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GRID APPROXIMATION OF A SINGULARLY PERTURBED PARABOLIC EQUATION WITH DEGENERATING CONVECTIVE TERM AND DISCONTINUOUS RIGHT-HAND SIDE

C. CLAVERO, J.L. GRACIA, G.I. SHISHKIN, L.P. SHISHKINA

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This paper is dedicated to the 60th birthday of Martin Stynes

Abstract. The grid approximation of an initial-boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation with a convective flux directed from the lateral boundary inside the domain in the case when the convective flux degenerates inside the domain and the right-hand side has the first kind discontinuity on the degeneration line. The high-order derivative in the equation is multiplied by ε^2, where ε is the perturbation parameter, $\varepsilon \in (0, 1]$. For small values of ε, an *interior layer* appears in a neighbourhood of the set where the right-hand side has the discontinuity. A finite difference scheme based on the standard monotone approximation of the differential equation in the case of uniform grids converges only under the condition $N^{-1} = o(\varepsilon)$, $N_0^{-1} = o(1)$, where $N+1$ and N_0+1 are the numbers of nodes in the space and time meshes, respectively. A finite difference scheme is constructed on a piecewise-uniform grid condensing in a neighbourhood of the interior layer. The solution of this scheme converges ε-uniformly at the rate $\mathcal{O}(N^{-1} \ln N + N_0^{-1})$. Numerical experiments confirm the theoretical results.

Key words. parabolic convection-diffusion equation, perturbation parameter, degenerating convective term, discontinuous right-hand side, interior layer, technique of derivation to *a priori* estimates, piecewise-uniform grids, finite difference scheme, ε-uniform convergence, maximum norm.

1. Introduction

At present, methods to construct special ε-uniformly convergent finite difference schemes for singularly perturbed elliptic and parabolic convection-diffusion equations are well developed for the case when the problem data are sufficiently smooth and the convective term in the equations preserves the sign (e.g., strictly positive) everywhere in the domain (see, e.g., [2, 8, 10, 13, 16] and the references therein). Special methods for singularly perturbed problems with discontinuous data and degenerating convective terms are less developed; see, e.g., the case with discontinuous data in differential equations (coefficients and the right-hand side) in [9, 13], the case with discontinuous boundary conditions in [4, 16], the case with a convective term degenerating on the domain boundary for a parabolic convection-diffusion equation in [3]. Special schemes for problems with a convective term degenerating inside the domain and discontinuous data in equations were not considered earlier.

In the present paper the grid approximation of an initial-boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation with a convective flux directed from the lateral boundary inside the domain in the

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case when the convective term, being sufficiently smooth, degenerates inside the domain and the right-hand side has the first kind discontinuity on the degeneration line. For small values of the perturbation parameter ε, an interior layer arises in a neighbourhood of the set where the right-hand side has the discontinuity; the interior layer does not arise in the case of the smooth right-hand side (see Sect. 3).

For the initial-boundary value problem, a finite difference scheme is constructed on a piecewise-uniform grid condensing in a neighbourhood of the interior layer. The solution of this scheme converges ε-uniformly in the maximum norm at the rate $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$, where $N+1$ and N_0+1 are the numbers of nodes in the space and time meshes, respectively. Note that in the case of a parabolic convection-diffusion equation with sufficiently smooth problem data and positive diffusion coefficient on the definition domain, the finite difference scheme converges at the same convergence rate (see, e.g., [16] and the references therein).

Contents of the paper. The formulation of the initial-boundary value problem and the aim of the research are given in Section 2. *A priori* estimates used to construct and justify developed schemes are exposed in Section 3. Finite difference schemes on uniform and piecewise-uniform grids are studied, respectively, in Section 4 and 5. Numerical experiments to investigate the constructed schemes are shown in Section 6. Conclusions are given in Section 7.

Notation. We denote by $C^{k,k/2}$ the space of functions $u(x,\,t)$ with continuous derivatives in x up to order k and continuous derivatives in t up to order $k/2$. Henceforth, M, M_i (or m) denote sufficiently large (small) positive constants that are independent of the parameter ε and of the discretization parameters. Finally, the notation $L_{(j.k)}$ ($\overline{G}_{(j.k)}, M_{(j.k)}$) means that this operator (domain, constant) is introduced in formula $(j.k)$.

2. Problem formulation and aim of the research

2.1. On the set $\overline{G}$ with the boundary S

$$\overline{G} = G \bigcup S, \quad G = D \times (0,T], \quad D = (-d,d), \tag{2.1}$$

we consider the initial-boundary value problem for the singularly perturbed parabolic convection-diffusion equation

$$L\,u(x,t) \equiv \left\{\varepsilon^2\,\frac{\partial^2}{\partial x^2} + x\,\frac{\partial}{\partial x} - \frac{\partial}{\partial t} - 1\right\} u(x,t) = f(x,t), \quad (x,t) \in G \setminus S^{\pm}, \tag{2.2a}$$

where the function $f(x,t)$ is continuous on $\overline{G}$ for $x<0$ and $x>0$ and it has a discontinuity of the first kind on the set

$$S^{\pm} = \{x=0\} \times (0,T];$$

and thus on the set $S^{\pm}$, the following conjunction condition for the first-order derivative in x is imposed:

$$l^{\pm}u(x,t) \equiv \varepsilon\,\left[\frac{\partial}{\partial x}u(x+0,\,t) - \frac{\partial}{\partial x}u(x-0,\,t)\right] = 0, \quad (x,t) \in S^{\pm}. \tag{2.2b}$$

On the boundary S, the boundary condition is prescribed

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2c}$$

As a solution of the initial-boundary value problem (2.2), (2.1) in the case of the right-hand side having the first kind discontinuity on the set $S^{\pm}$, denoted by

$$[f(x^*,t)]^j \equiv f(x^*+0,t) - f(x^*-0,t) \neq 0, \quad (x^*,t) \in S^{\pm}, \tag{2.3}$$

we mean a function u, which is continuous on $\overline{G}$ and it has continuous derivatives up to second order in x and continuous derivative in t, satisfying the differential equation (2.2a) on $G \setminus S^{\pm}$. The first-order derivative in x is continuous on G, moreover, the conjunction condition (2.2b) holds on the set $S^{\pm}$ and the condition (2.2c) is fulfilled on the boundary S.

The coefficient multiplying the first-order derivative in x in the differential equation (2.2a) corresponds to the convective flux; the velocity of the convective flux vanishes on the set $S^{\pm}$. The source function $f(x,t)$ is continuous on $\overline{G}^{\,+}$ and $\overline{G}^{\,-}$, where

$$\overline{G}^{\,-} = [-d,\, 0] \times [0,\, T], \quad \overline{G}^{\,+} = [0,\, d] \times [0,\, T].$$

and it has a discontinuity on $S^{\pm}$.

Continuity of the first-order derivative in x on the interface boundary $S^{\pm}$ in the conjunction condition (2.2b), associates with continuity of diffusion flux across this boundary. Possible physical interpretations of the problem (2.2), (2.1), (2.3) are given in [1, 17].

Considering the problem (2.2), (2.1), we assume that the problem data satisfy conditions that guarantee the required smoothness of the solution on the sets $\overline{G}^{\,+}$ and $\overline{G}^{\,-}$. The function $f(x,t)$ is assumed to be sufficiently smooth on the sets $\overline{G}^{\,+}$ and $\overline{G}^{\,-}$. Also we assume that the function $\varphi(x,t)$ is smooth on the sets S^L and S_0^+, S_0^- (the lateral and lower parts of the boundary S), where $S = S^L \bigcup S_0$, $S_0 = S_0^+ \bigcup S_0^-$, $S^L = S^l \bigcup S^r$ (S^l and S^r are the left and right parts of the boundary S^L), and, thus,

$$S_0^+ = [0,\, d] \times \{t = 0\}, \quad S_0^- = [-d,\, 0] \times \{t = 0\},$$
$$S^l = \{x = -d\} \times (0,\, T], \quad S^r = \{x = d\} \times (0,\, T].$$

Moreover, on the set $S^c = \overline{S}^L \bigcap S_0$ of corner points, and on the set $S^{\pm\, c} = \overline{S}^{\pm} \bigcap S_0$ of interior corner points, compatibility conditions are satisfied that guarantee the required smoothness of the solution in neighborhoods of these points.

Let such conditions ensure the continuity of the problem solution on $\overline{G}$ together with the derivatives in t up to order $k_0 \geq 2$ and with the first-order derivative in x; the second-order derivative in x has a discontinuity on the set $S^{\pm}$.

Characteristics of the reduced equation are tangent to the set $S^{\pm}$ on which the right-hand side has a discontinuity that causes appearance of an interior layer in the solution, when $\varepsilon \to 0$. However, characteristics of the reduced equation enter into the domain $\overline{G}$, and they are not tangent to the lateral boundary $S^l \cup S^r$; therefore, boundary layers in the problem (2.2), (2.1) do not arise (see Section 3 and Remark 8 therein).

Problems of the type considered here could arise when heat transfer in sufficiently compound moving flows is studied. For example, in [12][Ch. 9], a problem on a laminar flat jet streaming into the half-space from a long narrow slit is examined. The streamline pattern in this case is analogous to the behavior of characteristics in equation (2.2a) for $\varepsilon = 0$. The consideration of heat transfer processes in the jet when heat sources on opposite sides from the jet center are different could bring us to a problem similar to that studied in the present paper.

