# A SEMIDISCRETE APPROXIMATION SCHEME FOR NEUTRAL DELAY-DIFFERENTIAL EQUATIONS

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**Abstract.** We consider an approximation scheme for systems of linear delay-differential equations of neutral type. The finite dimensional approximating systems are constructed with basis functions defined using linear splines, extending to neutral equations a scheme which had previously been defined only for retarded equations. A Trotter-Kato semigroup convergence result is proved, and numerical results are given to illustrate the qualitative behavior of the scheme.

Key words. neutral delay equation, semidiscrete approximation, semigroup theory

### 1. Introduction

In this paper we consider semidiscrete approximation schemes for linear autonomous neutral delay differential equations. In particular, the neutral equation is formulated as a linear system on an infinite dimensional Hilbert space, and this system is approximated by a sequence of linear differential equations on finite dimensional Hilbert spaces. A Trotter-Kato type theorem is used to argue convergence.

The idea of using this type of semigroup-theoretic finite dimensional approximation for delay differential equations has been known for some time. It has often been the case that an approximation scheme is developed first for retarded equations and later extended to neutral equations (this has often been the case for the development of other parts of the theory for delay differential equations as well). Perhaps the earliest paper with a rigorous implementation of this idea (that is, rigorous justification of both well-posedness as well as Trotter-Kato type semigroup convergence) is [1]. There Banks and Burns prove convergence of the so-called averaging approximation scheme (the basis functions are piecewise constant) for linear retarded delay equations, and the scheme is applied to a control problem. The averaging scheme for retarded equations was extended to neutral equations by Kappel and Kunisch in [19] and [23]. Meanwhile in [3] Banks and Kappel construct an approximation scheme for retarded delay equations which uses certain splines (in particular, splines which are restricted to be in the domain of the infinitesimal generator of the semigroup associated with the equation) as basis functions for the finite dimensional approximation spaces. They show this scheme obtains better convergence rates than the averaging scheme when applied to retarded equations. This spline based scheme was extended to neutral equations by Kappel and Kunisch in [19] and [20]. Later in [2] Banks, Ito, and Rosen observed numerically that this spline based scheme performed poorly (that is, worse than the numerically observed performance of the averaging scheme) when used to approximate feedback gains in an optimal control problem for a retarded equation. The authors in [2] conjectured that the spline based scheme of [3] did not yield convergence for the adjoint semigroup, and this conjecture was confirmed by Burns, Ito, and Propst

Received by the editors June 12, 2012.

<sup>2000</sup> Mathematics Subject Classification. 34K06, 34K28, 34K40, 47D06, 65L03.

in [8]. In [21] Kappel and Salamon developed an improved spline based approximation scheme for retarded delay equations (in particular, the domain restriction on the splines is removed), and the scheme yielded both semigroup convergence and adjoint semigroup convergence. In the present paper we extend this improved spline scheme to neutral delay equations, and prove semigroup convergence. This extends results in [14] which considered the scalar, single delay neutral equation. The adjoint semigroup convergence has been verified for the scalar single delay neutral case in [10], but the extension to the general multiple delay neutral systems under consideration in this paper is a subject for further investigation. An additional contribution of this paper is a variational version of the Trotter-Kato theorem which is especially suitable for our construction. This paper is concerned only with semidiscrete approximation of linear neutral delay equations. For fully discrete and other approaches to approximation of delay equations, we refer to [4], [5], [26], and the references therein.

We now define the system of linear neutral delay-differential equations under consideration. Given  $\eta_0 \in \mathbb{C}^n$  and  $\phi_0 \in L^2(-r, 0; \mathbb{C}^n)$ , consider the initial value problem

(1) 
$$\frac{d}{dt}[x(t) + \sum_{k=1}^{m} C_k x(t - r_k)] = Ax(t) + \sum_{k=1}^{m} B_k x(t - r_k),$$
$$x(0) + \sum_{k=1}^{m} C_k x(-r_k) = \eta_0, \quad x(\theta) = \phi_0(\theta), \quad -r \le \theta < 0,$$

where  $B_k$ ,  $C_k$ , k = 1, ..., m and A are  $n \times n$  matrices with complex entries, and  $0 = r_0 < r_1 < \cdots < r_m \equiv r$ . In a standard fashion first described in [7], the initial value problem (1) can be reformulated as an abstract Cauchy problem on a Hilbert space. In particular, define the Hilbert space  $X = \mathbb{C}^n \times L^2(-r, 0; \mathbb{C}^n)$  endowed with the norm

