh. In particular, the latter is particularly suited for problems with singularities. For both meshes, the fast collocation method reduces the computational cost from $O(N^3)$ to $O(N \log^2 N)$ and memory requirement from $O(N^2)$ to $O(N)$.

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(2) Only one-fold of integration is needed in the evaluation of the stiffness matrix, which further reduces the computational cost.

The rest of the paper is organized as follows. In §2 we present a non-local diffusion model and review its Galerkin finite element approximations. In §3 we develop a collocation method for the non-local diffusion model. In §4 we develop a fast collocation method on a uniform mesh. In §5 we develop a fast collocation method on a geometrically decreasing mesh. In §6 we conduct numerical experiments to investigate the computational benefits of the fast methods.

2. A non-local diffusion model and its Galerkin finite element approximation

In this section we briefly discuss the non-local diffusion model and its Galerkin finite element approximation.

2.1. A non-local diffusion model. A linear steady-state non-local diffusion model for microelastic materials on a finite bar is given by the following pseudo-differential equation [7, 8]

$$\int_\alpha^\beta \frac{u(x)-u(y)}{|x-y|^r}dy = b(x), \quad x \in (\alpha,\beta). \tag{1}$$

Here $b(x)$ represents the prescribed forcing term and $u(x)$ presents the displacement of the material. $r \geq 0$ is a parameter that characterizes the influence or decaying property of the kernel function.

By the symmetry of x and y the bilinear form $a(u,v)$ defined by

$$a(u,v) := \int_\alpha^\beta v(x) \int_\alpha^\beta \frac{u(x)-u(y)}{|x-y|^r}dydx \tag{2}$$

can be rewritten as

$$\begin{aligned} a(u,v) &= \int_\alpha^\beta \int_\alpha^\beta \frac{v(x)(u(x)-u(y))}{|x-y|^r}dydx \\ &= \int_\alpha^\beta \int_\alpha^\beta \frac{v(y)(u(y)-u(x))}{|x-y|^r}dydx. \end{aligned} \tag{3}$$

which concludes that

$$\int_\alpha^\beta \int_\alpha^\beta \left(\frac{v(x)\big(u(x)-u(y)\big) - v(y)\big(u(y)-u(x)\big)}{|x-y|^r} \right) dydx = 0. \tag{4}$$

The numerator of the integrand in (4) can be decomposed as

$$\begin{aligned} &v(x)\big(u(x)-u(y)\big) - v(y)\big(u(y)-u(x)\big) \\ &\quad = v(x)\big[\big(u(x)-u(y)\big) - \big(u(y)-u(x)\big)\big] \\ &\qquad -\big(v(y)-v(x)\big)\big(u(y)-u(x)\big) \\ &\quad = 2v(x)\big(u(x)-u(y)\big) - \big(v(y)-v(x)\big)\big(u(y)-u(x)\big). \end{aligned} \tag{5}$$

We incorporate (5) into (4) to derive an alternative expression for $a(u,v)$

$$a(u,v) = \int_\alpha^\beta \int_\alpha^\beta \frac{(u(x)-u(y))(v(x)-v(y))}{2|x-y|^r}dydx. \tag{6}$$

The following theoretical results were proved previously [7, 10]: If $r < 1$, i.e., the kernel is integrable, then $a(u,v)$ is a (semi-) positive-definite and bounded bilinear form on $L^2(\alpha,\beta) \times L^2(\alpha,\beta)$. If $r = 1 + 2s$ with $s > 0$, then $a(u,v)$ is a (semi-)

positive-definite and bounded bilinear form on $H^s(\alpha,\beta)\times H^s(\alpha,\beta)$. For $s>1/2$, i.e., $r>2$, then a standard boundary condition of the form

$$u(\alpha)=u_\alpha \qquad u(\beta)=u_\beta \tag{7}$$

can be specified to close the model (1). However, for $r<1$ or $r=1+2s$ with $s<1/2$, a trace-type of boundary condition of the form (7) cannot be specified. Instead, a domain-type of boundary condition of the form

$$u(x)=u_\alpha(x),\quad x\in(\alpha-\delta,\alpha), \qquad u(x)=u_\beta(x),\quad x\in(\beta,\beta+\delta) \tag{8}$$

is imposed for some $\delta>0$. The border case $r=1$ has also been used in the non-local diffusion model [3]. In this paper we consider problem (1) with $r=1$, imposed with the boundary condition (7). The boundary condition (8) can be treated similarly.