2.2. Problems for partial differential equations with discontinuous data in the differential equations (i.e., diffraction problems) in the case of regular equations are discussed, e.g., in [5, 6, 7]. Numerical methods for such regular problems with discontinuous data in the differential equations are considered e.g., in [11]; see also the bibliography therein.

Difference schemes for singularly perturbed elliptic and parabolic equations with sufficiently smooth data, which converge ε-uniformly in the maximum norm, are developed in [2, 8, 10, 13, 16]; see also the bibliography therein. However, the behaviour of solutions to singularly perturbed problems with discontinuous data is more complicated that generates an interest to develop special ε-uniformly convergent schemes for the problems with discontinuous data.

ε-uniformly convergent difference schemes for singularly perturbed reaction-diffusion elliptic and parabolic equations with discontinuous data and right-hand sides and concentrated sources were developed in [13].

A difference scheme for a singularly perturbed parabolic equation with the convective term degenerating on the domain boundary is considered in [3]. A technique for a numerical analysis of ε-uniform convergence of schemes on piecewise-uniform grids for a singularly perturbed elliptic equation with the convective term degenerating on the domain boundary is considered in [2].

Numerical methods for the singularly perturbed problem (2.2), (2.1), i.e., the problem with the convective term degenerating inside a domain and with the discontinuous right-hand side, were not considered early.

As it is shown in Sections 4 and 6, a classical difference scheme for the problem (2.2), (2.1) does not converge ε-uniformly.

Thus, our **aim** for the problem (2.2), (2.1) is to construct a finite difference scheme that converges ε-uniformly in the maximum norm.

3. *A priori* estimates

Here we give *a priori* estimates for solutions and derivatives of the initial-boundary value problem (2.2), (2.1), (2.3).

3.1. In this and following subsections, we give *a priori* estimates for the solution and the derivatives of the initial-boundary value problem (2.2), (2.1), in the case when the function $f(x,t)$ is continuous on $\overline{G}$, i.e., it satisfies the condition

$$[f(x^*,t)]^j \equiv 0, \quad (x^*,t) \in S^{\pm}. \tag{3.1}$$

Moreover, $f(x,t)$ is sufficiently smooth on the set $\overline{G}$; the derivation of these estimates is similar to that in [13, 16].

Applying the majorant function technique (see, e.g., [7]), we find the estimate

$$|u(x,t)| \le M, \quad (x,t) \in \overline{G}. \tag{3.2}$$

For sufficiently smooth data of the problem (2.2), (2.1), (3.1) (on the set $\overline{G}$) and under compatibility conditions for the derivatives $(\partial^{k_0}/\partial t^{k_0})u(x,t)$, $2k_0 \le K$, $K > 0$, (on the set S^c), providing the inclusion $u \in C^{K,K/2}(\overline{G})$, we find the estimate for the derivatives written in the variables $\xi = \varepsilon^{-2}x$, $\tau = \varepsilon^{-2}t$. In the original variables, we have the estimate (see, e.g., [13, 16])

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t) \right| \le M\, \varepsilon^{-2k-2k_0}, \quad (x,t) \in \overline{G}, \quad k + 2k_0 \le K. \tag{3.3}$$

It is not difficult to write out compatibility conditions for the derivatives $\partial^{k_0}/\partial t^{k_0} u(x,t)$, $2k_0 \le K$, $K > 0$ on the set S^c, e.g., in the case when the initial function $\varphi(x,0)$ together with its derivatives vanish on the set S^c. In particular, for smooth data of the problem (2.2), (2.1), (3.1), the condition

$$\frac{\partial^k}{\partial x^k}\,\varphi(x,t)=0,\quad \frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t)=0,\quad k+2k_0\le l_0, \tag{3.4}$$

$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x,t)=0,\quad k+2k_0\le l_0-2,\quad (x,t)\in S^c,$$

where $l_0>0$ is even, provides compatibility conditions for the derivatives in t up to order $l_0/2$ on the set S^c (see [7]). The condition (3.4) is sufficient for the inclusion $u\in C^{K,K/2}(\overline{G})$, and also for the estimate (3.3) for $K=l_0$.

Theorem 1. *Let the data of the initial-boundary value problem* (2.2), (2.1), (3.1) *satisfy the conditions* $f\in C^{l_1-2,\,(l_1-2)/2}(\overline{G})$, $\varphi\in C^{l_1}(S_0)\cap C^{l_1/2}(\overline{S}^L)$, $l_1=l_0+\alpha$, $l_0=l_{0\,(3.4)}>0$, $\alpha>0$, *and also* (3.4). *Then, the solution of the initial-boundary value problem satisfies the estimates* (3.2), (3.3), *where* $K=l_{0\,(3.4)}$.

The estimate (3.3) for the solution derivatives does not allow us to establish the ε-uniform convergence of the constructed schemes.

3.2. Now we shall improve the estimate (3.3). For this, we impose additional condition on the data of problem (2.2), (2.1) in order to improve smoothness of its solution.

We represent the problem solution as an expansion with respect to the parameter ε, in the form

$$u(x,t)=u_0(x,t)+\varepsilon^2u_1(x,t)+\ldots+\varepsilon^{2n}u_n(x,t)+v_u(x,t),\quad (x,t)\in\overline{G}. \tag{3.5}$$

Here the functions $u_i(x,t)$ for $i=0,1,\ldots,n$ are current terms of the expansion, and the function $v_u(x,t)$, $(x,t)\in\overline{G}$, is the remainder term. The functions $u_i(x,t)$ are solutions of the problems

$$\begin{aligned} L_0\,u_0(x,t)=L_{(2.2)}(\varepsilon=0)\,u_0(x,t) &\equiv \left\{x\,\frac{\partial}{\partial x}-\frac{\partial}{\partial t}-1\right\}u_0(x,t)= \\ &= f(x,t),\quad (x,t)\in G, \\ u_0(x,t) &= \varphi(x,t),\quad (x,t)\in S; \\ L_0\,u_i(x,t) &= -\frac{\partial^2}{\partial x^2}\,u_{i-1}(x,t),\quad (x,t)\in G, \\ u_i(x,t) &= 0,\quad (x,t)\in S,\quad i=1,\ldots,n. \end{aligned} \tag{3.6}$$

The function $v_u(x,t)$ is the solution of the problem

$$\begin{aligned} L_{(2.2)}\,v_u(x,t) &= -\varepsilon^{2n+2}\,\frac{\partial^2}{\partial x^2}\,u_n(x,t),\quad (x,t)\in G, \\ v_u(x,t) &= 0,\quad (x,t)\in S. \end{aligned} \tag{3.7}$$

Smoothness of the components in the expansion (3.5) is provided by the sufficient smoothness of the data to the problem (2.2), (2.1) and by the appropriate compatibility conditions for the data of the problems $\{(2.2),(2.1)\}$, $\{(3.6),(2.1)\}$ and $\{(3.7),(2.1)\}$ on the set S^c. For the solution of the initial-boundary value problem (2.2), (2.1) and its derivatives we obtain the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,u(x,t)\right|\le M\,[1+\varepsilon^{2(n+1-k-k_0)}],\quad (x,t)\in\overline{G},\quad k+2k_0\le K. \tag{3.8}$$

In particular, the conditions imposed on the functions $f(x,t)$ and $\varphi(x,t)$, $(x,t) \in S^c$:

$$\frac{\partial^k}{\partial x^k}\,\varphi(x,t)=0, \quad \frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t)=0, \qquad (3.9)$$

$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x,t)=0, \quad k+k_0 \le l_0, \quad (x,t) \in S^c,$$

where $l_0 = l_0^* + 2n$, $l_0^* > 0$ is even, $n \ge 0$, ensure smoothness of components in the expansion (3.5) and continuity on $\overline{G}$ of derivatives of the solution to problem (2.2), (2.1), (3.1). This condition is sufficient for the inclusion $u \in C^{K,K/2}(\overline{G})$, and also for the estimate (3.8) for $K = l_0^*$.

Theorem 2. *Let for the data of the initial-boundary value problem* (2.2), (2.1), (3.1) *the conditions* $f \in C^{l_1-2}(\overline{G})$, $\varphi \in C^{l_1}(S_0) \cap C^{l_1}(\overline{S}^L)$, $l_1 = l_0 + \alpha$, $l_0 = l_{0\,(3.9)} = l_0^* + 2n > 0$, $n \ge 0$, $\alpha > 0$, *and also* (3.9) *be satisfied. Then, the solution of the initial-boundary value problem satisfies the estimate* (3.8), *where* $K = l^*_{0\,(3.9)}$.

Remark 3. From the estimate (3.8) for $n=1$ it follows that the derivatives of the solution to the initial-boundary value problem (2.2), (2.1), (3.1), which are involved into the differential equation, are ε-uniformly bounded; the solution of the problem (2.2), (2.1), (3.1) with a sufficiently smooth right-hand side $f(x,t)$ is regular. □

3.3. Consider now the initial-boundary value problem (2.2), (2.1), (2.3), i.e., the problem with a discontinuous right-hand side $f(x,t)$.