(2) 
$$\|(\eta,\phi)\|_X^2 = \|\eta\|^2 + \int_{-r}^0 \overline{\phi(\theta)}^T G(\theta)\phi(\theta) \, d\theta.$$

Here and throughout the paper we use the unsubscripted norm notation  $\|\cdot\|$  to denote the standard Euclidean norm on  $\mathbb{C}^n$  or its induced matrix norm on  $n \times n$  matrices. It is clear that the norm  $\|\cdot\|_X$  depends on the choice of the  $n \times n$  matrix-valued weight function G, although we do not indicate this explicitly in the notation. However we take as a standing assumption that the weight function G is chosen so the norm  $\|\cdot\|_X$  is equivalent to the usual energy norm which corresponds to  $G(\theta) \equiv I$ . (Later we shall impose further restrictions on the function G so as to obtain an important dissipative inequality). We may also write the norm as

$$\|(\eta,\phi)\|_{X}^{2} = \|\eta\|^{2} + \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} \overline{\phi(\theta)}^{T} G(\theta)\phi(\theta) \, d\theta$$

with compatible inner product

(3) 
$$\langle (\eta,\phi), (\xi,\psi) \rangle_X = \overline{\xi}^T \eta + \sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} \overline{\psi(\theta)}^T G(\theta) \phi(\theta) \, d\theta.$$

Next define the linear operator  $\mathcal{A}$ : dom  $\mathcal{A} \subset X \to X$  on the domain

dom 
$$\mathcal{A} = \{(\eta, \phi) \in X : \phi \in H^1(-r, 0; \mathbb{C}^n), \eta = \phi(0) + \sum_{k=1}^m C_k \phi(-r_k) \},\$$

by

(4) 
$$\mathcal{A}(\eta,\phi) = (A\phi(0) + \sum_{k=1}^{m} B_k \phi(-r_k), \phi').$$

It is well known that  $\mathcal{A}$  is the infinitesimal generator of a strongly continuous semigroup T(t) on X, and if we make the identification

$$z(t) = (x(t) + \sum_{k=1}^{m} C_k x(t - r_k), x(t + \theta))$$

then as introduced in [7] equation (1) can be reformulated as the Cauchy problem

(5) 
$$\frac{d}{dt}z(t) = \mathcal{A}z(t),$$
$$z(0) = (\eta_0, \phi_0)$$

on X. It is within this semigroup theoretic setting that we wish to construct an approximation scheme and analyze its convergence properties. By a semidiscrete approximation scheme for (5) we mean a sequence  $\{\mathcal{A}^N, X^N\}_{N=1}^{\infty}$  of finite dimensional subspaces  $X^N \subset X$  and operators  $\mathcal{A}^N : X^N \to X^N$ . The operators  $\mathcal{A}^N$  define semigroups  $T^N(t) = e^{t\mathcal{A}^N}$  on  $X^N$ , and the subspaces  $X^N$  define orthogonal projections  $P^N : X \to X^N$ . Such an approximation scheme defines a sequence of finite dimensional Cauchy problems

(6) 
$$\frac{d}{dt}z^{N}(t) = \mathcal{A}^{N}z^{N}(t),$$
$$z^{N}(0) = P^{N}z_{0},$$

 $z^{**}(0) = P^{**}z_0$ , on  $X^N$ . A typical convergence result involves showing that  $P^N \to I$  strongly and that  $T^N(t)P^N \to T(t)$  in the Trotter-Kato sense. Such a convergence result justifies using (6) to approximate the dynamics of (5), and being finite dimensional, (6) can be solved on the computer. The remainder of the paper proceeds as follows. In the next section we consider the first order spline scheme found in [21], which was applied to retarded delay equations, and modify it to construct a scheme for neutral delay equations. We also prove Trotter-Kato type semigroup convergence for the scheme. In the final section we give some numerical examples illustrating the qualitative behavior of the approximation scheme.

## 2. A Semidiscrete Approximation Scheme

In [19] Kappel constructed semidiscrete approximation schemes for the neutral system (5) by extending the ideas and basis functions which had been used for retarded systems in [3]. In [20] Kappel and Kunisch extended these ideas to higher order spline schemes for neutral equations. Later in [21] and [22] Kappel and Salamon introduced a new approximation scheme (with different basis functions) for retarded systems and showed how the scheme improved upon that in [3]. In this section we extend the scheme in [21] to the neutral system (5). We note that in all of these references the authors make use of some version of the Trotter-Kato semigroup convergence theorem which is suitable for their construction. Likewise we shall make use of the following semigroup convergence theorem, which is particularly suitable to our construction (the proof is deferred to the appendix). The theorem is similar in theme to the theorems found in [3], [16], [18], [20], and [21]. The hypotheses of our theorem are stated in a variational form which is suitable for applications in delay equations, and are used to verify the so-called stability and consistency properties of the Trotter-Kato theorem. In this respect the motivation

for our result is similar to the motivation for the results in [16], in which Ito and Kappel discuss the issue of how to establish stability and consistency.