2.2. Galerkin finite element methods for the non-local diffusion model. Previously continuous and discontinuous Galerkin finite element methods were developed to solve the non-local diffusion model [3]. We take the continuous linear Galerkin finite element method as an example to discuss. The following error estimate was proved in [10] for the linear finite element method under the assumption that the true solution $u\in H^2(\alpha,\beta)$. For any $0<\varepsilon<<1$, there exists a positive constant $C=C(\varepsilon)$ which is independent of h such that

$$\|u-u^h\|_{L^2(\alpha,\beta)}\le Ch^{2-\varepsilon}\|u\|_{H^2(\alpha,\beta)}. \tag{9}$$

The estimate (9) shows that the finite element method to the non-local diffusion model (1) reaches a quasi-optimal order convergence rate of $O(h^{2-\varepsilon})$. But the coefficient matrix of the finite element method of the non-local diffusion model is dense or full. Thus, the computational cost of inverting the stiffness matrix of the finite element method is $O(N^3)$ while the memory requirement of the finite element method is of $O(N^2)$. This represents a significant increase of computational cost and memory requirement of the finite element method compared to its analogue for the classical differential equation models.

A simplified non-local diffusion model was proposed to reduce its computational cost, by introducing a parameter δ that describes the horizon of the material and further assuming $\delta=Mh$ with M being a fixed positive integer [3]. The benefit of this assumption is that it significantly reduces the computational cost of the finite element method to $O(M^2N)$ and the memory requirement to (MN). However, this simplification does not seem to be physically reasonable, as the horizon of the material is a physical property of the material of the finite bar which should be independent of the computational mesh size h. This inconsistency is also reflected in the error estimate of the corresponding finite element method, which now reduced to the following sub-optimal order error estimate

$$\|u-u^h\|_{L^2(\alpha,\beta)}\le Ch\|u\|_{H^2(\alpha,\beta)}. \tag{10}$$

3. A linear collocation method for the non-local diffusion model

In this section we develop a linear collocation method for the non-local diffusion model (1). The motivation is to replace all the double layer integrals in the continuous and discontinuous Galerkin finite element methods by the single layer integrals in the collocation method.

Let N be a positive integer. Then introduce a spatial partition on $[\alpha,\beta]$

$$\alpha=:x_0<x_1<\ldots<x_i<\ldots<x_N:=\beta. \tag{11}$$

Let $\{\phi_j\}_{j=0}^N$ be the standard "hat" basis functions associated with the node x_j. Then the approximation u_h to the true solution of the non-local diffusion model can be expressed as

$$u(x) = \sum_{j=0}^{N} u_j \phi_j(x) \tag{12}$$

with $u_0 := u_\alpha$ and $u_N := u_\beta$. If we choose x_i $(i = 1, \ldots, N-1)$ to be the collocation points, then a collocation method can be formulated as follows

$$\int_\alpha^\beta \frac{u_i - u(y)}{|x_i - y|} dy = b(x_i), \quad 1 \le i \le N-1. \tag{13}$$

Let $u := [u_1, u_2, \ldots, u_{N-1}]^T$, $f := [f_1, f_2, \ldots, f_{N-1}]^T$, and $A := [A_{i,j}]_{i,j=1}^{N-1}$. Then the collocation method (13) can be expressed in a matrix form

$$Au = f \tag{14}$$

with $f_i = b(x_i)$ and $A_{i,j}$ being defined by

$$A_{i,j} = \int_\alpha^\beta \frac{-\phi_j}{|x_i - y|} dy, \ i \ne j, \qquad A_{i,i} = \int_\alpha^\beta \frac{1 - \phi_i}{|x_i - y|} dy, \ 1 \le i \le N-1. \tag{15}$$

