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THE FINITE ELEMENT METHOD OF A EULER SCHEME FOR STOCHASTIC NAVIER-STOKES EQUATIONS INVOLVING THE TURBULENT COMPONENT

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Abstract. In this paper we study the finite element approximation for stochastic Navier-Stokes equations including a turbulent part. The discretization for space is derived by finite element method, and we use the backward Euler scheme in time discretization. We apply the generalized L_2-projection operator to approximate the noise term. Under suitable assumptions, strong convergence error estimations with respect to the fully discrete scheme are well proved.

Key words. stochastic Navier-Stokes equations, finite element method, discrete scheme, and error estimation.

1. Introduction

Let $\Omega \in R^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$. In this paper, we consider finite element approximation of stochastic Navier-Stokes equations with the turbulent term

$$\partial_t u = \triangle u - (u \cdot \nabla)u - \nabla p + f(u) + [(\sigma \cdot \nabla)u - \nabla \tilde{p} + g(u)]\dot{W}, \tag{1}$$

$$u(0) = u_0, \quad \nabla \cdot u = 0, \tag{2}$$

on Ω in a finite time interval $[0, T]$. The turbulent term is driven by the white noise $\dot{W}$. In this article, $\dot{W}$ denotes a time derivative of a Hilbert space valued Wiener process. Assumptions on other functions will be specified later.

The stochastic Navier-Stokes equation, which displays the behavior of a viscous velocity field of an incompressible liquid, is widely regarded as one of the most fascinating problems of fluid mechanics, see [1]. A. Bensoussan and R. Temam generally analyze Navier-Stokes equations driven by white noise type random force in [2]. Later, the existence, the uniqueness and other properties of the generalized solutions with respect to stochastic equations have been extensively researched by many authors; see [4], [9], [15], [18], etc. An overview of some developments in the ergodic theory of the stochastically forced Navier-Stokes equations are presented by Jonathan C. Mattingly in [6].

The effective researches about unsteady incompressible stochastic Navier-Stokes equations driven by white noise are considered by R. Mikulevicius and B. L. Rozovskii; see [13] and [14] with a review of relevant recent work. Under some basic assumptions, the existence of a global weak (martingale) solution of the unsteady incompressible stochastic Navier-Stokes equation (1.1) with Cauchy problem is well proved in [17]. Furthermore, R. Mikulevicius and B. L. Rozovskii consider the corresponding fluid dynamics modeled by a stochastic flow in [16].

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The finite element method, which is a common technique for partial differential equations, is widely used to obtain finite dimensional approximations. The ideas based on finite element approximation to investigate stochastic differential equations are well studied in many literatures; see, [3], [10], [12], [19], [21], [22], [23] for some previous work. Yubin Yan consider the semidiscrete Galerkin approximation of a stochastic parabolic partial differential equation in [25]. Later, the fully finite element method for stochastic parabolic partial differential equations driven by white noise is proved and optimal strong convergence error estimates are given in [24]. Semidiscrete finite element approximation of the linear stochastic wave equation with additive noise is well studies in [11]. However, numerical analysis of unsteady incompressible stochastic Navier-Stokes equations has not been thoroughly considered. The major purpose of our paper is to study finite element approximation for unsteady incompressible stochastic Navier-Stokes equations involving the complex turbulent component. In our paper, stochastic Navier-Stokes equations are taken in the generalized sense.

The plan of this paper is as follows. In section 2 useful notations and related properties are introduced. Some important preliminaries are given. The regularity in time of the solution is deduced. In section 3 we consider finite element approximation of stochastic Navier-Stokes equations with turbulent term. The semidiscrete form and the fully discretization are obtained. In section 4 we deduce the main error estimations with respect to the fully discretization of the stochastic equations. Using above-mentioned techniques, we finally complete the proofs of strong convergence error estimates. Section 5 are our conclusions of this paper.

2. Notations and preliminaries

In this section we will introduce some useful notations and some important preliminaries.

Let H be the Hilbert space of real vector functions in $L_2(\Omega)$ with the inner product $(\cdot,\cdot)$. Given integer $m \geq 0$ and $1 \leq p < \infty$, define

$$W^{m,p}(\Omega) = \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega), \forall \alpha, 0 \leq |\alpha| \leq m\},$$

equipped with the norm

$$||u||_{W^{m,p}(\Omega)} = ||u||_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p}, \ 1 \leq p < \infty.$$

Set

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) : u|_{\partial\Omega} = 0\}.$$

Obviously $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ stand for Sobolev spaces on Ω. More explicitly we can write $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and denote $W_0^{m,2}(\Omega)$ as $H_0^m(\Omega)$. It is easy to verify that $W^{0,p}(\Omega) = L_p(\Omega)$.

Moreover we can come to the conclusion that $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces, and $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces. The relevant inner conduct is

$$(u, v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}, \ u, v \in H^m(\Omega).$$

Obviously, the above-mentioned spaces can be extended to vector functions.

As usual, $(\Omega, \mathcal{F}, \mathbf{P})$ denotes a normal filtered probability space with a normal right continuous filtration $(\mathcal{F}_t)$. In our paper, W is a cylindrical Wiener process in a separable Hilbert space H on a normal filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If $\{e_i\}$ is a complete orthonormal basis of H, we can represent

$$W(t) = \sum_{i=1}^{\infty} W^i(t) e_i,$$

where $\{W^i(t)\}$ are mutually independent standard scalar Wiener processes.

Let $\mathbf{E}$ denote the expectation. The following isometry property for the noise holds:

$$\mathbf{E}\Big\| \int_0^s \psi(t) dW(t) \Big\| = \mathbf{E} \int_0^s \|\psi(t)\|^2 dt.$$

Since $\nabla \cdot u = 0$, we consider the Helmholtz decomposition of vector fields. The set of real vector functions φ such that $\nabla \cdot \varphi = 0$ and $\varphi \in C_0^\infty(\Omega)$ is denote by $\varphi \in C_{0,\sigma}^\infty(\Omega)$. Moreover, H_σ denotes the closure of $\varphi \in C_{0,\sigma}^\infty(\Omega)$ in H. Let P be the divergence free projection operator of the Helmholtz decomposition. Obviously, the following lemma holds.

Lemma 2.1. [14] *The divergence free projection operator P can be extended continuously to all $H_2^s(\Omega), s \in (-\infty, \infty)$: there is a constant $L_p > 0$ so that for all $v \in H_2^s(\Omega)$,*

$$\|Pv\|_{s,2} \le L_p \|v\|_{s,2}.$$

More summaries of the Helmholtz decomposition can be found in literatures [13] and [14].