**Theorem 1.** Suppose V and X are Hilbert spaces, with V densely and continuously embedded in X, and let  $c_V > 0$  satisfy

(7) 
$$\|x\|_X \le c_V \|x\|_V \quad \forall x \in V$$

Assume  $\mathcal{A}$ : dom  $\mathcal{A} \subset V \subset X \to X$  is the infinitesimal generator of a  $C_0$ -semigroup T(t) on X, and there is a sesquilinear form  $\sigma : V \times V \to \mathbb{C}$  and a fixed  $\omega \in \mathbb{R}$  satisfying

(8) 
$$\sigma(u,v) = \langle \mathcal{A}u, v \rangle_X \quad \forall u \in \mathrm{dom}\mathcal{A}, v \in V,$$

and

(9) 
$$\operatorname{Re}\sigma(u,u) \le \omega \|u\|_X^2 \quad \forall u \in V.$$

Let  $\{X^N\}_{N=1}^{\infty}$  be a sequence of finite dimensional subspaces of V, and let  $P^N$  denote the orthogonal projection of X onto  $X^N$ . For each N define the operator  $\mathcal{A}^N$ :  $X^N \to X^N$  by

(10) 
$$\langle \mathcal{A}^N u, v \rangle_X = \sigma(u, v) \qquad \forall u, v \in X^N.$$

If there are constants  $s \ge 1$  and L > 0 such that for all  $v \in \text{dom } \mathcal{A}^s$  and all  $N = 1, 2, \ldots$ , there exists  $v^N \in X^N$  satisfying

(11) 
$$|\sigma(u,v-v^N)| \le L \, \|u\|_X \, \|v-v^N\|_V \quad \forall u \in V,$$

and

(12) 
$$\lim_{N \to \infty} \|v - v^N\|_V = 0,$$

then  $T^N(t)P^N \to T(t)$  strongly on X. Here  $T^N(t) = e^{t\mathcal{A}^N}$  is the semigroup on  $X^N$  generated by  $\mathcal{A}^N$ .

In order to apply this result we must first construct a suitable sesquilinear form  $\sigma$ . Thus we define the Hilbert space V by

$$V = \mathbb{C}^n \times H^1(-r, 0; \mathbb{C}^n)$$

endowed with the usual norm

(13) 
$$\|(\eta,\phi)\|_{V}^{2} = \|\eta\|^{2} + \int_{-r}^{0} (\|\phi(\theta)\|^{2} + \|\phi'(\theta)\|^{2}) \, d\theta.$$

Clearly V is densely and continuously embedded in X for any choice of weight function G under consideration in this paper. For  $u = (\eta, \phi), v = (\xi, \psi) \in V$  define the sesquilinear form  $\sigma : V \times V \to \mathbb{C}$  by

$$\sigma(u,v) = \overline{\xi}^T \left[ A\eta - A \sum_{k=1}^m C_k \phi(-r_k) + \sum_{k=1}^m B_k \phi(-r_k) \right]$$
  
+ 
$$\sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} \overline{\psi(\theta)}^T G(\theta) \phi'(\theta) d\theta$$
  
+ 
$$\overline{\psi(0)}^T G(0) \left[ \eta - \phi(0) - \sum_{k=1}^m C_k \phi(-r_k) \right].$$

The form  $\sigma$  is dependent upon the choice of weight function G. However, assuming the same weight function is used in the definition of  $\sigma$  and the norm  $\|\cdot\|_X$ , it is straightforward to check that

$$\sigma(u,v) = \langle \mathcal{A}u, v \rangle_X$$

for all  $u \in \text{dom}\mathcal{A}$ ,  $v \in V$ , so (8) holds. We turn next to verification of (9), a dissipative inequality for the sesquilinear form  $\sigma$ . Is has been noted in [20] that such a dissipative inequality for the operator  $\mathcal{A}$  is generally not possible without further restrictions on the weight function G, and it is reasonable to expect similar restrictions for (9). Thus let us further assume