For the problem (2.2), (2.1), (2.3), the following comparison theorem holds (similar to one established in [13]):

Theorem 4. *Let the functions* $u^1(x,t)$, $u^2(x,t)$ *satisfy the relations*

$$\begin{aligned} L\,u^1(x,\,t) &\le L\,u^2(x,t), && (x,\,t) \in G \setminus S^{\pm},\\ l^{\pm}\,u^1(x,\,t) &\le l^{\pm}\,u^2(x,\,t), && (x,\,t) \in S^{\pm},\\ u^1(x,\,t) &\ge u^2(x,t), && (x,\,t) \in S. \end{aligned}$$

Then

$$u^1(x,t) \ge u^2(x,t), \quad (x,t) \in \overline{G}.$$

Using Theorem 4, we obtain the following estimate for the solution of problem (2.2), (2.1), (2.3):

$$|u(x,\,t)| \le M, \quad (x,\,t) \in \overline{G}. \qquad (3.10)$$

Now we estimate the derivatives of problem solutions on the sets $\overline{G}^+$ and $\overline{G}^-$. Under sufficiently smooth problem data and appropriate compatibility conditions on the sets S^c and $S^{\pm\,c}$, the problem (2.2), (2.1), (2.3) admits differentiation in t; we find the derivatives in t on the set S_0 by virtue of the differential equation. The initial-boundary value problems written out for the derivatives in t are similar to the problem (2.2), (2.1), (2.3). For the derivatives in t we obtain the estimate

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\,u(x,t)\right| \le M, \quad (x,t) \in \overline{G}, \quad 2k_0 \le K. \qquad (3.11)$$

The following condition that guarantees the compatibility condition for the derivatives in t up to order $l_0/2$ where $l_0 > 0$ is even:

$$\begin{aligned} &\frac{\partial^k}{\partial x^k}\,\varphi(x,t)=0, \quad \frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t)=0, && k+2k_0 \le l_0,\\ &\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x,t)=0, && k+2k_0 \le l_0-2, \quad (x,t) \in S^c; \end{aligned} \qquad (3.12a)$$

$$\begin{aligned} &\frac{\partial^k}{\partial x^k}\,\varphi(x\pm 0,t)=0, && k\le l_0,\\ &\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x\pm 0,t)=0, && k+2k_0\le l_0-2, \quad (x,t)\in S^{\pm\,c}, \end{aligned} \tag{3.12b}$$

is sufficient for the estimate (3.11) for $K = l_0$ (existence of solutions for regular parabolic equations with discontinuous data and properties of the solutions were discussed in [7], Ch. 3).

By virtue of the condition (3.12b), we have

$$\frac{\partial^{k_0}}{\partial t^{k_0}}\,u(x\pm 0,t)=0,\quad (x,t)\in S^{\pm\,c},\quad 2k_0\le K,\quad K=l_0. \tag{3.13}$$

Thus, in the point $(0,0)$, which is the left corner point of the set $\overline{G}^{+}$ and the right corner point of the set $\overline{G}^{-}$, by virtue of the conditions (3.12b) and (3.13), compatibility conditions for the derivatives in t up to order $K/2$ are fulfilled.

Now we consider initial-boundary value problems on the sets $\overline{G}^{+}$ and $\overline{G}^{-}$, where by virtue of the estimate (3.11) the derivatives in t are ε-uniformly bounded; moreover, by virtue of the conditions (3.12) and (3.13), compatibility conditions in the corner points of the sets $\overline{G}^{+}$ and $\overline{G}^{-}$ are fulfilled. Thus, $u\in C^{K,K/2}(\overline{G}^{+})\cap C^{K,K/2}(\overline{G}^{-})$, and for the solution of the problem (2.2), (2.1), (2.3) we have the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,u(x,t)\right|\le M\,\varepsilon^{-2k-2k_0},\quad (x,t)\in\overline{G}^{+}\cup\overline{G}^{-},\quad k+2k_0\le K,$$

which is similar to the estimate (3.3). Taking into account the estimate (3.11) leads to the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,u(x,t)\right|\le M\,\varepsilon^{-2k},\quad (x,t)\in\overline{G}^{+}\cup\overline{G}^{-},\quad k+2k_0\le K, \tag{3.14}$$

where $K=l_{0\,(3.12)}$.

Theorem 5. *Let for the data of the initial-boundary value problem* (2.2), (2.1), (2.3), *the conditions* $f\in C^{l_1-2,\,(l_1-2)/2}(\overline{G}^{+})\cap C^{l_1-2,\,(l_1-2)/2}(\overline{G}^{-})$, $\varphi\in C^{l_1}(S_0^{+})\cap C^{l_1}(S_0^{-})\cap C^{l_1/2}(\overline{S}^{L})$ *for* $l_1=l_0+\alpha$, $l_0=l_{0\,(3.12)}>0$, $\alpha>0$, *and also* (3.12) *be satisfied. Then, the solution of the initial-boundary value problem satisfies the estimates* (3.10) *and* (3.14), *where* $K=l_{0\,(3.12)}$.

3.4. Now we shall improve the estimate, using the decomposition of the solution into the regular and singular components.

We represent the solution of problem (2.2), (2.1), (2.3) on each of the sets $\overline{G}^{\,+}$ and $\overline{G}^{\,-}$ as the sums of the functions

$$u(x,\,t)=U^{+}(x,\,t)+V^{+}(x,\,t),\quad (x,\,t)\in\overline{G}^{\,+}, \tag{3.15}$$

$$u(x,\,t)=U^{-}(x,\,t)+V^{-}(x,\,t),\quad (x,\,t)\in\overline{G}^{\,-},$$

where $U^{+}(x,\,t)$, $U^{-}(x,\,t)$ and $V^{+}(x,\,t)$, $V^{-}(x,\,t)$ are regular and singular components of the solution.

Further, we consider the components $U^{+}(x,\,t)$ and $V^{+}(x,\,t)$ on the set $\overline{G}^{\,+}$. Preliminary, we note that the function $u(x,\,t)$ considered on the set $\overline{G}^{+}$ is the solution of the following initial-boundary value problem:

$$
\begin{aligned}
L_{(2.2)}u(x,t) &= f(x,t), && (x,t)\in G^+,\\
u(x,t) &= \varphi(x,t), && (x,t)\in S_0^+\cup S^r,\\
u(x,t) &= \varphi_u(x,t), && (x,t)\in S^\pm,
\end{aligned} \tag{3.16}
$$

where $\varphi_u(x,t)=u_{\{(2.2),(2.1),(2.3)\}}(x,t)$, $(x,t)\in S^\pm$, and $u_{\{(2.2),(2.1),(2.3)\}}(x,t)$ is the solution of the problem (2.2), (2.1), (2.3). Derivatives in t of the function $\varphi_u(x,t)$, $(x,t)\in S^\pm$, by virtue of the estimate (3.11), are ε-uniformly bounded, moreover, by virtue of the estimates (3.12), (3.13) compatibility conditions are fulfilled in the corner points of the set $\overline{G}^+$.

The function $U^+(x,t)$, $(x,t)\in\overline{G}^+$ is the restriction to $\overline{G}^+$ of the function $U^{+\,e}(x,t)$, $(x,t)\in\overline{G}^{+\,e}$, i.e., $U^+(x,t)=U^{+\,e}(x,t)$, $(x,t)\in\overline{G}^+$. The function $U^e(x,t)$, $(x,t)\in\overline{G}^{\,e}$ is the solution of the initial-boundary value problem

$$
\begin{aligned}
L^eU^{+\,e}(x,t) &= f^{+\,e}(x,t), && (x,t)\in G^{+\,e},\\
U^{+\,e}(x,t) &= \varphi^{+\,e}(x,t), && (x,t)\in S^{+\,e},
\end{aligned} \tag{3.17a}
$$

which is an extension of the problem (3.15) beyond the set $S^\pm$; here we take

$$
\overline{G}^{+\,e}=\overline{G}_{(2.1)}. \tag{3.17b}
$$

The data of the problem (3.17) (the functions $f^{+\,e}(x,t)$, $(x,t)\in\overline{G}^+$ and $\varphi^{+\,e}(x,t)$, $(x,t)\in S_0^+\cup S^r$) on the set $\overline{G}^+$ are the data of the problem (2.2), (2.1), and on the set $\overline{G}^-$ they are smooth extensions of the data to the problem (2.2), (2.1) prescribed on $\overline{G}^+$. The function $V^+(x,t)$, $(x,t)\in\overline{G}^+$ is the solution of the problem

$$
\begin{aligned}
L_{(2.2)}\,V^+(x,t) &= 0, && (x,t)\in G^+,\\
V^+(x,t) &= \varphi_{V^+}(x,t), && (x,t)\in S^\pm,\\
V^+(x,t) &= 0, && (x,t)\in S_0^+\cup S^r,
\end{aligned} \tag{3.18}
$$

where $\varphi_{V^+}(x,t)=\varphi_u(x,t)-U^+(x,t)$, $(x,t)\in S^\pm$.