Motivated by the fact that the coefficient matrix of the finite element method for the non-local diffusion model is weakly ill-conditioned, we expect that a Krylov subspace type of iterative method should perform more efficiently than a direct solver. We look at the conjugate gradient method for (14) which is as follows [1]: Let u_0 be an initial guess, then compute $r_0 := f - Au_0$ and $d_1 := r_0$

for $k = 2, 3, \ldots$

$\gamma_k := r_{k-1}^T r_{k-1} / r_{k-2}^T r_{k-2}$

$d_k := r_{k-1} + \gamma_k d_{k-1}$

$\omega_k := r_{k-1}^T r_{k-1} / d_k^T A d_k$

$u_k := u_{k-1} + \omega_k d_k$

$r_k := r_{k-1} - \omega_k A d_k$

Check for convergence; continue if necessary

end

$u := u_k$

In the CG formulation, the evaluation of the matrix-vector multiplication Ad_k requires $O(N^2)$ of computational work. Furthermore, the storage of the coefficient matrix A also requires $O(N^2)$ of memory requirement. All other computations in the CG formulation require only $O(N)$ of computational work and memory. Therefore, we need only to accelerate the matrix-vector multiplication Ad for any vector d and to store A efficiently. This of course depends on the structure of the matrix A, which in turn relies on the underlying partition (11). The rest of the paper is devoted to develop such methods.

4. A fast collocation method on a uniform partition

In this section we develop a fast collocation method on a uniform partition. In this case, the partition (11) reduces to

$$x_i = \alpha + ih, \qquad i = 0, 1, \ldots, N \qquad h := (\beta - \alpha)/N. \tag{16}$$

We go through algebraic manipulations to find out that the stiffness matrix A is symmetric and its upper triangular part is of the form

$$\begin{aligned} A_{i,i} &= \ln(i) + \ln(N-i) + 2 \qquad 1 \le i \le N-1, \\ A_{i,i+1} &= -2\ln 2, \\ A_{i,i+m} &= (m-1)\ln\left(\frac{m}{m-1}\right) + (m+1)\ln\left(\frac{m}{m+1}\right). \end{aligned} \tag{17}$$

Theorem 1 *The $N-1$-by-$N-1$ stiffness matrix A can be stored in $2N-3$ of memory. Ad can be evaluated in $O(N\log N)$ of operations for any vector $d \in \mathbb{R}^{N-1}$.*

Proof. We split the matrix A as the sum of a diagonal matrix and a Toeplitz matrix with zero diagonal entries

$$A = \operatorname{diag}(A_{i,i})_{i=1}^{N-1} + A_o. \tag{18}$$

Thus, instead of storing $N(N-1)/2$ entries, we need only to store $2N-3$ nonzero entries of the matrix A.

The Toeplitz matrix A_o can then be embedded into a $2(N-1)\times 2(N-1)$ circulant matrix C as follows [2, 6]

$$C := \begin{pmatrix} A_o & B \\ B & A_o \end{pmatrix} \qquad B := \begin{pmatrix} 0 & q_{N-2} & \cdots & q_2 & q_1 \\ q_{N-2} & 0 & q_{N-2} & \cdots & q_2 \\ \vdots & q_{N-2} & 0 & \ddots & \vdots \\ q_2 & \vdots & \ddots & \ddots & q_{N-2} \\ q_1 & q_2 & \cdots & q_{N-2} & 0 \end{pmatrix}. \tag{19}$$

The circulant matrix C can be diagonalized by the discrete Fourier transform matrix as follows [4, 6]

$$C = F^{-1}\ \operatorname{diag}(Fc)\ F \tag{20}$$

where c is the first column vector of C and F is the $2(N-1)\times 2(N-1)$ discrete Fourier transform matrix. It is well known that the matrix-vector multiplication Fw for $w \in \mathbb{R}^{2(N-1)}$ can be carried out in $O(N\log N)$ operations via the fast Fourier transform (FFT). (20) shows that Cw can be evaluated in $O(N\log N)$ operations. So Eq. (19) implies that Ad and so the conjugate gradient method can be evaluated in $O(N\log N)$ operations per iteration! □

Remark 1 *The numerical experiments in §5 indicate that the number of iterations is $O(\log N)$, which suggests that the overall computational cost of the fast conjugate gradient method is $O(N\log^2 N)$.*

5. A fast collocation method on a geometrically decreasing mesh

Peridynamics model intends to handle deformation of a continuous body involving discontinuity or other singularity. In this case a geometrically decreasing mesh is often used. We hence develop a fast collocation method on such a mesh. We consider the case that the displacement has a singularity at the right-end point β. In this case, the grids are defined by

$$x_0 := \alpha, \qquad x_i := \alpha + \sum_{k=1}^{i} \frac{\beta-\alpha}{2^k}, \quad 1 \le i \le N-1, \qquad x_N := \beta. \tag{21}$$

The coefficient matrix A in the matrix equation (14) has a more complex structure.