$$G(\theta) = g_k(\theta)I$$
 for  $-r_k \le \theta \le -r_{k-1}$ ,

where I is the  $n \times n$  identity matrix and the scalar functions  $g_k$  are positive and continuously differentiable on  $[-r_k, -r_{k-1}]$  for k = 1, 2, ..., m. With this assumption we may write the norm as

$$\|(\eta,\phi)\|_X^2 = \|\eta\|^2 + \sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} g_k(\theta) \, \|\phi(\theta)\|^2 \, d\theta,$$

the inner product as

(14) 
$$\langle (\eta, \phi), (\xi, \psi) \rangle_X = \overline{\xi}^T \eta + \sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} g_k(\theta) \overline{\psi(\theta)}^T \phi(\theta) \, d\theta,$$

and the sesquilinear form  $\sigma$  as

$$\sigma((\eta,\phi),(\xi,\psi)) = \overline{\xi}^T \left[ A\eta - A \sum_{k=1}^m C_k \phi(-r_k) + \sum_{k=1}^m B_k \phi(-r_k) \right]$$
$$+ \sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} g_k(\theta) \overline{\psi(\theta)}^T \phi'(\theta) \, d\theta$$
$$+ \overline{\psi(0)}^T g_1(0) \left[ \eta - \phi(0) - \sum_{k=1}^m C_k \phi(-r_k) \right].$$

We have the following result, which essentially says that if the weight functions  $g_k$  have appropriate jump discontinuities at the interior delays  $r_1, \ldots, r_{m-1}$ , then (9) holds.

**Lemma 2.** If the scalar weight functions  $g_k$ , k = 1, ..., m are positive and continuously differentiable on  $[-r_{k-1}, -r_k]$  and satisfy

(15) 
$$\frac{1}{2}g_m(-r_m) - g_1(0)m\|C_m\|^2 \ge 0$$

(16)  $g_k(-r_k) - g_{k+1}(-r_k) - 1 - 2g_1(0)m \|C_k\|^2 \ge 0, \quad k = 1, \dots, m-1,$ 

then there exists  $\omega \in \mathbb{R}$  so that (9) holds. That is,

(17) 
$$\operatorname{Re}\sigma(u,u) \le \omega \|u\|_X^2 \quad \forall u \in V.$$

Proof: For  $u = (\eta, \phi) \in V$ , we have

$$\operatorname{Re} \sigma(u, u) = \operatorname{Re} \left\{ \overline{\eta}^{T} A \eta + \sum_{k=1}^{m} \overline{\eta}^{T} (B_{k} - AC_{k}) \phi(-r_{k}) + \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} g_{k}(\theta) \overline{\phi(\theta)}^{T} \phi'(\theta) \, d\theta + g_{1}(0) \overline{\phi(0)}^{T} [\eta - \phi(0) - \sum_{k=1}^{m} C_{k} \phi(-r_{k})] \right\}$$

$$= \operatorname{Re} \left\{ \overline{\eta}^{T} A \eta + \sum_{k=1}^{m} \overline{\eta}^{T} (B_{k} - AC_{k}) \phi(-r_{k}) \right\}$$

$$+ \frac{1}{2} \sum_{k=1}^{m} \left( g_{k}(-r_{k-1}) \| \phi(-r_{k-1}) \|^{2} - g_{k}(-r_{k}) \| \phi(-r_{k}) \|^{2} \right)$$

$$- g_{1}(0) \| \phi(0) \|^{2} - \frac{1}{2} \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} g'_{k}(\theta) \| \phi(\theta) \|^{2} \, d\theta$$

$$+ g_{1}(0) \operatorname{Re} \{ \overline{\phi(0)}^{T} [\eta - \sum_{k=1}^{m} C_{k} \phi(-r_{k})] \}.$$

Next we apply the Cauchy-Schwarz inequality  $\operatorname{Re} \overline{y}^T x \leq \frac{\epsilon}{2} ||x||^2 + \frac{1}{2\epsilon} ||y||^2$  to several of the above terms. We have

Re 
$$\sum_{k=1}^{m} \overline{\eta}^T (B_k - AC_k) \phi(-r_k) \le \frac{m\widetilde{B}}{2} \|\eta\|^2 + \frac{1}{2} \sum_{k=1}^{m} \|\phi(-r_k)\|^2$$

where  $\widetilde{B} = \max_{k} ||B_k - AC_k||^2$ . Also

$$g_1(0) \operatorname{Re} \overline{\phi(0)}^T \eta \le g_1(0) \|\eta\|^2 + \frac{1}{4} g_1(0) \|\phi(0)\|^2,$$

and

$$-g_1(0)\operatorname{Re}\overline{\phi(0)}^T \sum_{k=1}^m C_k \phi(-r_k) \le \frac{1}{4}g_1(0) \|\phi(0)\|^2 + g_1(0)m \sum_{k=1}^m \|C_k\|^2 \|\phi(-r_k)\|^2.$$