We represent the function $U^{+e}(x,t)$, which is the solution of the problem (3.17), as an expansion similar to (3.5), in the form

$$
U^{+e}(x,t)=U_0(x,t)+\varepsilon^2U_1(x,t)+\cdots+\varepsilon^{2n}U_n(x,t)+v_U(x,t),\quad (x,t)\in\overline{G}. \tag{3.19}
$$

The functions $U_i(x,t)$ are the solution of the problems

$$
\begin{aligned}
L_0U_0(x,t)=L^e_{(3.17)}(\varepsilon=0)\,U_0(x,t) &\equiv \left\{x\frac{\partial}{\partial x}-\frac{\partial}{\partial t}-1\right\}\ U_0(x,t)= \\
&= f^{+e}(x,t), && (x,t)\in G,\\
U_0(x,t) &= \varphi^{+e}(x,t), && (x,t)\in S;\\
L_0U_i(x,t) &= -\frac{\partial^2}{\partial x^2}U_{i-1}(x,t), && (x,t)\in G,\\
U_i(x,t) &= 0, && (x,t)\in S,\\
& && i=1,...,n.
\end{aligned} \tag{3.20}
$$

The function $v_U(x,t)$ is the solution of the problem

$$
\begin{aligned}
L^e_{(3.17)}v_U(x,t) &= -\varepsilon^{2n+2}\,\frac{\partial^2}{\partial x^2}\,U_n(x,t), && (x,t)\in G,\\
v_U(x,t) &= 0, && (x,t)\in S.
\end{aligned} \tag{3.21}
$$

Smoothness of the components in the expansion (3.19) is provided by the sufficient smoothness of the data to the problem (3.17) and by the appropriate compatibility conditions for the data of the problems (3.17), (3.20), (3.21) on the set S^c. For the solution of the initial-boundary value problem (3.17) we obtain the estimate for derivatives of the function $U^{+e}(x,t)$ on $\overline{G}^{+e}$. For the function $U^+(x,t)$ on $\overline{G}^+$ we have the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U^+(x,t)\right| \le M\,[1+\varepsilon^{2(n+1-k-k_0)}], \quad (x,t)\in \overline{G}^+, \quad k+2k_0\le K. \tag{3.22}$$

In particular, the conditions imposed on the functions $f(x,t)$ and $\varphi(x,t)$, $(x,t)\in S^c$,

$$\frac{\partial^k}{\partial x^k}\,\varphi(x,t)=0, \quad \frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi(x,t)=0, \quad \frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x,t)=0, \tag{3.23}$$

$$k+k_0\le l_0, \quad (x,t)\in S^c, \quad l_0=l_0^*+2n, \quad l_0^*>0 \;\; is \;\; even,$$

guarantee smoothness of components in the expansion (3.19) and continuity on $\overline{G}^+$ of derivatives of the solution to problem (3.17). Moreover, these conditions are sufficient for the inclusion $U^+\in C^{K,K/2}(\overline{G}^+)$, and also for the estimate (3.22) for $K=l^*_{0\,(3.23)}$.

Consider the problem (3.18). Assume that the condition (3.23) is fulfilled on S^c, and also the following condition holds on $S^{\pm c}$:

$$\frac{\partial^k}{\partial x^k}\,\varphi(x\pm 0,t)=0, \qquad k\le l_0^{**}, \tag{3.24}$$

$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}\,f(x\pm 0,t)=0, \qquad k+2k_0\le l_0^{**}-2, \quad (x,t)\in S^{\pm\,c},$$

where $l_0^{**}=l^*_{0\,(3.23)}$.

In the case of conditions (3.23) on S^c and (3.24) on $S^{\pm c}$ for smooth data of problem (2.2), (2.1), (2.3) on $\overline{G}^+$ and $\overline{G}^-$, we have $u\in C^{K+\alpha,(K+\alpha)/2}(\overline{G}^+)$ for $K=l^*_{0\,(3.23)}$. Thus, $V^+\in C^{K+\alpha,(K+\alpha)/2}(\overline{G}^+)$, and by virtue of (3.11) and (3.22), we have

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\,\varphi_{V^+}(x,t)\right| \le M\left[1+\varepsilon^{2(n+1-k_0)}\right], \quad (x,t)\in S^{\pm}, \quad 2k_0\le K.$$

Using a standard comparison theorem with the majorant function

$$W(x,t)=M\left[1+\varepsilon^{2(n+1-k_0)}\right]\left\{1-\Phi\left(m_0^{1/2}\frac{x}{\varepsilon}\right)\right\}, \quad (x,t)\in\overline{G}^+,$$

where $\Phi(z)=\frac{2}{\pi^{1/2}}\int_0^z e^{-\zeta^2}d\,\zeta$ is the error integral (see, e.g., [17], Ch. 3), and m_0 is an arbitrary constant satisfying the inequality $m_0\le 1/2$, we obtain the estimate for the derivatives in t

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\,V^+(x,t)\right| \le M\left[1+\varepsilon^{2(n+1-k_0)}\right]\,\exp^{-m_1\frac{x^2}{\varepsilon^2}}, \quad (x,t)\in\overline{G}^+, \quad 2k_0\le K,$$

where m_1 is an arbitrary constant in the interval $(0,1/2)$. Taking into account this estimate and using the change of variables $\xi=\varepsilon^{-1}x$, $\tau=t$ in the problem (3.18), we find the estimate for derivatives of the solution written in the variables ξ, τ. In the original variables we obtain the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,V^+(x,t)\right| \le M\left[\varepsilon^{-k}+\varepsilon^{2(n+1-k_0)-k}\right]\,\exp^{-m_1\frac{x^2}{\varepsilon^2}}, \tag{3.25}$$

$$(x,t) \in \overline{G}^+, \quad k + 2k_0 \le K;$$

where $K = K_{(3.22)} = l^*_{0\,(3.23)}$ with $m_1 < 1/2$.

In a similar way, we find estimates for the components $U^-(x,t)$ and $V^-(x,t)$ on the set $\overline{G}^-$

$$\begin{aligned} \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, U^-(x,t) \right| &\le M \left[1 + \varepsilon^{2(n+1-k-k_0)} \right], \\ \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, V^-(x,t) \right| &\le M \left[\varepsilon^{-k} + \varepsilon^{2(n+1-k_0)-k} \right] \exp^{-m_1 \frac{x^2}{\varepsilon^2}}, \\ (x,t) \in \overline{G}^-, \ & k + 2k_0 \le K, \end{aligned} \tag{3.26}$$

where $K = K_{(3.25)}$ with $m_1 = m_{1\,(3.25)}$.

Theorem 6. *Let for the data of the initial-boundary value problem* (2.2), (2.1), (2.3) *the conditions* $f \in C^{l_1}(\overline{G}^+) \cap C^{l_1}(\overline{G}^-)$, $\varphi \in C^{l_1}(S_0^+) \cap C^{l_1}(S_0^-) \cap C^{l_1}(\overline{S}^L)$, *where* $l_1 = l_0 + \alpha$, $l_0 = l_0^* + 2n$, $l_0 = l_{0\,(3.23)} > 0$, $l_0^* > 0$, $n > 0$, $\alpha > 0$, *and also* (3.23), (3.24) *be satisfied. Then, for the components* $U^+(x,t)$, $V^+(x,t)$, $(x,t) \in \overline{G}^+$ *and* $U^-(x,t)$, $V^-(x,t)$, $(x,t) \in \overline{G}^-$ *in the representation* (3.15) *of the solution to the initial-boundary value problem, the estimates* (3.22), (3.25) *and* (3.26), *are valid, respectively, where* $K = l^*_{0\,(3.23)}$.

Remark 7. Let the hypotheses of Theorem 6 be fulfilled, where $l_0^* = 4$ and $n = 1$. Then, for the components $U^+(x,t)$, $V^+(x,t)$, $(x,t) \in \overline{G}^+$ and $U^-(x,t)$, $V^-(x,t)$, $(x,t) \in \overline{G}^-$ in the representation (3.15) of the solution to the initial-boundary value problem (2.2), (2.1), (2.3), the following estimates are valid:

$$\begin{aligned} \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, U^+(x,t) \right| &\le M \left[1 + \varepsilon^{2(n+1-k-k_0)} \right], && (x,t) \in \overline{G}^+; \\ \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, V^+(x,t) \right| &\le M \left[\varepsilon^{-k} + \varepsilon^{2(n+1-k_0)-k} \right] \exp^{-m_1 \frac{x^2}{\varepsilon^2}}, && (x,t) \in \overline{G}^+; \\ \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, U^-(x,t) \right| &\le M \left[1 + \varepsilon^{2(n+1-k-k_0)} \right], && (x,t) \in \overline{G}^-; \\ \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} \, V^-(x,t) \right| &\le M \left[\varepsilon^{-k} + \varepsilon^{2(n+1-k_0)-k} \right] \exp^{-m_1 \frac{x^2}{\varepsilon^2}}, && (x,t) \in \overline{G}^-, \\ & k + 2k_0 \le K, \end{aligned} \tag{3.27}$$

where $K = 4$ and $m_1 = m_{1\,(3.25)}$.