5.1. Structure of the coefficient matrix. Because the right-most cell has the same size as its left neighbor and so violates the geometrically decreasing pattern, the stiffness matrix A is naturally divided into two parts: the first $N-2$ columns and the $(N-1)$th column.

In the first part the entries of the main diagonal are

$$A_{i,i} = \ln\big(2^{i-1}-1\big) + \ln 2 + 2, \quad 1 \le i \le N-2 \tag{22}$$

The entries on the superdiagonal and subdiagonal are

$$\begin{aligned} A_{i,i+1} &= -3\ln\left(\frac{3}{2}\right), \quad 1 \le i \le N-3 \\ A_{i+1,i} &= -\frac{3}{2}\ln 3. \qquad\quad 1 \le i \le N-2 \end{aligned} \tag{23}$$

For $m \ge 2$, the entries on each diagonal are given by

$$\begin{aligned} A_{i,i+m} &= \big(2^{m+1}-1\big)\ln\left(\frac{2(2^m-1)}{2^{m+1}-1}\right) + 2\big(2^{m-1}-1\big)\ln\left(\frac{2^m-1}{2(2^{m-1}-1)}\right) \\ A_{i+m,i} &= \left(2-\frac{1}{2^m}\right)\ln\Big(\frac{2^m-1}{2^{m+1}-1}\Big) + \left(1-\frac{1}{2^{m-1}}\right)\ln\Big(\frac{2^m-1}{2^{m-1}-1}\Big), \\ &\quad 1 \le i \le N-1-m. \end{aligned} \tag{24}$$

The second part is $A(i, N-1)$, for $1 \le i \le N-1$,

$$\begin{aligned} A_{i,N-1} &= 2^{N-i-1}\ln\Big(\frac{2^{N-i-1}-1}{2^{N-i-1}}\Big) + \big(2^{N-i-1}-2\big)\ln\Big(\frac{2^{N-i-1}-1}{2^{N-i-1}-2}\Big) \\ A_{N-2,N-1} &= -2\ln 2; \\ A_{N-1,N-1} &= \ln(2^{N-1}-1) + 2. \end{aligned} \tag{25}$$

5.2. Efficient storage and fast matrix-vector multiplication. In this subsection we prove the following theorem

Theorem 2 *The $N-1$-by-$N-1$ stiffness matrix A can be stored in $2N-3$ of memory. Ad can be evaluated in $O(N\log N)$ of operations for any vector $d \in \mathbb{R}^{N-1}$.*

Proof. A careful examination of these entries reveals that except for the main diagonal and the last column the stiffness matrix A has a Toeplitz structure

$$A_{i,j} = a_{j-i} \qquad |j-i| \ge 1, \quad j \ne N-1. \tag{26}$$

Here a_k can be defined by the entries of A in first row (except for the last one in the column) or in first column, namely

$$\begin{aligned} a_k &:= A_{1,1+k}, \quad 1 \le k \le N-3, \\ a_{-k} &:= A_{1+k,1}, \quad 1 \le k \le N-2. \end{aligned} \tag{27}$$

We accordingly define a Toeplitz matrix T as follows

$$\begin{aligned} &T_{i,i} := 0, \qquad 1 \le i \le N-1, \quad T_{i,j} := a_{j-i}, \quad j < i \le N-1, \\ &T_{i,j} := a_{j-i}, \quad i < j \le N-2, \quad T_{1,N-1} := 0. \end{aligned} \tag{28}$$