It follows that (because the  $\|\phi(0)\|^2$  terms cancel to zero)

$$\operatorname{Re} \sigma(u, u) \leq \|\eta\|^2 (\|A\| + \frac{m\widetilde{B}}{2} + g_1(0)) + \frac{1}{2} \sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} |g'_k(\theta)| \|\phi(\theta)\|^2 d\theta + T_1 + T_2,$$

where

$$T_{1} = -\|\phi(-r_{m})\|^{2} \left(\frac{1}{2}g_{m}(-r_{m}) - g_{1}(0)m\|C_{m}\|^{2}\right)$$
  

$$T_{2} = -\sum_{k=1}^{m-1} \frac{1}{2}\|\phi(-r_{k})\|^{2} \left(g_{k}(-r_{k}) - g_{k+1}(-r_{k}) - 1 - 2g_{1}(0)m\|C_{k}\|^{2}\right).$$

But from the assumptions (15)-(16) on the functions  $g_k$  we get  $T_1 \leq 0$  and  $T_2 \leq 0$ . Thus

(18) 
$$\operatorname{Re} \sigma(u, u) \le T_3 \|\eta\|^2 + T_4 \sum_{k=1}^m \int_{-r_k}^{-r_{k-1}} |g_k(\theta)| \|\phi(\theta)\|^2 d\theta,$$

where

$$T_3 = ||A|| + \frac{mB}{2} + g_1(0), \qquad T_4 = \max_{\substack{1 \le k \le m, \\ \theta \in [-r_{k-1}, -r_k]}} \frac{1}{2} \frac{|g'_k(\theta)|}{g_k(\theta)}.$$

Hence (17) follows with  $\omega = \max\{T_3, T_4\}$ .

We make several remarks. First, it is straightforward to construct a piecewise linear weight function G with jumps at the delays  $-r_1, -r_2, \ldots, -r_{m-1}$  which satisfies (15)-(16). For example this construction was done in [19] and [20]. Second, as observed in [20] and by others, it appears that for the case of multiple delays the jump discontinuities in G at each interior delay  $-r_1, -r_2, \ldots, -r_{m-1}$  are necessary to obtain such a dissipative inequality. Finally, we have not attempted here to find a weight function which gives the 'best' (smallest) value of the dissipative constant  $\omega$  - the argument in the proof yields a rather crude estimate. Indeed if  $\omega < 0$  then (17) guarantees that the semigroup generated by  $\mathcal{A}$  is exponentially stable and that the finite dimensional semigroups to be constructed using  $\sigma$  are exponentially stable uniformly in the discretization parameter. In certain cases such a weight function has been successfully constructed. For example, for single delay neutral equations see [11], [13], and for retarded systems with multiple delays see [12]. For the general multiple delay neutral equations considered here, the construction of weight functions which yield such dissipative inequalities is open for further investigation.

We turn next to the construction of the sequence of finite dimensional subspaces  $V^N$  needed to apply the theorem. We are motivated by [21] and use basis functions similar to the basis functions found in that paper. For a discretization parameter  $N = 1, 2, \ldots$ , choose the meshpoints

$$\theta_{k,j}^N = -r_{k-1} - j\frac{R_k}{N}, \quad k = 1, 2, \dots, m, \quad j = 0, 1, \dots, N,$$

where  $R_k = r_k - r_{k-1}$ . Thus each delay is a meshpoint and although the meshpoints are not equally spaced across the whole interval, they are equally spaced between each delay. With this notation we see that  $\theta_{k,N}^N = \theta_{k+1,0}^N$ , so there are Nm + 1distinct meshpoints. Next we define a set of first order splines across the entire interval [-r, 0] by combining splines defined on each subinterval, taking care to avoid overlap at the endpoints of each subinterval. Thus on the first subinterval (k = 1), define the following first order splines:

$$b_{1,0}^{N}(\theta) = \begin{cases} \frac{N}{R_{1}}(\theta - \theta_{1,1}^{N}) & \text{if } \theta_{1,1}^{N} \leq \theta \leq \theta_{1,0}^{N} \\ 0 & \text{otherwise,} \end{cases}$$
$$b_{1,j}^{N}(\theta) = \begin{cases} -\frac{N}{R_{1}}(\theta - \theta_{1,j-1}^{N}) & \text{if } \theta_{1,j}^{N} \leq \theta \leq \theta_{1,j-1}^{N} \\ \frac{N}{R_{1}}(\theta - \theta_{1,j+1}^{N}) & \text{if } \theta_{1,j+1}^{N} \leq \theta \leq \theta_{1,j}^{N} \\ 0 & \text{otherwise,} \end{cases}$$

for j = 1, ..., N - 1. On the remaining subintervals, for k = 2, ..., m, define

$$b_{k,0}^{N}(\theta) = \begin{cases} -\frac{N}{R_{k-1}}(\theta - \theta_{k-1,N-1}^{N}) & \text{if } \theta_{k-1,N}^{N} \leq \theta \leq \theta_{k-1,N-1}^{N} \\ \\ \frac{N}{R_{k}}(\theta - \theta_{k,1}^{N}) & \text{if } \theta_{k,1}^{N} \leq \theta \leq \theta_{k,0}^{N} \\ \\ 0 & \text{otherwise,} \end{cases}$$

$$b_{k,j}^{N}(\theta) = \begin{cases} -\frac{N}{R_{k}}(\theta - \theta_{k,j-1}^{N}) & \text{if } \theta_{k,j}^{N} \leq \theta \leq \theta_{k,j-1}^{N} \\\\ \frac{N}{R_{k}}(\theta - \theta_{k,j+1}^{N}) & \text{if } \theta_{k,j+1}^{N} \leq \theta \leq \theta_{k,j}^{N} \\\\ 0 & \text{otherwise,} \end{cases}$$

for j = 1, ..., N - 1. On the last subinterval of the last interval (k = m, j = N) define

$$b_{m,N}^{N}(\theta) = \begin{cases} -\frac{N}{R_{m}}(\theta - \theta_{m,N-1}^{N}) & \text{if } -r \leq \theta \leq \theta_{m,N-1}^{N} \\ 0 & \text{otherwise.} \end{cases}$$

There are Nm + 1 of these first order splines, the so-called 'hat functions' corresponding to the distinct mesh points. Here we point out a difference between our construction and the one in [21]. We use mN + 1 splines, but in [21] they use m(N+1) splines. The reason is that at each meshpoint corresponding to an interior delay they have one more spline than we do. In fact all of our splines are the same as those in [21] except those which are nonzero at the interior delays. In particular, instead of our  $b_{k,0}^N(\theta)$  they use two distinct splines essentially corresponding to the two distinct parts of  $b_{k,0}^N(\theta)$  on either side of the delay. Since there are m-1 interior delays, they have m-1 more first order splines than we do. We could still apply our method using their splines, but it would require a differently defined Hilbert space V, and we do not pursue the idea here. It is an issue for further investigation, and we note that it is only an issue for the multiple delay case - for the single delay case our splines are exactly the same as those in [21]. To complete our construction, let  $e_i \in \mathbb{C}^n$  denote the standard Euclidean basis vector (entries all zero except for the value 1 in the *i*th position). Define

$$\mathcal{E}_{l} = \begin{cases} (e_{l}, 0) & \text{if } l = 1, 2, \dots, n \\\\ (0, b_{k,j}^{N}(\theta) e_{i}) & \text{if } l = n + (k-1)Nn + jn + i, \\\\ (0, b_{m,N}^{N}(\theta) e_{i}) & \text{if } l = (mN+1)n + i, \end{cases}$$

for k = 1, 2, ..., m, j = 0, 1, ..., N - 1, and i = 1, 2, ..., n. Then define

$$X^N = \operatorname{span} \left\{ \mathcal{E}_l \right\}_{l=1}^{n[mN+2]},$$

and we observe that  $X^N$  is a subspace of V of dimension n[mN+2]. (Note that because of our earlier remark the subspace  $X^N$  constructed in [21] has dimension n[(N+1)m+1]).

We note that it is possible to arrange the basis functions in a different order. For example, we can define

$$\widetilde{\mathcal{E}}_{l} = \begin{cases} (e_{i}, 0) & \text{if } l = (i-1)(Nm+2) + 1 \\ (0, b_{k,j}^{N}(\theta) e_{i}) & \text{if } l = (i-1)(Nm+2) + (k-1)N + j + 2, \\ (0, b_{m,N}^{N}(\theta) e_{i}) & \text{if } l = i(Nm+2), \end{cases}$$

for k = 1, 2, ..., m, j = 0, 1, ..., N - 1, and i = 1, 2, ..., n. We still have  $X^N = \text{span} \{ \widetilde{\mathcal{E}}_l \}_{l=1}^{n[mN+2]}.$ 

The choice of ordering for the basis functions  $(\mathcal{E}_l \text{ or } \widetilde{\mathcal{E}}_l)$  affects the computation of matrix representations.