For the components $V^+(x,\,t)$ and $V^-(x,\,t)$ we have also the estimates

$$\begin{aligned} \left| V^+(x,\,t) \right| &\le M \, \exp(-m\varepsilon^{-1}x), \quad (x,\,t) \in \overline{G}^{\,+}, \\ \left| V^-(x,\,t) \right| &\le M \, \exp(m\varepsilon^{-1}x), \qquad (x,\,t) \in \overline{G}^{\,-}, \end{aligned} \tag{3.28}$$

where m is a positive arbitrary constant satisfying the condition $m \le 1$. □

Remark 8. From the bounds (3.27) and (3.28) it follows that only an interior layer appears in the solution of the boundary value problem in a neighbourhood of the degeneration line, but boundary layers do not arise. The reason is that the convective term is negative in G^- and positive in G^+ and therefore the characteristics of the reduced equation enter into the domain $\overline{G}$, and they are not tangent to the lateral boundary of the domain $\overline{G}$. □

Remark 9. In a similar way, it is possible to obtain estimates analogous to those (3.27), (3.28) for the following problem with variable coefficients:

$$\begin{aligned} L\,u(x,t) \quad &\equiv \quad \left\{\varepsilon^2\,\frac{\partial^2}{\partial x^2} + xb(x)\,\frac{\partial}{\partial x} - p(x,t)\frac{\partial}{\partial t} - c(x,t)\right\} u(x,t) = f(x,t), \\ &\qquad\qquad (x,t) \in G \setminus S^{\pm}, \\ u(x,t) \quad &= \quad \varphi(x,t), \quad (x,t) \in S, \end{aligned}$$

where b, c, p are smooth functions such that $b(x) > 0$, $c(x,t) \geq 0$, $p(x,t) > 0$, and the function $f(x,t)$ has a discontinuity. □

4. Classical finite difference scheme

Here we consider a finite difference scheme that is constructed based on a classical approximation of the problem (2.2), (2.1), (2.3) on a uniform grid.

4.1. Construct a finite difference scheme for the problem (2.2), (2.1), (2.3).

On the set $\overline{G}$ we introduce the rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the intervals $[-d,\, d]$ and $[0,\, T]$ such that $x = 0 \in \overline{\omega}$. The mesh points in $\overline{\omega}$ are denoted by x^i, $i = 0, 1, \cdots, N$ and the step sizes by $h^i = x^{i+1} - x^i$, $i = 0, 1, \cdots, N-1$. On the other hand, $\overline{\omega}_0$ is a uniform mesh with the time step $\tau = T/N_0$. Here $N+1$ and N_0+1 are the numbers of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

Outside the set $S_h^{\pm} = S^{\pm} \cap G_h$, we approximate the problem (2.2), (2.1), (2.3) by the finite difference scheme

$$\Lambda\, z(x,t) \equiv \left\{\varepsilon^2\, \delta_{\overline{x}\widehat{x}} + x\, \delta_x^* - \delta_{\overline{t}} - 1\right\} z(x,t) = f(x,t), \quad (x,t) \in G_h \setminus S^{\pm}, \tag{4.2a}$$

and also we impose the continuity of the discrete flux in $S_h^{\pm}$, i.e.,

$$\Lambda^{\pm}\, z(x,t) \equiv \varepsilon \left[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)\right] = 0, \quad (x,t) \in S_h^{\pm}, \tag{4.2b}$$

and on the boundary S_h we have the condition

$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{4.2c}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$, $S_h^{\pm} = S^{\pm} \cap \overline{G}_h$,

$$\delta_x^*\, z(x,t) = \begin{cases} \delta_x\, z(x,t), & x > 0, \\ \delta_{\overline{x}}\, z(x,t), & x < 0, \end{cases}$$

is the monotone approximation of the derivative $\dfrac{\partial}{\partial x}\, u(x,t)$ in the differential equation, $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the central second-order difference derivative on a nonuniform grid, given by

$$\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2\left(h^i + h^{i-1}\right)^{-1}\left[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)\right], \quad (x,t) = (x^i,t) \in G_h.$$

$\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$ are the first-order (forward and backward) difference derivatives

$$\begin{aligned} \delta_x\, z(x,t) &= \left(h^i\right)^{-1}\left[z(x^{i+1},t) - z(x,t)\right], \qquad (x,t) = (x^i,t) \in G_h, \\ \delta_{\overline{x}}\, z(x,t) &= (h^{i-1})^{-1}\left[z(x^i,t) - z(x^{i-1},t)\right], \quad (x,t) = (x^i,t) \in G_h, \end{aligned}$$

and $\delta_{\overline{t}}\, z(x,t)$ is the backward difference given by

$$\delta_{\overline{t}}\, z(x,t) = (\tau)^{-1}\left[z(x^i,t) - z(x^i,t-\tau)\right], \quad (x^i,t), (x^i,t-\tau) \in G_h.$$

The finite difference scheme (4.2), (4.1) is monotone ε-uniformly (see, e.g., [11], and also monotone grid approximations in [13], Ch. 1, §2).

4.2. Consider convergence of the difference scheme (4.2), (4.1) in the case of a uniform grid.

The following comparison theorem is valid.

Theorem 10. *Let the functions $z^1(x,t)$, $z^2(x,t)$, $(x,t) \in \overline{G}_h$ satisfy the conditions*

$$\begin{array}{lcll} \Lambda z^1(x,\,t) & \le & \Lambda z^2(x,t), & (x,\,t) \in G_h \setminus S^{\pm}, \\ \Lambda^{\pm}\, z^1(x,\,t) & \le & \Lambda^{\pm}\, z^2(x,\,t), & (x^*,\,t) \in S_h^{\pm}, \\ z^1(x,\,t) & \ge & z^2(x,\,t), & (x,\,t) \in S. \end{array}$$

Then

$$z^1(x,\,t) \ge z^2(x,\,t), \quad (x,t) \in \overline{G}_h.$$

Consider the difference scheme on the uniform grid (in space and time variables)

$$\overline{G}_h = \overline{G}_h^{\,u} \equiv \overline{\omega}^{\,u} \times \overline{\omega}_0^u. \tag{4.3}$$

Taking into account the *a priori* estimates (3.27) in Remark 7, using the comparison Theorem 10, and applying a standard technique from [16], based on appropriate bounds for the truncation error and the discrete maximum principle, we obtain the estimate

$$|u(x,t) - z(x,t)| \le M \,\left[\left(\varepsilon + N^{-1}\right)^{-1}\, N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h^{\,u}. \tag{4.4}$$

Thus, the scheme (4.2), (4.3) converges under the condition $N^{-1} \ll \varepsilon$, or more precisely

$$N^{-1} = o(\varepsilon), \quad N_0^{-1} = o(1). \tag{4.5}$$

Theorem 11. *Let the solution $u(x,t)$ of the problem* (2.2), (2.1), (2.3) *satisfy the estimates* (3.27) *in Remark* 7*. Then, the condition* (4.5) *is sufficient for convergence of the difference scheme* (4.2), (4.3)*. The solution of the difference scheme* (4.2), (4.3) *satisfies the estimate* (4.4).

Remark 12. Necessary conditions for convergence of difference scheme (4.2) on the uniform grid (4.3) are discussed in Subsection 6.4. □

5. Special finite difference scheme

Here we consider a special finite difference scheme which converges ε-uniformly. On the set $\overline{G}$ we introduce so called Shishkin grid condensed in a neighbourhood of the interior layer (see, e.g., [13, 14, 15] and the bibliography there)

$$\overline{G}_h = \overline{G}_h^{\,s} \equiv \overline{\omega}^{\,s} \times \overline{\omega}_0^u, \tag{5.1a}$$

where $\overline{\omega}^{\,s}$ is a piecewise-uniform mesh constructed in the following way. We divide the interval $[-d,\,d]$ into three parts $[-d,\,-\sigma]$, $[-\sigma,\,\sigma]$ and $[\sigma,\,d]$. The step-sizes in the mesh $\overline{\omega}^{\,s}$ are $h^{(1)} = 4\,\sigma\,N^{-1}$ on $[-\sigma,\,\sigma]$ and $h^{(2)} = 4\,(d-\sigma)\,N^{-1}$ on $[-d,\,-\sigma]$ and $[\sigma,\,d]$. Here σ is defined by the relation

$$\sigma = \sigma(\varepsilon,\,N) = \min\left[4^{-1}\,d,\,m^{-1}\,\varepsilon\,\ln N\right], \tag{5.1b}$$

where m is an arbitrary number from the interval $(0, m_2)$ with $m_2 = m_{(3.28)}$. The mesh $\overline{\omega}^{\,s}$ and the grid $\overline{G}_h^{\,s}$ are constructed.

For the solution of the difference scheme (4.2) on the grid (5.1), using a standard technique from [16], we obtain the estimate

$$|u(x,t) - z(x,t)| \le M \,\left\{N^{-1}\,\min\,\left[\varepsilon^{-1},\,\ln N\right] + N_0^{-1}\right\}, \quad (x,t) \in \overline{G}_h^{\,s}, \tag{5.2a}$$

and also the ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M\,\left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h^{\,s}. \tag{5.2b}$$

The difference scheme (4.2), (5.1) converges ε-uniformly with the first accuracy order in time and with the first order up to a logarithmic factor in space. For fixed values of ε, the scheme converges with the first order in time and space.

Theorem 13. *Let the solution $u(x,t)$ of the problem* (2.2), (2.1), (2.3) *satisfy the estimates* (3.27) *in Remark* 7. *Then, the difference scheme* (4.2), (5.1) *converges ε-uniformly. The solution of the difference scheme* (4.2), (5.1) *satisfies the estimate* (5.2).

6. Numerical experiments

In this section we present results of numerical experiments that illustrate the theoretical results.