Then we denote the self-adjoint operator in H_σ formally given by $A = -P\Delta$, and let $B(u,v) = P(u \cdot \nabla)v$. The set of real vector functions φ such that $\nabla \cdot \varphi = 0$ and $\varphi \in H_0^1(\Omega)$ is denote by $H_{0,\sigma}^1(\Omega)$. Equivalently the relation $Au = w$ is true if and only if

$$(\nabla u, \nabla v) = (w, v), \ \forall \, v \in H_{0,\sigma}^1(\Omega).$$

It is easy to verify that $\mathscr{D}(A^{1/2}) = H_{0,\sigma}^1(\Omega)$ and $\|A^{1/2}u\| = \|\nabla u\|$. The operator A is strictly positive. For the sake of simplicity, we shall use the same notations $f(u)$ and $g(u)$ instead of $Pf(u)$ and $Pg(u)$. More summaries of the divergence free projection operator can be found in literatures [5], [14], and so on.

Thus, by applying the divergence free projection operator P to equation (1.1), and taking account of the other equations, we can obtain the following abstract problem:

$$\partial_t u + Au + B(u,u) = f(u) + \left[B\left(\sigma(u), u\right) + g(u)\right] \dot{W}. \tag{1}$$

Let $E(t) = e^{-tA}$ $(t \ge 0)$ be the analytic semigroup generated by $-A$. Then the stochastic abstract problem (1) admits the following abstract integral equation:

$$\begin{aligned} u(t) \quad &= \quad E(t)u_0 - \int_0^t E(t-s)B(u,u)ds + \int_0^t E(t-s)f(u)ds \\ &\quad + \int_0^t E(t-s)\left[B(\sigma(u),u) + g(u)\right] dW(s). \end{aligned} \tag{2}$$

Now we collect some regularity analytic semigroup properties in the following lemma; see V. Thomee [20] for more details.

Lemma 2.2. [20] *For any $0 \le \mu \le e = 2.718...$, $0 < \nu < 1$, there are positive constants C_1 and C_2 that*

$$\|A^\mu E(t)\| \le C_1 t^{-\mu}, \quad for\ t > 0,$$

and

$$\|A^{-\nu}(I-E(t))\| \le C_2 t^{\nu}, \quad \textit{for } t>0.$$

In our paper, we assume that $f(u)$ is a measurable function, and there is a constant $L_f > 0$ such that:

$$\|f(u)-f(v)\| \le L_f\|u-v\|,$$

and

$$\|f(u)\| \le L_f(1+\|u\|).$$

Meanwhile we assume that $g(u)$ and $\sigma(u)$ satisfy the above-mentioned similar properties. Moreover, σ is bounded and $\nabla\cdot\sigma = 0$. More basic assumptions about the functions f, g, σ are the same as [14] and [17].

Throughout the paper, c and C denote generic positive constants independent of h, not necessarily the same at different occurrences. Moreover, $A \le CB$ is abbreviated as $A \lesssim B$.

The existence of a global martingale solution of the Cauchy problem for the stochastic Navier-Stokes equations (1.1) is obtained in the following theorem. The detailed proofs can be found in R. Mikulevicius and B. L. Rozovskii [14], too. Herein we have the same basic assumptions for the coefficients as paper [14].

Theorem 2.1. [14] *There exist a probability space $(\Omega,\mathcal{F},\boldsymbol{P})$ with a right continuous filtration $(\mathcal{F}_t)$ of σ-algebras, a cylindrical $(\mathcal{F}_t)$-adapted Wiener process W in H, and L_2-valued weakly continuous $(\mathcal{F}_t)$-adapted process $u(t)$ such that*

$$\boldsymbol{E}\left[\sup_{s\le T}|u(s)|_2^2 + \int_0^T |\nabla u(s)|_2^2 ds\right] < \infty.$$

Moreover, $u(t)$ is (strongly) continuous in t.

In order to prove our main conclusion of this paper, we need the following regularity in time of the solution.

Lemma 2.3. *Let u be the mild solution defined in* (2). *Then for any $t_1,t_2\in[0,T]$ and $0\le\gamma<1$, there is a positive constant C that*

$$\|u(t_1)-u(t_2)\|_{L_2(\Omega;H)} \le C(t_1-t_2)^{\gamma}.$$

Proof. For any $t_1,t_2\in[0,T]$ $(t_1>t_2)$, there exists

$$\begin{aligned}
& u(t_1)-u(t_2)\\
= \; & (E(t_1)-E(t_2))\,u_0 + \int_0^{t_1} E(t_1-s)f(u(s))ds - \int_0^{t_2} E(t_2-s)f(u(s))ds\\
& + \int_0^{t_1} E(t_1-s)g(u(s))dW(s) - \int_0^{t_2} E(t_2-s)g(u(s))dW(s)\\
& - \int_0^{t_1} E(t_1-s)B(u(s),u(s))ds + \int_0^{t_2} E(t_2-s)B(u(s),u(s))ds\\
& + \int_0^{t_1} E(t_1-s)B(\sigma(u(s)),u(s))dW(s)\\
& - \int_0^{t_2} E(t_2-s)B(\sigma(u(s)),u(s))dW(s)\\
= \; & L_1+L_2+L_3,
\end{aligned}$$

where

$$L_1 = (E(t_1) - E(t_2))\, u_0 + \int_0^{t_1} E(t_1 - s) f(u(s)) ds - \int_0^{t_2} E(t_2 - s) f(u(s)) ds + \int_0^{t_1} E(t_1 - s) g(u(s)) dW(s) - \int_0^{t_2} E(t_2 - s) g(u(s)) dW(s),$$

$$L_2 = -\int_0^{t_1} E(t_1 - s) B(u(s), u(s)) ds + \int_0^{t_2} E(t_2 - s) B(u(s), u(s)) ds,$$

$$L_3 = \int_0^{t_1} E(t_1 - s) B(\sigma(u(s)), u(s)) dW(s) - \int_0^{t_2} E(t_2 - s) B(\sigma(u(s)), u(s)) dW(s).$$

For $0 \le \gamma < 1$, based on the similar proof for Proposition 3.4 in [7], it is simple to verify that

$$\|L_1\|_{L_2(\Omega;H)} \lesssim \left(1 + \sup_{s\in[0,T]} \mathbf{E}\|u(s)\|\right)(t_1 - t_2)^\gamma.$$

Now we deal with the part L_2.

$$\begin{aligned} & -\int_0^{t_1} E(t_1 - s) B(u(s), u(s)) ds + \int_0^{t_2} E(t_2 - s) B(u(s), u(s)) ds \\ = & -\int_0^{t_2} (E(t_1 - s) - E(t_2 - s))\, B(u(s), u(s)) ds - \int_{t_2}^{t_1} E(t_1 - s) B(u(s), u(s)) ds. \end{aligned}$$