Now that we have constructed the finite dimensional spaces  $X^N \subset V$ , we use (10) to define the operators  $A^N$ . In particular, for each  $N = 1, 2, \ldots$ , define the operator  $\mathcal{A}^N : X^N \to X^N$  by

$$\langle \mathcal{A}^N u, v \rangle_X = \sigma(u, v)$$
 for all  $u, v \in X^N$ .

These operators define semigroups  $T^N(t) = e^{t\mathcal{A}^N}$ , and in order to conclude that  $T^N(t)P^N \to T(t)$  strongly on X we must verify (11) and (12). To do this we will take advantage of properties of interpolating linear splines. In particular, for  $N = 1, 2, \ldots$ , and any  $f \in H^3(-r, 0)$  let  $f_I^N$  be the linear spline which interpolates f at the meshpoints  $\theta_{k,j}^N$ ,  $k = 1, \ldots, m$ ,  $j = 0, \ldots, N$ . The interpolating spline has the property that (see [25])

(19) 
$$||f - f_I^N||_{L^2(-r,0)} \leq \mathcal{O}(\frac{1}{N^2})$$

(20) 
$$\|\frac{d}{d\theta}(f - f_I^N)\|_{L^2(-r,0)} \leq \mathcal{O}(\frac{1}{N})$$

and hence  $||f - f_I^N||_{H^1(-r,0)} \to 0$  as  $N \to \infty$ . Next let us define the following space which is the span of our first order splines:

$$Y^{N} = \operatorname{span}\left\{ \{b_{m,N}^{N}(\theta)\} \cup \bigcup_{k=1}^{m} \bigcup_{j=0}^{N-1} b_{k,j}^{N}(\theta) \right\},\$$

and observe that the interpolating spline is an element of  $Y^N$ . In Theorem 1 we take s = 3 and observe that if  $v \in \text{dom}\mathcal{A}^3$ , then

$$v = (\psi(0) + \sum_{k=1}^{m} C_k \psi(-r_k), \psi).$$

Here we have

$$\psi(\theta) = (\psi_1(\theta), \dots, \psi_n(\theta)),$$
  
where  $\psi_i \in H^3(-r, 0)$  for  $i = 1, 2, \dots, n$ . For each  $N = 1, 2, \dots$ , let us define  
 $\psi_I^N(\theta) = ((\psi_1)_I^N, \dots, (\psi_n)_I^N),$ 

where  $(\psi_i)_I^N$  is the linear spline which interpolates  $\psi_i(\theta)$ . Then define

$$v^{N} = \left(\psi_{I}^{N}(0) + \sum_{k=1}^{m} C_{k}\psi_{I}^{N}(-r_{k}), \psi_{I}^{N}\right) \in X^{N}.$$

Because each delay is also a meshpoint, we have

$$(\psi_i)_I^N(-r_k) = \psi_i(-r_k)$$
  
..., *n* and  $k = 0, ..., m$ . From this it follows that

 $v = 0, \dots, m.$  From this it follow  $v - v^N = \left(0, \psi - \psi_I^N\right).$ 

Therefore

for i = 1, .

$$\begin{aligned} \|v - v^N\|_V^2 &= \|\psi - \psi_I^N\|_{H^1(-r,0;\mathbb{C}^n)}^2 \\ &= \sum_{i=1}^n \|\psi_i - (\psi_i)_I^N\|_{H^1(-r,0)}^2 \\ &\to 0 \text{ as } N \to \infty \end{aligned}$$

by (19)-(20). Thus, (12) holds, and it only remains to verify (11). To see that (11) holds, note that for  $u = (\eta, \phi) \in V$  we have