6.1. In this subsection we discuss the qualitative behaviour of the solutions to problem (2.2), (2.1).

The first example that we consider is given by

$$\begin{cases} \varepsilon^2 u_{xx} - u_t + x u_x - u &= f(x,t), & (x,t) \in G, \\ u_x(x+0,t) - u_x(x-0,t) &= 0, & (x,t) \in S^{\pm}, \\ u(x,t) &= 0, & (x,t) \in S, \end{cases} \tag{6.1}$$

where $\overline{G} = [-1,1] \times [0,1]$, $S^{\pm} = \{x = 0\} \times (0,1]$, and

$$f(x,t) = \begin{cases} -(t^3 + x\sin^3 t), & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{6.2}$$

which solution is unknown.

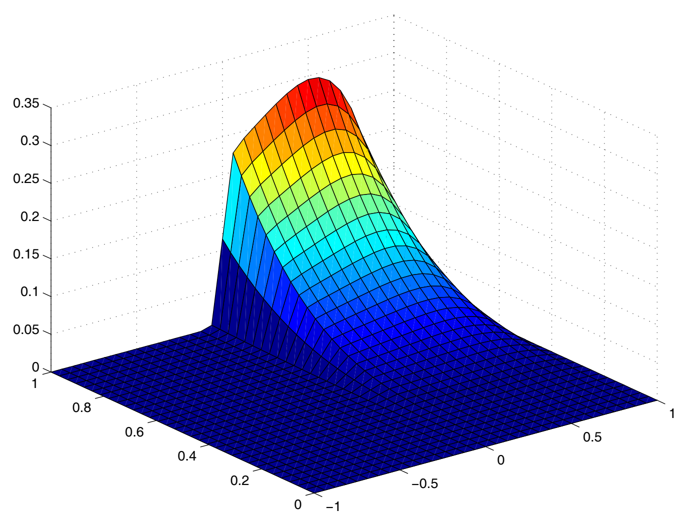

FIGURE 1. Numerical solution of problem (6.1), (6.2) for $\varepsilon = 2^{-5}$ with $N = N_0 = 32$ on a uniform grid

Plots of discrete solutions of the problem (6.1), (6.2) are given on Figure 1 and Figure 2, on a uniform and the piecewise-uniform Shishkin grids, respectively. From

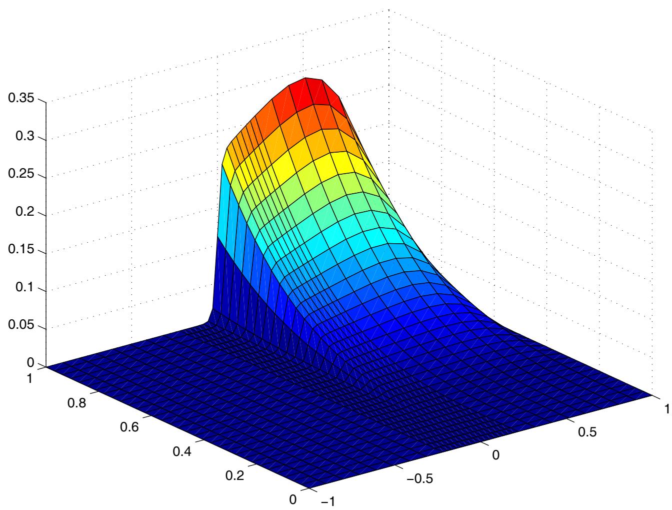

FIGURE 2. Numerical solution of problem (6.1), (6.2) for $\varepsilon = 2^{-5}$ with $N = N_0 = 32$ on the Shishkin grid

Figure 1 it is seen that the scheme on a uniform grid "smears out" the interior layer. On the other hand, see Figure 2, the piecewise-uniform grid resolves the interior layer, i.e., visually, the interior layer on the uniform grid is seen more wider than it is on the piecewise-uniform grid. Also these figures show that there are no boundary layer according with Remark 8.

To show better the differences between the uniform and Shishkin grids, we include Figure 3 showing a zoom of the numerical solution near $x = 0$ for the final time $t = 1$, on both meshes; from them we clearly see that the interior layer on the uniform grid is more wider than on the piecewise-uniform grid.

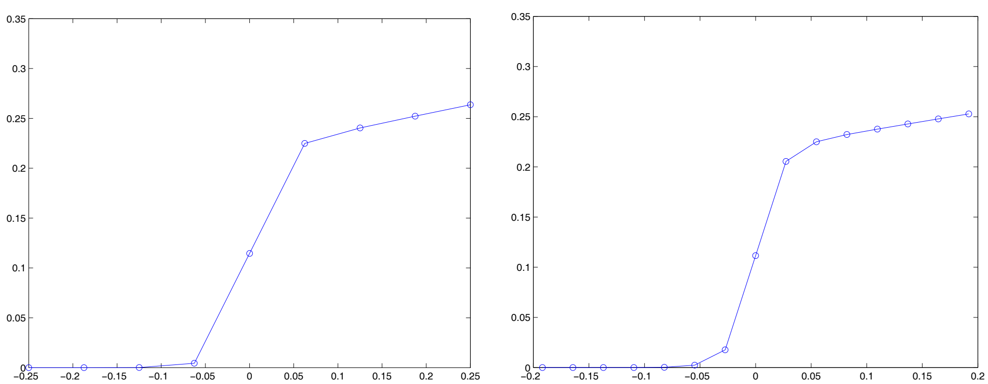

FIGURE 3. Zoom of the numerical solution of problem (6.1), (6.2) at $t = 1$ for $\varepsilon = 2^{-5}$ on the uniform grid (from the left) and on the Shishkin grid (from the right)

In the second example, we consider the problem (6.1) with a continuous function $f(x, t)$. Plot of a discrete solution of the problem (6.1) is given on Figure 4 in the case of the continuous right-hand side prescribed by

$$f(x,t) = -t^2, \quad x \in [-1,1], \tag{6.3}$$

for a difference scheme on a uniform grid. From Figure 4 it is observed that in the

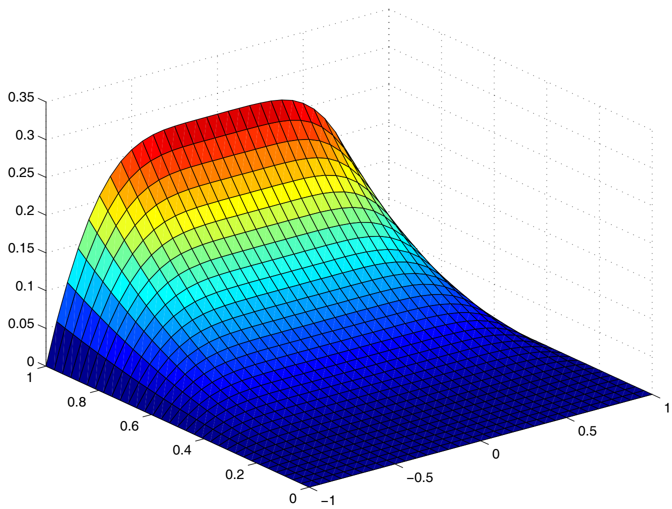

FIGURE 4. Numerical solution of problem (6.1), (6.3) for $\varepsilon = 2^{-5}$ with $N = N_0 = 32$ on uniform grid

problem under consideration, when the right-hand side is a continuous function, an interior layer does not arise!

6.2. It is of interest also to reveal the influence of the value of the parameter ε on the "real" error of the discrete solution. To estimate the computed errors for a scheme on a *piecewise-uniform grid* with $N = N_0$, we use a variant of the double mesh technique (see, e.g., [2], Ch. 8). Consequently, at times $t_n = n\tau$ for each point $x_j,\ j = 0, 1, \ldots, N$, for the perturbation parameter ε, the error $D_{j,n}^{\varepsilon,N,N_0}$ is defined by using

$$D_{j,n}^{\varepsilon,N,N_0} = \left| U_{j,n}^{\varepsilon,N,N_0} - \overline{U}_{j,n}^{\varepsilon,2N,2N_0} \right|,$$

computed on the nodes $(x_j, t_n) \in \overline{G}_h$, where $U_{j,n}^{\varepsilon,N,N_0}$ is the numerical solution obtained using the constant time step $\tau = 1/N_0$ and $(N+1)$ points in the spatial mesh, and $\overline{U}_{j,n}^{\varepsilon,2N,2N_0}$ is the piecewise linear interpolant of the solution $U_{j,n}^{\varepsilon,2N,2N_0}$, which is computed using $\frac{\tau}{2}$ as a time step and $(2N+1)$ points in the spatial mesh but with the same transition parameters as in the original mesh. For each fixed value of ε, the maximum global error D^{ε,N,N_0} is estimated by

$$D^{\varepsilon,N,N_0} = \max_{j,n} D_{j,n}^{\varepsilon,N,N_0},$$

and therefore, in a standard way, the computed order of convergence q is given by

$$q = q(\varepsilon, N, N_0) = \frac{\log\left(D^{\varepsilon,N,N_0}/D^{\varepsilon,2N,2N_0}\right)}{\log 2}.$$

From these values we obtain the ε-uniform error D^{N,N_0} and the ε-uniform order of convergence q_{uni} in the standard way

$$D^{N,N_0} = \max_{\varepsilon} D^{\varepsilon,N,N_0}, \quad q_{uni} = q_{uni}(N,N_0) = \frac{\log\left(D^{N,N_0}/D^{2N,2N_0}\right)}{\log 2}. \tag{6.4}$$

The results of the numerical experiments for the example (6.1), (6.2) in the case of the piecewise-uniform Shishkin grid with $N = N_0$ are given in Table 1. To specify the transition parameter σ in (5.1b), we have taken the constant $m = 1/2$.