Considering the property of $E(t)$ in Lemma 2.2,

$$\begin{aligned} & \left\| \int_0^{t_2} (E(t_1 - s) - E(t_2 - s))\, B(u(s), u(s)) ds \right\| \\ \lesssim & \int_0^{t_2} \|A^\gamma E(t_2 - s)\| \; \left\|A^{-\gamma}(E(t_1 - t_2) - I)\right\| \|B(u(s), u(s))\| ds \\ \lesssim & (t_1 - t_2)^\gamma \int_0^{t_2} \frac{1}{(t_2 - s)^\gamma} \|B(u(s), u(s))\| ds \\ \lesssim & \sup_{s\in[0,T]} \|u(s)\|_1^2 \cdot (t_1 - t_2)^\gamma. \end{aligned}$$

Similarly, there exists

$$\left\| \int_{t_2}^{t_1} E(t_1 - s) B(u(s), u(s)) ds \right\| \lesssim \sup_{s\in[0,T]} \|u(s)\|_1^2 \cdot (t_1 - t_2)^\gamma.$$

For the stochastic term L_3, we can obtain

$$\begin{aligned} L_3 &= \int_0^{t_1} E(t_1 - s) B(\sigma(u(s)), u(s)) dW(s) - \int_0^{t_2} E(t_2 - s) B(\sigma(u(s)), u(s)) dW(s) \\ &= \int_0^{t_2} (E(t_1 - s) - E(t_2 - s))\, B(\sigma(u(s)), u(s)) dW(s) \\ &\quad + \int_{t_2}^{t_1} E(t_1 - s) B(\sigma(u(s)), u(s)) dW(s) \\ &= L_{31} + L_{32}. \end{aligned}$$

For L_{31}, considering the property of the function $\sigma(u)$,

$$\|L_{31}\|^2_{L_2(\Omega;H)}$$

$$
\begin{aligned}
&= \mathbf{E}\left\|\int_0^{t_2}\left(E(t_1-s)-E(t_2-s)\right)B(\sigma(u(s)),u(s))dW(s)\right\|^2 \\
&= \int_0^{t_2}\mathbf{E}\left\|\left(E(t_1-s)-E(t_2-s)\right)B(\sigma(u(s)),u(s))\right\|^2 ds \\
&\lesssim \int_0^{t_2}\|A^{\gamma}E(t_2-s)\|^2\ \left\|A^{-\gamma}\left(E(t_1-t_2)-I\right)\right\|^2\ \mathbf{E}\|B(\sigma(u(s)),u(s))\|^2 ds \\
&\lesssim (t_1-t_2)^{2\gamma}\int_0^{t_2}\frac{1}{(t_2-s)^{2\gamma}}\ \mathbf{E}\|B(\sigma(u(s)),u(s))\|^2 ds \\
&\lesssim \sup_{s\in[0,T]}\mathbf{E}\|\sigma(u(s))\|_1^2\cdot\sup_{s\in[0,T]}\mathbf{E}\|u(s)\|_1^2\cdot(t_1-t_2)^{2\gamma}.
\end{aligned}
$$

Similarly, for L_{32}, we can get

$$
\begin{aligned}
\|L_{32}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\int_{t_2}^{t_1}E(t_1-s)B(\sigma(u(s)),u(s))dW(s)\right\|^2 \\
&= \int_{t_2}^{t_1}\mathbf{E}\left\|E(t_1-s)B(\sigma(u(s)),u(s))\right\|^2 ds \\
&\lesssim \sup_{s\in[0,T]}\mathbf{E}\|\sigma(u(s))\|_1^2\cdot\sup_{s\in[0,T]}\mathbf{E}\|u(s)\|_1^2\cdot(t_1-t_2)^{2\gamma}.
\end{aligned}
$$

Eventually, with the above-mentioned estimations, we complete the proof. □

3. Discretization of the stochastic problem

In this section we study the finite element method for stochastic Navier-Stokes equations. The discretization with respect to time is done by backward Euler method. With the fully discretization scheme and abstract integral equation, the main error estimations are obtained, and strong convergence are well proved here.

Let Ω^h be a polygonal approximation to Ω with the boundary $\partial\Omega^h$. We consider a family $\{T_h\}_{h>0}$ of triangulations composed of triangular elements such that $\overline{\Omega^h} = \bigcup_{K\in T^h}\overline{K}$ for all $h>0$. Each element has at most one edge on $\partial\Omega^h$, and the nonempty intersection of any two elements is either only a vertex or a complete edge. Let h_K denote the maximum diameter of the element K in T_h, and let $h=\max\limits_{K\in T^h}h_K$. Assume the family $\{T_h\}_{h>0}$ to be shape regular.

Assume that $\Omega^h=\Omega\subset R^2$ for simplicity. In this article, we construct a triangulation T_hh of $\bar{\Omega}$, and approximate the velocity on each element K by a polynomial of

$$\mathscr{P}_1(K)=[P_1\oplus span\{\lambda_1\lambda_2\lambda_3\}]^2$$

and the pressure by a polynomial of P_1. Now we choose the following finite element spaces (more details can be found in [19]):

$$
\begin{aligned}
S_h &= \{v\in\mathscr{C}^0(\bar{\Omega})^2; v|_K\in\mathscr{P}_1(K),\forall K\in T_h, v|_{\partial\Omega}=0\},\\
Q_h &= \{q\in\mathscr{C}^0(\bar{\Omega})\cap L_0^2(\bar{\Omega}); q|_K\in P_1,\forall K\in T_h\}.
\end{aligned}
$$

We always assume that $S_h\subset H^1_{0,\sigma}$.

Now we consider the generalized L_2-projection operator P_h defined by

$$< P_hv,\mathcal{X} >=< v,\mathcal{X} >,\quad \forall\mathcal{X}\in S_h.$$

More details about the projection operator P_h can be found in previous work; see K. Chrysafinos and L. S. Hou [8]. Moreover, for $\forall f\in H^l(\Omega)$,

$$\|(I-P_h)f\|\le ch^l\|f\|_l,\quad l=0,1,2. \tag{1}$$

Let $A_h : S_h \to S_h$ denote the discrete analogue of the operator A, i.e.

$$< A_h\psi, \mathcal{X} >=< \nabla\psi, \nabla\mathcal{X} >, \quad \forall\psi, \mathcal{X} \in S_h.$$

Then the semidiscrete problem corresponding to stochastic abstract problem (1) is to find the process $\{u_h(t)\} \in S_h$ such that

$$(2)\quad \partial_t u_h + A_h u_h + P_h B(u_h, u_h) = P_h f(u_h) + \left[P_h B\left(\sigma(u_h), u_h\right) + P_h g(u_h)\right]\dot{W},$$

with $u_h(0) = P_h u_0$.