Then it is clear that $S := A - T$ is a sparse matrix. As a matter of fact, we conclude from the definition of T that

$$
\begin{aligned}
&S_{1,N-1} = A_{1,N-1}, \qquad S_{i,N-1} = A_{i,N-1} - A_{1,N-i}, \quad 2 \le i \le N-2,\\
&S_{i,i} = A_{i,i}, \quad 1 \le i \le N-1, \qquad S_{i,j} = 0, \quad \text{otherwise.}
\end{aligned} \tag{29}
$$

In summary, the stiffness matrix A can be decomposed as the sum of a Toeplitz matrix T and a sparse matrix S. We conclude from (28) that only the $2N-5$ nonzero entries of the $N-1$-by-$N-1$ Toeplitz matrix T needs to be stored. As for the sparse matrix S, only the entries on the main diagonal or the $(N-1)$th column are nonzero and thus have to be stored. These are $2N-3$ entries. Therefore, to store A we need to store totally $4N-8$ entries.

For any vector $d \in \mathbb{R}^{N-1}$,

$$
Ad = Sd + Td. \tag{30}
$$

Td can be evaluated in $O(N \log N)$ operations, while Sd can be evaluated in $O(N)$ operations. Thus, we prove the theorem. □

We note that the matrix A is non-symmetric, so a nonsymmetric conjugate gradient type of method should be used. In this paper we use a generalized minimum residual (GMRES) method to solve the problem (14), in which the matrix is stored and the matrix-vector multiplication is evaluated in the manner proved in Theorem 2.

Let u_0 be an initial guess, then compute $r_0 := f - Au_0$ and $h_{10} := \|r_0\|_2$

while $(h_{k+1,k} > 0)$

$q_k := r_k / h_{k+1,k}$

$k := k + 1$

$r_k := Aq_k$

for $i = 1{:}k$

$h_{ik} := q_i^T r_k$

$r_k := r_k - h_{ik} q_i$

end

$h_{k+1,k} = \|rr_k\|_2$

$u_k = u_0 + Q_k y_k$ where $\|h_{10} e_1 - \tilde{H}_k y_k\|_2 = min$

end

$u := u_k$

Here,

$$
\tilde{H}_k := \begin{pmatrix}
h_{11} & h_{12} & \dots & \dots & h_{1k} \\
h_{21} & h_{22} & \dots & \dots & h_{2k} \\
0 & \vdots & \vdots & & \ddots \\
\ddots & & \vdots & \vdots & \ddots \\
0 & \dots & \dots & h_{k,k-1} & h_{k,k} \\
0 & 0 & \dots & \dots & h_{k+1,k}
\end{pmatrix}. \tag{31}
$$

is an upper Hessenberg matrix.

6. Numerical Experiments

In this section we carry out numerical experiments to observe the performance of the fast collocation method developed in this paper. All the numerical methods were implemented using Matlab.

6.1. Numerical experiments with a uniform mesh. In this example, the spatial domain $(\alpha, \beta) = (0, 1)$. The true solution is chosen to be $u(x) = x^2(1-x)^2$, which corresponds to

$$b(x) = \frac{25}{6}x^4 - \frac{25}{3}x^3 + \frac{9}{2}x^2 - \frac{1}{3}x - \frac{1}{12}. \tag{32}$$

We use Gaussian elimination, conjugate gradient (CG) method, and the fast conjugate gradient (FCG) method to solve the system (14) to investigate the performance of these methods. In Table 1 we present the L^2 and L^∞ errors of the numerical solutions solved by these methods with gradually decreasing mesh size h from $h = 2^{-6}$ to $h = 2^{-14}$. These results show that the three methods generate numerical solutions with the same accuracy. A linear regression is used to fit the convergence rates and the associated constants

$$\|u_h - u\|_{L^p(\alpha,\beta)} \le M_\gamma h^\gamma, \qquad p = 2, \infty, \tag{33}$$

which shows the convergence rate of the collocation method seems to be close to 1.5 that is half order less accurate than that of the correponding Galerkin finite element method [3, 10]. Further, the results in Table 1 show that the condition number of the stiff matrix get doubled each time the mesh size is reduced by half. This seems to indicate that the condition number of the stiffness matrix depends inversely on the grid mesh size. These results seems also indicate that the number of iterations in the conjugate gradient method and the fast conjugate gradient method increases logarithmically with respect to the number of unknowns. Finally, these numerical results show that the conjugate gradient method performs better than the Gauss elimination, while the fast conjugate gradient method performs the most efficiently. For example, at the finest mesh size $h = 2^{-14}$, the CG method is about 150 times more efficient than Gauss elimination, while the fast CG method is about 240 times more efficient than the CG method.