Corollary 14. From Table 1 it follows that the discrete solution of the classical difference scheme on the piecewise-uniform Shishkin grid converges ε-uniformly with the accuracy order close to one when N grows, in agreement with the theoretical results (see statements of Theorem 13). □

TABLE 1. Maximum errors and orders of convergence on the Shishkin grid for problem (6.1), (6.2) with $N = N_0$

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
2^{-5}	0.106E-1 0.806	0.605E-2 0.970	0.309E-2 1.042	0.150E-2 1.082	0.709E-3 1.106	0.329E-3 1.124	0.151E-3
2^{-6}	0.122E-1 0.788	0.705E-2 0.888	0.381E-2 0.970	0.195E-2 1.007	0.969E-3 1.026	0.476E-3 1.036	0.232E-3
2^{-7}	0.129E-1 0.754	0.763E-2 0.873	0.417E-2 0.943	0.217E-2 0.981	0.110E-2 1.001	0.549E-3 1.010	0.272E-3
2^{-8}	0.132E-1 0.739	0.789E-2 0.864	0.434E-2 0.931	0.228E-2 0.971	0.116E-2 0.990	0.585E-3 0.999	0.293E-3
2^{-9}	0.133E-1 0.732	0.802E-2 0.857	0.443E-2 0.928	0.233E-2 0.965	0.119E-2 0.985	0.603E-3 0.995	0.303E-3
2^{-10}	0.134E-1 0.729	0.808E-2 0.853	0.447E-2 0.926	0.235E-2 0.962	0.121E-2 0.982	0.612E-3 0.992	0.308E-3
...	...	...	...	...	...	...	...
2^{-15}	0.135E-1 0.726	0.814E-2 0.850	0.451E-2 0.923	0.238E-2 0.960	0.122E-2 0.980	0.621E-3 0.990	0.312E-3
D^{N,N_0} q_{uni}	0.135E-1 **0.726**	0.814E-2 **0.850**	0.451E-2 **0.923**	0.238E-2 **0.960**	0.122E-2 **0.980**	0.621E-3 **0.990**	0.312E-3

6.3. Consider a scheme on a uniform grid, assuming that $N = N_0$. In this case, the value $D_{j,n}^{\varepsilon,N,N_0}$ is defined by the formula

$$D_{j,n}^{\varepsilon,N,N_0} = \left| U_{j,n}^{\varepsilon,N,N_0} - \overline{U}_{j,n}^{\varepsilon,2N,2N_0} \right|,$$

where the function $U_{j,n}^{\varepsilon,N,N_0}$ is computed on the uniform grid, and the interpolant $\overline{U}_{j,n}^{\varepsilon,2N,2N_0}$ is constructed using the function $U_{j,n}^{\varepsilon,2N,2N_0}$ computed on the piecewise-uniform grid.

The results of the numerical experiments for the scheme on a uniform grid with $N = N_0$ for the example (6.1), (6.2), are given in Table 2. Under parameters ε and N, for which the scheme on the uniform grid converges, i.e., under the condition $h \le \varepsilon$ (or $\varepsilon \ge 2\,N^{-1}$), we observe the *"self-similar nature" in the behavior of the errors* to discrete solutions with respect to $\varepsilon^{-1}h$, i.e., the errors arranged in diagonal elements of the table for $\varepsilon \ge h$ change unessentially as N grows; moreover, for fixed values of ε, orders of the convergence rate are close to one when N grows. For $\varepsilon < h$ and fixed N, and when ε decreases, the behaviour of the errors is *irregular*.

Corollary 15. From Table 2 it follows that the discrete solution of the classical difference scheme on the uniform grid does not converge ε-uniformly. Largest errors of order $\mathcal{O}(1)$ are observed for $\varepsilon \approx h$; the scheme converges under the condition $\varepsilon^{-1}h << 1$ ($\varepsilon^{-1}N^{-1} << 1$). For $\varepsilon^{-1}h \leq 1$, errors decrease when N grows as $\mathcal{O}(\varepsilon^{-1} N^{-1})$; the scheme converges with order close to one for fixed values of ε. For $\varepsilon^{-1} h > 1$, the behaviour of errors is not regular when N grows. Thus, these results are according to the statement of Theorem 11, when $N = N_0$. □

TABLE 2. Maximum errors and orders of convergence on a uniform grid for problem (6.1), (6.2) with $N = N_0$

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
2^{-5}	0.640E-2 0.472	**0.461**E-2 0.855	0.255E-2 1.297	0.104E-2 1.118	0.478E-3 0.826	0.270E-3 0.913	0.143E-3
2^{-6}	0.730E-2 1.039	0.355E-2 -0.577	**0.530E-2** 0.850	0.294E-2 1.293	0.120E-2 1.118	0.553E-3 1.115	0.255E-3
2^{-7}	0.779E-2 1.078	0.369E-2 0.059	0.354E-2 -0.658	**0.559E-2** 0.864	0.307E-2 1.248	0.129E-2 1.123	0.594E-3
2^{-8}	0.805E-2 1.053	0.388E-2 1.064	0.186E-2 -0.926	0.353E-2 -0.696	**0.571E-2** 0.859	0.315E-2 1.239	0.133E-2
2^{-9}	0.818E-2 1.040	0.398E-2 1.040	0.194E-2 0.969	0.990E-3 -1.831	0.352E-2 -0.716	**0.578E-2** 0.860	0.319E-2
2^{-10}	0.825E-2 1.033	0.403E-2 1.027	0.198E-2 1.033	0.967E-3 -0.018	0.980E-3 -1.843	0.352E-2 -0.727	**0.582E-2**
...	...	...	...	...	...	...	...
2^{-15}	0.832E-2 1.026	0.408E-2 1.014	0.202E-2 1.007	0.101E-2 1.005	0.501E-3 1.005	0.250E-3 1.007	0.124E-3
D^{N,N_0} q_{uni}	0.832E-2 0.850	0.461E-2 -0.200	0.530E-2 -0.076	0.559E-2 -0.032	0.571E-2 -0.018	0.578E-2 -0.009	0.582E-2

6.4. In Section 4 and in Corollary 15 it is shown theoretically and numerically that in the case of difference scheme (4.2) on uniform grid (4.3), condition (4.5) is sufficient for convergence of this scheme as N, $N_0 \to 0$. Necessity of condition (4.5) for convergence of this scheme can be established theoretically (on the base of analytical investigations) and also numerically that we show in this subsection, using numerical experiments.

For this it is sufficient to show that for a problem from the class of singularly perturbed problem under consideration, the following estimate from below is satisfied:

$$(6.5)\quad \max |u(x,t) - z(x,t)| \geq m \left[\varepsilon^{-1} N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h^u \ \ for\ \ \varepsilon \leq N^{-1}.$$

The results of the numerical experiments given in Table 2 show that in the domain of values of ε and N, for which the scheme converges, the convergence take place for $\varepsilon^{-1} N^{-1} \to 0$. Moreover, the *minimal order* of the convergence rate with respect to N is close to one that goes with the estimate

$$(6.6)\quad \max |u(x,t) - z(x,t)| \geq m\, \varepsilon^{-1} N^{-1}, \quad (x,t) \in \overline{G}_h^u \ \ for\ \ \varepsilon \leq N^{-1}, \ \ N = N_0.$$

For solutions depending only on the temporal variable, it is not difficult to write out a boundary value problem and a corresponding difference scheme, such that its numerical solution satisfies the estimate

$$(6.7)\qquad \max |u(x,t) - z(x,t)| \geq m\, N_0^{-1}, \quad (x,t) \in \overline{G}_h^u.$$

Corollary 16. From estimates (6.6) and (6.7) the estimate (6.5) follows. Thus, it is shown numerically that condition (4.5) is necessary for the convergence of the scheme (4.2) on uniform grid (4.3). □

6.5. Here we discuss influence of compatibility conditions on accuracy of the discrete solution. In the next example we just change the right-hand side of problem (6.1) by the following:

$$f(x,t) = \begin{cases} -(2t - t^2 + x/2), & \text{if } \ x > 0, \\ 0, & \text{if } \ x < 0, \end{cases} \tag{6.8}$$

and again the solution is unknown.

The plot of the discrete solution of problem (6.1), (6.8) on the piecewise-uniform Shishkin grid is given in Figure 5. The results of the numerical experiments are given in Table 3. From Table 3 it follows that the discrete solution of the classical difference scheme on the piecewise-uniform Shishkin grid (with the constant $m = 1/2$ for th parameter σ in (5.1b)) converges ε-uniformly with the accuracy order close to ≈ 0.7 when N grows.