Let $E_h(t) = e^{-tA_h}$ be the analytic semigroup on S_h generated by $-A_h$. Then the semidiscrete problem (2) admits the abstract integral equation given by

$$(3)\quad \begin{aligned} & u_h(t) \\ = \; & E_h(t)P_h u_0 - \int_0^t E_h(t-s)P_h B(u_h, u_h)ds + \int_0^t E_h(t-s)P_h f(u_h)ds \\ & + \int_0^t E_h(t-s)\left[P_h B\left(\sigma(u_h), u_h\right) + P_h g(u_h)\right]dW(s). \end{aligned}$$

Let k be a time step and $t_m = mk$ with $m \geq 1$. Now we apply the backward Euler method to generate a sequence of u^m $(m = 1, 2, ...)$ such that

$$(4)\quad \begin{aligned} & \frac{u_h^m - u_h^{m-1}}{k} + A_h u_h^m + P_h B(u_h^{m-1}, u_h^m) \\ = \; & P_h f(u_h^{m-1}) + \frac{1}{k}\int_{t_{m-1}}^{t_m} \left[P_h B\left(\sigma(u_h^{m-1}), u_h^{m-1}\right) + P_h g(u_h^{m-1})\right] dW(s). \end{aligned}$$

With the definition of $r(\lambda) = \frac{1}{1+\lambda}$, $E_{kh} = r(kA_h)$. Then we have $E_{kh}^m = r(kA_h)^m$, which represents the m-th power of E_{kh}. Now we can rewrite the e-quality (4) in the following form

$$(5)\quad \begin{aligned} u_h^m = \; & E_{kh}^m P_h u_0 - \sum_{j=1}^m E_{kh}^{m-j+1} k P_h B(u_h^{j-1}, u_h^j) + \sum_{j=1}^m E_{kh}^{m-j+1} k P_h f(u_h^{j-1}) \\ & + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1}\left[P_h B\left(\sigma(u_h^{j-1}), u_h^{j-1}\right) + P_h g(u_h^{j-1})\right] dW(s). \end{aligned}$$

4. Error estimates for the stochastic problem

In this section we will study the main error estimations of the solutions. Strong convergence estimations are well proved here.

It follows from the scheme (2) that

$$(1)\quad \begin{aligned} u(t_m) = \; & E(t_m)u_0 - \int_0^{t_m} E(t_m - s)B(u, u)ds + \int_0^{t_m} E(t_m - s)f(u)ds \\ & + \int_0^{t_m} E(t_m - s)\left[B(\sigma(u), u) + g(u)\right]dW(s). \end{aligned}$$

Define the main error $e^m = u_h^m - u(t_m)$. Thus from equations (5) and (1), we can obtain the following equality

$$\begin{aligned} e^m = \; & [E_{kh}^m P_h - E(t_m)]u_0 \\ & + \int_0^{t_m} E(t_m - s)B(u, u)ds - \sum_{j=1}^m E_{kh}^{m-j+1} k P_h B(u_h^{j-1}, u_h^j) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h f(u_h^{j-1}) - \int_0^{t_m} E(t_m - s) f(u) ds \\
&+\sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1} P_h g(u_h^{j-1}) dW(s) - \int_0^{t_m} E(t_m - s) g(u) dW(s) \\
&+\sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1} P_h B\left(\sigma(u_h^{j-1}), u_h^{j-1}\right) dW(s) \\
&-\int_0^{t_m} E(t_m - s) B(\sigma(u), u) dW(s).
\end{aligned}
$$

In order to obtain the error estimations thoroughly, we divide the main error e^m into five components which are denoted respectively by I_k $(k = 1, 2, 3, 4, 5)$ as follows.

$$
\begin{aligned}
I_1 &= \left[E_{kh}^m P_h - E(t_m)\right] u_0, \\
I_2 &= \int_0^{t_m} E(t_m - s) B(u, u) ds - \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h B(u_h^{j-1}, u_h^j), \\
I_3 &= \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h f(u_h^{j-1}) - \int_0^{t_m} E(t_m - s) f(u) ds, \\
I_4 &= \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1} P_h g(u_h^{j-1}) dW(s) - \int_0^{t_m} E(t_m - s) g(u) dW(s), \\
I_5 &= \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1} P_h B\left(\sigma(u_h^{j-1}), u_h^{j-1}\right) dW(s) - \int_0^{t_m} E(t_m - s) B(\sigma(u), u) dW(s).
\end{aligned}
$$

Furthermore, the equalities imply the following statement.

$$
\begin{aligned}
& \|e^m\|_{L_2(\Omega;H)} \\
\leq\; & C(\|I_1\|_{L_2(\Omega;H)} + \|I_2\|_{L_2(\Omega;H)} + \|I_3\|_{L_2(\Omega;H)} + \|I_4\|_{L_2(\Omega;H)} + \|I_5\|_{L_2(\Omega;H)}).
\end{aligned}
$$

In order to prove the main error estimations, we need the following useful conclusions for the corresponding deterministic problem; see Y.B. Yan [24] for more details.

Lemma 4.1. [24] *Let $F_n = E_{kh}^n P_h - E(t_n)$. Then for $0 \leq \beta \leq 1$,*

$$
\|F_n v\| \leq C\left(k^{\beta/2} + h^\beta\right) |v|_\beta, \quad v \in H_0^\beta,
$$

and

$$
\left(k \sum_{j=1}^{n} \|F_j v\|^2\right)^{1/2} \leq C\left(k^{\beta/2} + h^\beta\right) |v|_{\beta-1}, \quad v \in H_0^{\beta-1}.
$$

Using the above-mentioned conclusions and assumptions for corresponding functions and projector operators, we deduce the estimations of the five different components.

First of all, we consider the second term I_2 of the main error e^m.