6.2. Numerical experiments with a geometrically decreasing mesh. Since the one-dimensional non-local diffusion model describes the deformation of a finite bar involving singularity, it is natural to look at a problem with a singularity at one of the end points. For such a problem, a geometrically decreasing mesh is more efficient than a uniform mesh. Hence, in this example, we look at the performance of the fast collocation method on a geometrically decreasing mesh. In the example run, the spatial domain $(\alpha, \beta) = (0, 1)$. The true solution $u(x)$ and corresponding RHS function $b(x)$ are:

$$\begin{aligned} u(x) &= \frac{1}{1000(1.001 - x)} \\ b(x) &= \frac{\ln(1.001) + \ln(0.001) - 2\ln(1.001 - x)}{1.001 - x}. \end{aligned} \tag{34}$$

In this example, the true solution has a boundary layer near the right end point. We compare the performance of the fast collocation method with a geometrically decreasing mesh and that of the fast collocation method with a uniform mesh. We present the results in Table 2, which show that the fast collocation method with a geometrically decreasing mesh is more efficient as anticipated.

TABLE 1. Performance of the Gaussian elimination, the conjugate gradient (CG) method, and the fast conjugate gradient (FCG) method.

	h	L_2	L_∞	Iteration	Cond	cpuT
Gauss	2^{-6}	2.4779×10^{-4}	3.0678×10^{-4}	-	124.7	0.0469s
	2^{-7}	9.6034×10^{-5}	1.1345×10^{-4}	-	251.7	0.0781s
	2^{-8}	3.8711×10^{-5}	4.3739×10^{-5}	-	505.0	0.3281s
	2^{-9}	1.6231×10^{-5}	1.7655×10^{-5}	-	1010	3.6250s
	2^{-10}	7.0375×10^{-6}	7.4347×10^{-6}	-	2020	26.609s
	2^{-11}	3.1318×10^{-6}	3.2411×10^{-6}	-	4029	4min6s
	2^{-12}	1.4028×10^{-6}	1.4506×10^{-6}	-	8050	20min4s
	2^{-13}	6.5249×10^{-7}	6.6054×10^{-7}	-	16100	3h17min
	2^{-14}	3.0381×10^{-7}	3.0597×10^{-7}	-	32200	1day2h
CG	2^{-6}	2.4779×10^{-4}	3.0678×10^{-4}	23	124.7	0.0156s
	2^{-7}	9.6034×10^{-5}	1.1345×10^{-4}	28	251.7	0.0313s
	2^{-8}	3.8711×10^{-5}	4.3739×10^{-5}	31	505.0	0.3281s
	2^{-9}	1.6231×10^{-5}	1.7655×10^{-5}	34	1010	0.3750s
	2^{-10}	7.0375×10^{-6}	7.4347×10^{-6}	37	2020	0.8906s
	2^{-11}	3.1318×10^{-6}	3.2411×10^{-6}	40	4029	3.5313s
	2^{-12}	1.4028×10^{-6}	1.4506×10^{-6}	43	8050	36.890s
	2^{-13}	6.5249×10^{-7}	6.6054×10^{-7}	48	16100	2min45s
	2^{-14}	3.0381×10^{-7}	3.0597×10^{-7}	49	32200	9min47s
FCG	2^{-6}	2.4779×10^{-4}	3.0678×10^{-4}	21	124.7	0.0156s
	2^{-7}	9.6034×10^{-5}	1.1345×10^{-4}	27	251.7	0.0313s
	2^{-8}	3.8711×10^{-5}	4.3739×10^{-5}	31	505.0	0.0469s
	2^{-9}	1.6231×10^{-5}	1.7655×10^{-5}	34	1010	0.0938s
	2^{-10}	7.0375×10^{-6}	7.4347×10^{-6}	37	2020	0.2031s
	2^{-11}	3.1318×10^{-6}	3.2411×10^{-6}	40	4029	0.3096s
	2^{-12}	1.4028×10^{-6}	1.4506×10^{-6}	43	8050	0.5781s
	2^{-13}	6.5249×10^{-7}	6.6054×10^{-7}	48	16100	1.2656s
	2^{-14}	3.0381×10^{-7}	3.0597×10^{-7}	50	32200	2.4063s
M_γ		0.0622	0.0750	-	-	-
γ		1.3156	1.3737	-	-	-