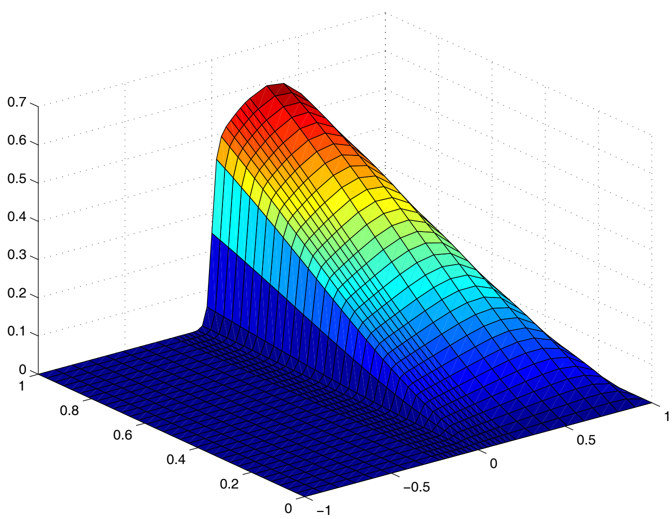

FIGURE 5. Numerical solution of problem (6.1), (6.8) for $\varepsilon = 2^{-5}$ with $N = N_0 = 32$ on the Shishkin grid

Remark 17. Note that the compatibility conditions imposed on the data of the problem in Theorem 6 and Remark 7 are sufficient for convergence of the difference scheme (4.2), (5.1) with the first accuracy order in t and close to one in x (but these conditions are not necessary). As it is seen from Table 1 to example (6.1), (6.2), the theoretical conditions can be weekend without reduction of the convergence rate of the scheme. So, for this example we have the following conditions:

$$\frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} f(\pm 0, 0) = 0, \quad 0 \le k + k_0 \le 2, \quad \frac{\partial^3}{\partial t^3} f(\pm 0, 0) \ne 0;$$

$$\frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} f(1, 0) = 0, \quad 0 \le k + k_0 \le 2, \quad \frac{\partial^3}{\partial t^3} f(1, 0) \ne 0;$$

which are less restrictive than the conditions in Remark 7, and the obtained accuracy order close to one. Similar results are obtained also, e.g., when the right-hand side takes the form $f(x,t) = -(t^2 + x \sin^2 t)$ for $x > 0$ and $f(x,t) = 0)$ for $x < 0$.

However, in the case of example (6.1), (6.8), we have only rather week compatibility conditions

$$f(\pm 0, 0) = 0, \quad \frac{\partial}{\partial x} f(\pm 0, 0) \neq 0, \quad \frac{\partial}{\partial t} f(\pm 0, 0) \neq 0; \quad f(1, 0) \neq 0,$$

and this brings to significant reduction of the convergence rate of the scheme (up to ≈ 0.7) that is seen from Table 3 (where again we have taken $m = 1/2$ in (5.1b) to define the mesh). Thus, compatibility conditions could not be too weakened without essential lost of accuracy. □

TABLE 3. Maximum errors and orders of convergence on the Shishkin grid for problem (6.1), (6.8) with $N = N_0$

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
2^{-5}	0.235E-1 0.693	0.145E-1 0.767	0.854E-2 0.827	0.481E-2 0.879	0.262E-2 0.935	0.137E-2 0.975	0.696E-3
2^{-6}	0.261E-1 0.687	0.162E-1 0.693	0.100E-1 0.735	0.603E-2 0.748	0.359E-2 0.759	0.212E-2 0.779	0.124E-2
2^{-7}	0.263E-1 0.606	0.173E-1 0.697	0.107E-1 0.709	0.652E-2 0.706	0.400E-2 0.692	0.247E-2 0.682	0.154E-2
2^{-8}	0.262E-1 0.585	0.175E-1 0.673	0.110E-1 0.702	0.674E-2 0.692	0.417E-2 0.675	0.261E-2 0.649	0.167E-2
2^{-9}	0.267E-1 0.581	0.178E-1 0.676	0.112E-1 0.704	0.685E-2 0.689	0.425E-2 0.670	0.267E-2 0.640	0.171E-2
2^{-10}	0.269E-1 0.580	0.180E-1 0.676	0.113E-1 0.701	0.692E-2 0.692	0.428E-2 0.668	0.270E-2 0.637	0.173E-2
...	...	...	...	...	...	...	...
2^{-15}	0.271E-1 0.582	0.181E-1 0.679	0.113E-1 0.700	0.696E-2 0.690	0.432E-2 0.668	0.272E-2 0.635	0.175E-2
D^{N,N_0} q_{uni}	0.271E-1 0.582	0.181E-1 0.679	0.113E-1 0.700	0.696E-2 0.690	0.432E-2 0.668	0.272E-2 0.635	0.175E-2

7. Summary

• Grid approximation of an initial-boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation with a convective flux which is directed from the lateral boundary inside the domain in the case when
a) the convective term degenerates inside the domain according to a linear law and
b) the right-hand side has the first kind discontinuity on the degeneration line.

• For small values of the parameter ε, an interior layer arises in the solution of this type of problem. When the right-hand side is a continuous function, an interior layer does not arise.

• If the finite difference scheme is constructed on a uniform grid then the method is not uniformly-convergent. On the other hand, if the finite difference scheme is constructed on a piecewise-uniform mesh condensing in a neighborhood of the interior layer, the solution of this scheme converges ε-uniformly at the rate $\mathcal{O}(N^{-1} \ln N + N_0^{-1})$, where $N+1$ and N_0+1 are the numbers of nodes in the space and time meshes, respectively. Results of some numerical experiments support the theoretical results.

• The developed technique to construct the ε-uniformly convergent scheme for the problem with the degenerating linearly convective term, is also applicable, in principle, to construct ε-uniformly convergent schemes in the case of problems where the convective term is given by $x^{2p+1} \dfrac{\partial}{\partial x} u(x,t)$, where $p \geq 1$ is a positive integer. We plan to study such a problem.

References

[1] B.M. Budak, A.A. Samarskii, A.N.Tikhonov. *Problems of Mathematical Physics*. Nauka, Moscow, 1980 (in Russian); Spanish Translation: *B.M. Budak, A.A. Samarskii, A.N.Tikhonov*. Problemas de la fisica matematica. Tomo 1, 2. Mir, Moscow, 1984.

[2] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin. *Robust Computational Techniques for Boundary Layers*. Applied Mathematics, **16**. Chapman and Hall/CRC, 2000.

[3] R.K. Dunne, E. O'Riordan, G.I. Shishkin. *A fitted mesh method for a class of singularly perturbed parabolic problems with a boundary turning point*. Computational Methods in Applied Mathematics, **3** (2003), pp. 361–372.

[4] P.W. Hemker, G.I. Shishkin. *Discrete approximation of singularly perturbed parabolic PDEs with a discontinuous initial condition*. Comput. Fluid Dynamics J., **2** (1994), pp. 375–392.

[5] O.A. Ladyzhenskaya, and N.N. Uraltseva. *Linear and quasilinear equations of elliptic type*. Nauka, Moscow, 1964 (in Russian); translation *O.A. Ladyzhenskaya, and N.N. Uraltseva*. Linear and Quasilinear Elliptic Equations, Academic Press, New York–London.

[6] O.A. Ladyzhenskaya, and N.N. Uraltseva. *Linear and Quasilinear Equations of Elliptic Type*. Nauka, Moscow, 1973 (in Russian).

[7] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Uraltseva. *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow, 1967. (in Russian); Translations of Mathematical Monographs, 1967, **23**, American Mathematical Society, Providence, R.I.

[8] J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. *Fitted Numerical Methods for Singular Perturbation Problems.* World Scientific, Singapore, 1996.

[9] E. O'Riordan, G.I. Shishkin. *Singularly perturbed parabolic problems with non-smooth data*. Journal of Computational and Applied Mathematics, **166** (2004), pp. 233–245.

[10] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical Methods for Singularly Perturbed Differential Equations. Convection-Diffusion-Reaction and Flow Problems, second edition.* In: Springer Series in Computational Mathematics, **24**, Springer-Verlag, Berlin, 2008.

[11] A.A. Samarskii. *Theory of Difference Schemes.* Nauka, Moscow, 1989 (in Russian).

[12] H. Schlichting. *Boundary Layer Theory.* McGraw-Hill, New York, seventh edition, 1979.

[13] G.I. Shishkin. *Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations.* Ural Branch of RAS, Ekaterinburg, 1992 (in Russian).

[14] G.I. Shishkin. *On numerical methods on adaptive meshes for a singularly perturbed reaction-diffusion equation with a moving concentrated source.* Finite Difference Schemes (2000), pp. 205–214, Lith. Acad. Sci., Inst. Math. Inform., Vilnius.

[15] G.I. Shishkin. *Grid approximation of a singularly perturbed parabolic equation on a composed domain with a moving interface containing a concentrated source*. Comput. Maths. Math. Phys., **43** (2003), pp. 1738–1755.

[16] G.I. Shishkin, L.P. Shishkina.*Difference Methods for Singular Perturbation Problems*. Series: Monographs & Surveys in Pure & Applied Math. Chapman and Hall/CRC, 2009.

[17] A.N. Tikhonov, A.A. Samarskii. *Equations of Mathematical Physics*. Dover Publications, Inc., New York, 1990.

Department of Applied Mathematics, University of Zaragoza, Zaragoza, Spain
E-mail: `clavero@unizar.es` and `jlgracia@unizar.es`

Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia
E-mail: `shishkin@imm.uran.ru` and `lida@convex.ru`