Lemma 4.2. *Let I_2 be defined in (2). For $0 < \gamma < 1$ and $0 \leq \beta \leq 1$, there exist positive constants C_{21} and C_{22} such that*

$$
\|I_2\|_{L_2(\Omega;H)} \leq C_{21} k^{min\{\gamma,\beta/2\}} + C_{22} h^\beta.
$$

Proof. By the definition of I_2, we study it as follows.

$$\begin{aligned}
& I_2 \\
= & \int_0^{t_m} E(t_m-s)B(u(s),u(s))ds - \sum_{j=1}^m E_{kh}^{m-j+1}kP_hB(u_h^{j-1},u_h^j) \\
= & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-s)B(u(s),u(s))ds - \sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-s)B(u(s_j),u(s_j)ds \\
& +\sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-s)B(u(s_j),u(s_j))ds - \sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-t_{j-1})B(u(s_j),u(s_j))ds \\
& +\sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-t_{j-1})B(u(s_j),u(s_j))ds - \sum_{j=1}^m E_{kh}^{m-j+1}kP_hB(u(s_j),u(s_j)) \\
& +\sum_{j=1}^m E_{kh}^{m-j+1}kP_hB(u(s_j),u(s_j)) - \sum_{j=1}^m E_{kh}^{m-j+1}kP_hB(u(s_{j-1}),u(s_j)) \\
& +\sum_{j=1}^m E_{kh}^{m-j+1}kP_hB(u(s_{j-1}),u(s_j)) - \sum_{j=1}^m E_{kh}^{m-j+1}kP_hB(u_h^{j-1},u_h^j) \\
= & I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4}+I_{2,5}.
\end{aligned}$$

We firstly deal with the estimation of $I_{2,1}$. It follows from Lemma 2.3 that

$$\begin{aligned}
& \|I_{2,1}\|^2_{L_2(\Omega;H)} \\
= & \mathbf{E}\left\|\sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-s)B(u(s),u(s))ds - \sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-s)B(u(s_j),u(s_j)ds\right\|^2 \\
= & \mathbf{E}\left\|\sum_{j=1}^m \int_{t_{j-1}}^{t_j} E(t_m-s)\left[B(u(s),u(s))-B(u(s_j),u(s_j))\right]ds\right\|^2 \\
\le & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \mathbf{E}\left\|E(t_m-s)\left[B(u(s),u(s))-B(u(s_j),u(s_j))\right]\right\|^2 ds \\
\le & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \mathbf{E}\left\|B(u(s),u(s))-B(u(s_j),u(s_j))\right\|^2 ds \\
\le & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \mathbf{E}\left\|B(u(s)-u(s_j),u(s))\right\|^2 ds + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \mathbf{E}\left\|B(u(s_j),u(s)-u(s_j))\right\|^2 ds \\
\lesssim & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (s-t_j)^{2\gamma} ds \lesssim k^{2\gamma}.
\end{aligned}$$

Considering $E(t)=e^{-tA}$, there exists

$$\begin{aligned}
& E(t_m-t_{j-1})-E(t_m-s)=e^{-(t_m-t_{j-1})A}-e^{-(t_m-s)A} \\
& =e^{-(t_m-t_{j-1})A}(I-e^{-(t_{j-1}-s)A})=E(t_m-t_{j-1})(I-E(t_{j-1}-s)),
\end{aligned}$$

where $s\in[t_{j-1},t_j]$. By Lemma 2.2,

$$\begin{aligned}
& \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \left\|E(t_m-t_{j-1})-E(t_m-s)\right\|^2 ds \\
\le & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \left\|A^{1/2}E(t_m-t_{j-1})\right\|^2 \left\|A^{-1/2}(I-E(t_{j-1}-s))\right\|^2 ds
\end{aligned}$$

$$\leq Ck\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left\|A^{1/2}E(t_m-t_{j-1})\right\|^2 ds$$

$$= Ck\left(\sum_{j=1}^{m}k\left\|A^{1/2}E(t_m-t_{j-1})\right\|^2\right)\lesssim k.$$

Then for $I_{2,2}$, we can obtain

$$\begin{aligned}
&\|I_{2,2}\|^2_{L_2(\Omega;H)}\\
=\;& \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E(t_m-s)B(u(s_j),u(s_j))ds-\int_{t_{j-1}}^{t_j}E(t_m-t_{j-1})B(u(s_j),u(s_j))ds\right\|^2\\
=\;& \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E(t_m-s)-E(t_m-t_{j-1})\right]B(u(s_j),u(s_j))ds\right\|^2\\
\leq\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\|E(t_m-s)-E(t_m-t_{j-1})\|^2 ds\|B(u(s_j),u(s_j))\|^2\lesssim k.
\end{aligned}$$

For $I_{2,3}$, considering Lemma 4.1,

$$\begin{aligned}
&\|I_{2,3}\|^2_{L_2(\Omega;H)}\\
=\;& \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E(t_m-t_{j-1})B(u(s_j),u(s_j))ds-\sum_{j=1}^{m}E_{kh}^{m-j+1}kP_hB(u(s_j),u(s_j))\right\|^2\\
=\;& \mathbf{E}\left\|\sum_{j=1}^{m}\left[\int_{t_{j-1}}^{t_j}E(t_m-t_{j-1})ds-E_{kh}^{m-j+1}P_hk\right]B(u(s_j),u(s_j))\right\|^2\\
\lesssim\;& \mathbf{E}\left\|k\sum_{j=1}^{m}\left[E(t_m-t_{j-1})-E_{kh}^{m-j+1}P_h\right]B(u(s_j),u(s_j))\right\|^2\\
=\;& \mathbf{E}\left\|k\sum_{j=1}^{m}F_{m-j+1}B(u(s_j),u(s_j))\right\|^2\lesssim k^{\beta}+h^{2\beta}.
\end{aligned}$$

For $I_{2,4}$, considering the stability of E_{kh}, there holds

$$\begin{aligned}
&\|I_{2,4}\|^2_{L_2(\Omega;H)}\\
=\;& \mathbf{E}\left\|\sum_{j=1}^{m}E_{kh}^{m-j+1}kP_hB(u(s_j),u(s_j))-\sum_{j=1}^{m}E_{kh}^{m-j+1}kP_hB(u(s_{j-1}),u(s_j))\right\|^2\\
=\;& \mathbf{E}\left\|k\sum_{j=1}^{m}E_{kh}^{m-j+1}P_h\left[B(u(s_j),u(s_j))-B(u(s_{j-1}),u(s_j))\right]\right\|^2\\
\leq\;& k\sum_{j=1}^{m}\mathbf{E}\left\|B(u(s_j)-u(s_{j-1}),u(s_j))\right\|^2\lesssim k.
\end{aligned}$$

For $I_{2,5}$, it follows from the inequality (1) that

$$\|I_{2,5}\|^2_{L_2(\Omega;H)}$$

$$
\begin{aligned}
&= \mathbf{E}\left\|\sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h B(u(s_{j-1}), u(s_j)) - \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h B(u_h^{j-1}, u_h^j)\right\|^2 \\
&= \mathbf{E}\left\|k \sum_{j=1}^{m} E_{kh}^{m-j+1} P_h \left[B(u(s_{j-1}), u(s_j)) - B(u_h^{j-1}, u_h^j)\right]\right\|^2 \\
&\leq \mathbf{E}\left\|k \sum_{j=1}^{m} E_{kh}^{m-j+1} P_h B\left(u(s_{j-1}) - u_h^{j-1}, u(s_j)\right)\right\|^2 \\
&\quad + \mathbf{E}\left\|k \sum_{j=1}^{m} E_{kh}^{m-j+1} P_h B\left(u_h^{j-1}, u(s_j) - u_h^j\right)\right\|^2 \\
&\leq \mathbf{E}\left\|k \sum_{j=1}^{m} B(e^{j-1}, u(s_j))\right\|^2 + \mathbf{E}\left\|k \sum_{j=1}^{m-1} B(u_h^{j-1}, e^j)\right\|^2 \\
&\lesssim k \sum_{j=1}^{m} \mathbf{E}\|e^{j-1}\|_1^2.
\end{aligned}
$$