TABLE 2. Compare the uniform and non-uniform grid methods

Uniform		Non-uniform	
Grids	$L_\infty - error$	Grids	$L_\infty - error$
8	0.6005	3	0.4678
16	0.5723	4	0.3937
32	0.5234	5	0.3150
64	0.4554	6	0.2418
128	0.3694	7	0.1780
256	0.2722	8	0.1248
512	0.1788	9	0.0824
1024	0.1050	10	0.0518
2048	0.0567	11	0.0328
4096	0.0297	12	0.0234

7. Concluding Remarks

The classical theory of solid mechanics assumes that all internal forces act through zeros distance and leads to mathematical models which are described by partial differential equations. These models do not properly describe problems with spontaneous formation of discontinuities or other singularities. Silling proposed a non-local diffusion model as a reformulation of solid mechanics [8], which does not explicitly involve the notion of deformation gradients and thus applies directly to problems with singularities. However, the non-local nature of the new model results in numerical difficulties that were not encountered in the context of classical models described by partial differential equations.

Galerkin finite element methods developed and analyzed for the non-local diffusion model suffer from two major shortcomings computationally, despite their theoretical advantages [3, 5, 7, 10]: (i) They generate a dense or even full stiffness matrix, which requires $O(N^3)$ of computational work and $O(N^2)$ of memory for storage. (ii) Each entry in the stiffness matrix involves two-folds of integration, which makes the assembly of the stiffness matrix more expensive. As a matter of fact, matrix assembly often takes more computational time than the solution of the discrete linear system does.

In this paper we develop a fast collocation method which possesses the following advantages: (i) Collocation method reduces the two-fold integration to one-fold integration for the non-local diffusion model in one space dimension. (ii) By exploring the structure of the stiffness matrix, the fast collocation method significantly reduces the number of entries to be computed in assembling the stiffness matrix. (iii) The fast collocation method significantly reduces the computational cost in the matrix-vector multiplication in the conjugate gradient method. (iv) The fast collocation method significantly reduces the memory requirement to store the stiffness matrix.

We conclude this paper by the following remarks: (i) Although we have focused on the development of the fast collocation method for piecewise linear approximations, the development also works for high-order approximations. (ii) We note that a piecewise-constant collocation method yields inaccurate results, this is in contrast to piecewise-constant Galerkin finite element method which has been shown to work well [3]. (iii) In this paper we take the advantage of the relatively simple geometry of the one space dimension and have evaluated the entries analytically in the fast collocation method. However, in a multidimensional analogue an analytical evaluation of the stiffness matrix is impossible in general. A numerical quadrature will have to be used. We refer the evaluation and discussion of singular integrals to the work in [5, 9, 10]. (iv) In this paper we have also developed a fast piecewise-linear collocation method on a geometrically decreasing grid, aiming at handling point singularities for one-dimensional problems. In this case the finest grid size size decreases rapidly to zero and consequently the stiffness matrix has a very mild size. It is not clear whether any statistically meanful information on the asymptotic behavior such as the number of iterations of the iterative method or the condition number of the stiffness matrix can be observed for such a small size problem. We believe this issue could be studied in a more meanful way for multidimensional problems in the future. (v) For a more realistic non-local diffusion model in multiple space dimensions, we feel that a domain decomposition approach probably should be used, with a coarse uniform mesh away from a singularity and a geometrically decreasing mesh near the singularity. (vi) An important issue in the study of any numerical method is the convergence estimate and the corresponding

error estimate. The fast collocation method developed in this paper calls for a rigorous theoretical analysis and convergence error estimates, which remains to be proved and will be investigated in the future.

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