With the above-mentioned estimations, we can obtain the following conclusion.

$$
\|I_2\|^2_{L_2(\Omega;H)} \leq \hat{C}_{21} k + \hat{C}_{22} k^{2\gamma} + \hat{C}_{23} k^{\beta} + \hat{C}_{24} h^{2\beta} + \hat{C}_{25} k \sum_{j=1}^{m} \mathbf{E}\|e^{j-1}\|_1^2.
$$

By the discrete Gronwall lemma, we can get

$$
\|I_2\|^2_{L_2(\Omega;H)} \leq \tilde{C}_{21} k^{min\{2\gamma,\beta\}} + \tilde{C}_{22} h^{2\beta},
$$

which implies that

$$
\|I_2\|_{L_2(\Omega;H)} \leq C_{21} k^{min\{\gamma,\beta/2\}} + C_{22} h^{\beta}.
$$

Here we complete the proof. □

Similarly we study the third term I_3 of the main error e^m.

Lemma 4.3. *Let I_3 be defined in* (2). *For $0 < \gamma < 1$ and $0 \leq \beta \leq 1$, there exist positive constants C_{31} and C_{32} such that*

$$
\|I_3\|_{L_2(\Omega;H)} \leq C_{31} k^{min\{\gamma,\beta/2\}} + C_{32} h^{\beta}.
$$

Proof. Similarly as the proof of I_2, we can obtain

$$
\begin{aligned}
I_3 &= \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h f(u_h^{j-1}) - \int_0^{t_m} E(t_m - s) f(u) ds \\
&= \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h f(u_h^{j-1}) - \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h f(u(s_{j-1})) \\
&\quad + \sum_{j=1}^{m} E_{kh}^{m-j+1} k P_h f(u(s_{j-1})) - \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1} P_h f(u(s)) ds \\
&\quad + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1} P_h f(u(s)) ds - \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} E(t_m - t_{j-1}) f(u(s)) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E(t_m-t_{j-1})-E(t_m-s)\right]f(u(s))ds\\
=\;& I_{3,1}+I_{3,2}+I_{3,3}+I_{3,4}.
\end{aligned}
$$

For $I_{3,1}$, considering the stability of E_{kh}, there is

$$
\begin{aligned}
\|I_{3,1}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{m}E_{kh}^{m-j+1}kP_hf(u_h^{j-1})-\sum_{j=1}^{m}E_{kh}^{m-j+1}kP_hf(u(s_{j-1}))\right\|^2\\
&= \mathbf{E}\left\|k\sum_{j=1}^{m}E_{kh}^{m-j+1}P_h\left[f(u_h^{j-1})-f(u(s_{j-1}))\right]\right\|^2\\
&\le Ck\sum_{j=1}^{m}\mathbf{E}\left\|u_h^{j-1}-u(s_{j-1})\right\|^2\lesssim k\sum_{j=1}^{m}\mathbf{E}\left\|e^{j-1}\right\|_1^2.
\end{aligned}
$$

For $I_{3,2}$, note that $f(u)$ satisfies global Lipschitz condition. It follows from Lemma 2.3 that

$$
\begin{aligned}
\|I_{3,2}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{m}E_{kh}^{m-j+1}kP_hf(u(s_{j-1}))-\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_hf(u(s))ds\right\|^2\\
&= \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_h\left[f(u(s_{j-1}))-f(u(s))\right]ds\right\|^2\\
&\le \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{m-j+1}P_h\left[f(u(s_{j-1}))-f(u(s))\right]\right\|^2ds\\
&\lesssim \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|u(s_{j-1})-u(s)\right\|^2\\
&\lesssim \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}(s-s_{j-1})^{2\gamma}ds\lesssim k^{2\gamma}.
\end{aligned}
$$

For $I_{3,3}$, considering Lemma 4.1, we can get

$$
\begin{aligned}
&\|I_{3,3}\|^2_{L_2(\Omega;H)}\\
=\;&\mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_hf(u(s))ds-\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E(t_m-t_{j-1})f(u(s))ds\right\|^2\\
=\;&\mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E_{kh}^{m-j+1}P_h-E(t_m-t_{j-1})\right]f(u(s))ds\right\|^2\\
=\;&\mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}F_{m-j+1}f(u(s))ds\right\|^2\lesssim k^{\beta}+h^{2\beta}.
\end{aligned}
$$

Similar as the technique we used for proving $I_{2,2}$, there holds

$$\|I_{3,4}\|^2_{L_2(\Omega;H)} = \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E(t_m-s)-E(t_m-t_{j-1})\right]f(u(s))ds\right\|^2 \lesssim k.$$

Therefore, we obtain the following conclusion.

$$\|I_3\|^2_{L_2(\Omega;H)} \le \hat{C}_{31}k^{2\gamma}+\hat{C}_{32}k^{\beta}+\hat{C}_{33}h^{2\beta}+\hat{C}_{34}k\sum_{j=1}^{m}\mathbf{E}\|e^{j-1}\|_1^2.$$

By the discrete Gronwall lemma, we can get

$$\|I_3\|^2_{L_2(\Omega;H)} \le \tilde{C}_{31}k^{min\{2\gamma,\beta\}}+\tilde{C}_{32}h^{2\beta},$$

which implies that

$$\|I_3\|_{L_2(\Omega;H)} \le C_{31}k^{min\{\gamma,\beta/2\}}+C_{32}h^{\beta}.$$

Here we complete the proof. □

Now we study the fourth term I_4 of the main error e^m.

Lemma 4.4. *Let I_4 be defined in* (2)*. For $0<\gamma<1$ and $0\le\beta\le1$, there exist positive constants C_{41} and C_{42} such that*

$$\|I_4\|_{L_2(\Omega;H)} \le C_{41}k^{min\{\gamma,\beta/2\}}+C_{42}h^{\beta}.$$

Proof. Now we consider the estimation of I_4.

$$\begin{aligned}
I_4 &= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_h g(u_h^{j-1})dW(s)-\int_0^{t_m}E(t_m-s)g(u)dW(s)\\
&= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_h\left[g(u_h^{j-1})-g(u(s_{j-1}))\right]dW(s)\\
&\quad+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_h\left[g(u(s_{j-1}))-g(u(s))\right]dW(s)\\
&\quad+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E_{kh}^{m-j+1}P_h-E(t_m-t_{j-1})\right]g(u(s))dW(s)\\
&\quad+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E(t_m-t_{j-1})-E(t_m-s)\right]g(u(s))dW(s)\\
&= I_{4,1}+I_{4,2}+I_{4,3}+I_{4,4}.
\end{aligned}$$

For $I_{4,1}$, considering the isometry property, we can get

$$\begin{aligned}
\|I_{4,1}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}E_{kh}^{m-j+1}P_h\left[g(u_h^{j-1})-g(u(s_{j-1}))\right]dW(s)\right\|^2\\
&= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{m-j+1}P_h\left[g(u_h^{j-1})-g(u(s_{j-1}))\right]\right\|^2ds\\
&\le \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|g(u_h^{j-1})-g(u(s_{j-1}))\right\|^2ds\\
&\lesssim k\sum_{j=1}^{m}\mathbf{E}\|u_h^{j-1}-u(s_{j-1})\|^2 \lesssim k\sum_{j=1}^{m}\mathbf{E}\|e^{j-1}\|_1^2.
\end{aligned}$$

For $I_{4,2}$, it follows from Lemma 2.3 that

$$\begin{aligned}\|I_{4,2}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1}P_h\left[g(u(s_{j-1}))-g(u(s))\right]dW(s)\right\|^2\\ &= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{m-j+1}P_h\left[g(u(s_{j-1}))-g(u(s))\right]\right\|^2 ds\\ &\lesssim \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|u(s_{j-1})-u(s)\right\|^2 ds\\ &\lesssim \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}(s-s_{j-1})^{2\gamma}ds \lesssim k^{2\gamma}.\end{aligned}$$

For $I_{4,3}$, by Lemma 4.1 there holds

$$\begin{aligned}\|I_{4,3}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E_{hk}^{m-j+1}P_k-E(t_m-t_{j-1})\right]g(u(s))dW(s)\right\|^2\\ &= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|\left[E_{hk}^{m-j+1}P_k-E(t_m-t_{j-1})\right]g(u(s))\right\|^2 ds\\ &= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|F_{m-j+1}g(u(s))\right\|^2 ds \lesssim k^{\beta}+h^{2\beta}.\end{aligned}$$

For $I_{4,4}$, it is easy to verify that

$$\begin{aligned}\|I_{4,4}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E(t_m-t_{j-1})-E(t_m-s)\right]g(u(s))dW(s)\right\|^2\\ &= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|\left[E(t_m-t_{j-1})-E(t_m-s)\right]g(u(s))\right\|^2 ds \lesssim k.\end{aligned}$$

It follows from above estimations that

$$\|I_4\|^2_{L_2(\Omega;H)} \le \hat{C}_{41}k^{2\gamma}+\hat{C}_{42}k^{\beta}+\hat{C}_{43}h^{2\beta}+\hat{C}_{44}k+\hat{C}_{45}k\sum_{j=1}^{m}\mathbf{E}\|e^{j-1}\|_1^2.$$

By the discrete Gronwall lemma, we can get

$$\|I_4\|^2_{L_2(\Omega;H)} \le \tilde{C}_{41}k^{min\{2\gamma,\beta\}}+\tilde{C}_{42}h^{2\beta},$$

which implies that

$$\|I_4\|_{L_2(\Omega;H)} \le C_{41}k^{min\{\gamma,\beta/2\}}+C_{42}h^{\beta}.$$

Here we complete the proof. □

Then we study the fifth term I_5 of the main error e^m.

Lemma 4.5. *Let I_5 be defined in (2). For $0<\gamma<1$ and $0\le\beta\le1$, there exist positive constants C_{51} and C_{52} such that*

$$\|I_5\|_{L_2(\Omega;H)} \le C_{51}k^{min\{\gamma,\beta/2\}}+C_{52}h^{\beta}.$$

Proof. Now we consider the estimation of I_5.

$$\begin{aligned}
&I_5\\
=\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1}P_h B\left(\sigma(u_h^{j-1}),u_h^{j-1}\right)dW(s)-\int_0^{t_m}E(t_m-s)B(\sigma(u),u)dW(s)\\
=\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1}P_h\left[B\left(\sigma(u_h^{j-1}),u_h^{j-1}\right)-B\left(\sigma(u(s_{j-1})),u(s_{j-1})\right)\right]dW(s)\\
&+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E_{kh}^{m-j+1}P_h-E(t_m-t_{j-1})\right]B\left(\sigma(u(s_{j-1})),u(s_{j-1})\right)dW(s)\\
&+\int_0^{t_m}\left[E(t_m-t_{j-1})-E(t_m-s)\right]B\left(\sigma(u(s_{j-1})),u(s_{j-1})\right)dW(s)\\
&+\int_0^{t_m}E(t_m-s)\left[B\left(\sigma(u(s_{j-1})),u(s_{j-1})\right)-B\left(\sigma(u(s)),u(s)\right)\right]dW(s)\\
=\;& I_{5,1}+I_{5,2}+I_{5,3}+I_{5,4}.
\end{aligned}$$

For $I_{5,1}$, considering the global Lipschitz property of $\sigma(u)$, we can obtain

$$\begin{aligned}
&\|I_{5,1}\|^2_{L_2(\Omega;H)}\\
=\;& \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j} E_{kh}^{m-j+1}P_h\left[B\left(\sigma(u_h^{j-1}),u_h^{j-1}\right)-B\left(\sigma(u(s_{j-1})),u(s_{j-1})\right)\right]dW(s)\right\|^2\\
=\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{m-j+1}P_h\left[B\left(\sigma(u_h^{j-1}),u_h^{j-1}\right)-B\left(\sigma(u(s_{j-1})),u(s_{j-1})\right)\right]\right\|^2ds\\
\le\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{m-j+1}P_h\left[B\left(\sigma(u_h^{j-1})-\sigma(u(s_{j-1})),u_h^{j-1}\right)\right]\right\|^2ds\\
&+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{m-j+1}P_h\left[B\left(\sigma(u(s_{j-1})),u_h^{j-1}-u(s_{j-1})\right)\right]\right\|^2ds\\
\le\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|B\left(\sigma(u_h^{j-1})-\sigma(u(s_{j-1})),u_h^{j-1}\right)\right\|^2ds\\
&+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|B\left(\sigma(u(s_{j-1})),u_h^{j-1}-u(s_{j-1})\right)\right\|^2ds\\
\lesssim\;& \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|B\left(u_h^{j-1}-u(s_{j-1}),u_h^{j-1}\right)\right\|^2ds\\
&+\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|B\left(u(s_{j-1}),u_h^{j-1}-u(s_{j-1})\right)\right\|^2ds\\
\le\;& k\sum_{j=1}^{m}\mathbf{E}\left\|B\left(e^{j-1},u_h^{j-1}\right)\right\|^2+k\sum_{j=1}^{m}\mathbf{E}\left\|B\left(u(s_{j-1}),e^{j-1}\right)\right\|^2\\
\le\;& k\sum_{j=1}^{m}\mathbf{E}\left\|e^{j-1}\right\|_1^2.
\end{aligned}$$

For $I_{5,2}$, it follows from Lemma 4.1 that

$$\|I_{5,2}\|^2_{L_2(\Omega;H)}$$

$$
\begin{aligned}
&= \mathbf{E}\left\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\left[E_{kh}^{m-j+1}P_h - E(t_m - t_{j-1})\right]B\left(\sigma(u(s_{j-1})), u(s_{j-1})\right)dW(s)\right\|^2 \\
&= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|\left[E_{kh}^{m-j+1}P_h - E(t_m - t_{j-1}))\right]B\left(\sigma(u(s_{j-1})), u(s_{j-1}))\right\|^2 ds \\
&= \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|F_{m-j+1}B\left(\sigma(u(s_{j-1})), u(s_{j-1}))\right\|^2 ds \lesssim k^{\beta} + h^{2\beta}.
\end{aligned}
$$

For $I_{5,3}$, it is easy to prove that

$$
\begin{aligned}
&\|I_{5,3}\|^2_{L_2(\Omega;H)} \\
&= \mathbf{E}\left\|\int_0^{t_m}\left[E(t_m - t_{j-1}) - E(t_m - s)\right]B\left(\sigma(u(s_{j-1})), u(s_{j-1})\right)dW(s)\right\|^2 \\
&= \int_0^{t_m}\mathbf{E}\left\|\left[E(t_m - t_{j-1}) - E(t_m - s)\right]B\left(\sigma(u(s_{j-1})), u(s_{j-1}))\right\|^2 ds \lesssim k.
\end{aligned}
$$

For $I_{5,4}$, considering Lemma 2.3, we can similarly get

$$
\begin{aligned}
&\|I_{5,4}\|^2_{L_2(\Omega;H)} \\
&= \mathbf{E}\left\|\int_0^{t_m}E(t_m - s)\left[B\left(\sigma(u(s_{j-1})), u(s_{j-1})\right) - B\left(\sigma(u(s)), u(s)\right)\right]dW(s)\right\|^2 \\
&= \int_0^{t_m}\mathbf{E}\left\|E(t_m - s)\left[B\left(\sigma(u(s_{j-1})), u(s_{j-1})\right) - B\left(\sigma(u(s)), u(s)\right)\right]\right\|^2 ds \\
&\leq \int_0^{t_m}\mathbf{E}\left\|B\left(\sigma(u(s_{j-1})), u(s_{j-1})\right) - B\left(\sigma(u(s)), u(s)\right)\right\|^2 ds \\
&\leq \int_0^{t_m}\mathbf{E}\left\|B\left(\sigma(u(s_{j-1})) - \sigma(u(s)), u(s_{j-1})\right)\right\|^2 ds \\
&\quad + \int_0^{t_m}\mathbf{E}\left\|B\left(\sigma(u(s)), u(s_{j-1}) - u(s)\right)\right\|^2 ds \\
&\leq \int_0^{t_m}\mathbf{E}\left\|B\left(u(s_{j-1}) - u(s), u(s_{j-1})\right)\right\|^2 ds \\
&\quad + \int_0^{t_m}\mathbf{E}\left\|B\left(u(s), u(s_{j-1}) - u(s)\right)\right\|^2 ds \lesssim k^{2\gamma}.
\end{aligned}
$$

Then we can come to the conclusion that

$$
\|I_5\|^2_{L_2(\Omega;H)} \leq \hat{C}_{51}k + \hat{C}_{52}k^{\beta} + \hat{C}_{53}k^{2\gamma} + \hat{C}_{54}h^{2\beta} + \hat{C}_{55}k\sum_{j=1}^{m}\mathbf{E}\|e^{j-1}\|_1^2.
$$

By the discrete Gronwall lemma, we can get

$$
\|I_5\|^2_{L_2(\Omega;H)} \leq \tilde{C}_{51}k^{min\{2\gamma,\beta\}} + \tilde{C}_{52}h^{2\beta},
$$

which implies that

$$
\|I_5\|_{L_2(\Omega;H)} \leq C_{51}k^{min\{\gamma,\beta/2\}} + C_{52}h^{\beta}.
$$

Here we complete the proof. □

Finally we prove the conclusions of the main error e^m.

Theorem 4.1. *Let u_h^m and $u(t_m)$ be the solution of equations* (5) *and* (1), *respectively. For $0 < \gamma < 1$ and $0 \leq \beta \leq 1$, there exist positive constants C_1 and C_2 such that*

$$\|e^m\|_{L_2(\Omega;H)} \lesssim C_1 k^{min\{\gamma,\beta/2\}} + C_2 h^\beta.$$

Proof. First of all we give the estimation of I_1 defined in (2). It follows from Lemma 4.1 that

$$\|I_1\|^2_{L_2(\Omega;H)} = \mathbf{E}\,\|[E^m_{kh}P_h - E(t_m)]\,u_0\|^2 \lesssim k^\beta + h^{2\beta},$$

which implies that

$$\|I_1\|_{L_2(\Omega;H)} \lesssim k^{\beta/2} + h^\beta.$$

Then considering Lemma 4.2 - Lemma 4.5, for $0 < \gamma < 1$ and $0 \leq \beta \leq 1$, we can come to the conclusion that

$$\begin{aligned} & \|e^m\|_{L_2(\Omega;H)} \\ \lesssim & \|I_1\|_{L_2(\Omega;H)} + \|I_2\|_{L_2(\Omega;H)} + \|I_3\|_{L_2(\Omega;H)} + \|I_4\|_{L_2(\Omega;H)} + \|I_5\|_{L_2(\Omega;H)} \\ \lesssim & C_1 k^{min\{\gamma,\beta/2\}} + C_2 h^\beta. \end{aligned}$$

The main proof is now completed. □

5. Conclusions

In this paper, we give the finite element approximation of backward Euler scheme for stochastic Navier-Stokes equations with turbulent component. With the projection operators, we apply finite element method to discrete the space. For the time discretization, we consider the backward Euler method. The semidiscrete and fully discrete scheme are obtained here. For the main error for the fully discretization, we divide it into five parts, and prove corresponding error estimations, respectively. Finally, strong convergence error estimations for the main error are well proved completely.

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