$$\begin{split} |\sigma(u, v - v^{N})| &= |\sigma((\eta, \phi), (0, \psi - \psi_{I}^{N}))| \\ &= \left| \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} g_{k}(\theta) \overline{|\psi(\theta) - \psi_{I}^{N}(\theta)|}^{T} \phi'(\theta) \, d\theta \right| \\ &= \left| \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} \frac{d}{d\theta} \left( g_{k}(\theta) \overline{|\psi(\theta) - \psi_{I}^{N}(\theta)|}^{T} \right) \phi(\theta) \, d\theta \right| \\ &\leq \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} |g_{k}(\theta)| \left| \frac{d}{d\theta} \left( \overline{|\psi(\theta) - \psi_{I}^{N}(\theta)|}^{T} \right) \phi(\theta) \right| \, d\theta \\ &+ \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} |g_{k}'(\theta)| \left| \overline{|\psi(\theta) - \psi_{I}^{N}(\theta)|}^{T} \right) \phi(\theta) \right| \, d\theta \\ &\leq K \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} \left| \frac{d}{d\theta} \left( \overline{|\psi(\theta) - \psi_{I}^{N}(\theta)|}^{T} \right) \phi(\theta) \right| \, d\theta \\ &+ K \sum_{k=1}^{m} \int_{-r_{k}}^{-r_{k-1}} \left| \overline{|\psi(\theta) - \psi_{I}^{N}(\theta)|}^{T} \phi(\theta) \right| \, d\theta, \end{split}$$

where K depends only on the weight functions  $g_k$ . Moreover

$$\begin{aligned} |\sigma(u,v-v^{N})| &\leq K \int_{-r}^{0} \left| \frac{d}{d\theta} \left( \overline{[\psi(\theta) - \psi_{I}^{N}(\theta)]}^{T} \right) \phi(\theta) \right| d\theta \\ &+ K \int_{-r}^{0} \left| \overline{[\psi(\theta) - \psi_{I}^{N}(\theta)]}^{T} \phi(\theta) \right| d\theta, \\ &\leq K \|\phi\|_{L^{2}(-r,0;\mathbb{C}^{n})} \|\psi - \psi_{I}^{N}\|_{H^{1}(-r,0;\mathbb{C}^{n})} \\ &\leq L \|u\|_{X} \|v - v^{N}\|_{V}, \end{aligned}$$

which verifies that (11) holds. It follows from Theorem 1 that  $T^N(t)P^N \to T(t)$  strongly, which is the desired Trotter-Kato semigroup convergence.

In the next section we illustrate the implementation of the scheme in some examples. We close this section by noting that our construction extends in a straightforward manner to higher order splines. In fact, as long as the interpolating splines satisfy properties analogous to (19)-(20), the convergence proof will be virtually unchanged.

Ν	4	8	16	32	64
$\epsilon_S^N$	.0444	.0139	.0040	-	-
$\epsilon_H^N$	.0940	.0230	.0046	-	-
$\epsilon^N$	.0673	.0196	.0063	.0042	.0024

TABLE 1. Approximation error for Example 1

# 3. Examples

In this section we present examples which illustrate implementation of the approximation scheme presented in this paper. In particular, we calculate the matrix representations of (6) and solve the resulting system of differential equations using standard differential equations solvers available in Matlab.

Example 1. Consider the simple scalar equation

$$\dot{x}(t) + \frac{1}{4}\dot{x}(t-1) = x(t) + x(t-1), \quad t \ge 0$$
  
$$x(t) = -t \quad \text{for } -1 \le t \le 0.$$

This example is considered in [20], and the true solution is given by

$$\begin{aligned} x(t) &= h(t) = t - \frac{1}{4} + \frac{1}{4}e^t, & t \in [0, 1], \\ x(t) &= h(t) - \frac{5}{4} - 2(t-1) + \frac{5}{4}e^{t-1} + \frac{3}{16}(t-1)e^{t-1}, & t \in [1, 2]. \end{aligned}$$

In [20] Kappel and Kunisch implement one approximation scheme using cubic splines and another using cubic Hermite splines. The main and significant difference between these schemes and the one constructed in this paper is that in [20] the semidiscrete problems (6) evolve on spaces  $X^N$  which satisfy  $X^N \subset \text{dom }\mathcal{A}$ , and our scheme does not have this restriction involving dom  $\mathcal{A}$ . For this example we have

$$\begin{aligned} \sigma((\eta,\phi),(\xi,\psi)) &= \eta \overline{\xi} + \frac{3}{4}\phi(-1)\overline{\xi} + \int_{-1}^{0} g(\theta)\phi'(\theta)\overline{\psi(\theta)} \, d\theta, \\ &+ g(0) \left[\eta - \phi(0) - \frac{1}{4}\phi(-1)\right]\overline{\psi(0)}. \end{aligned}$$

In the results reported in Table 1 we use a weight function  $g(\theta) \equiv 1$  and for various values of the discretization parameter N we compare  $x^{N}(t)$  with the true solution x(t). In particular, for  $t_{i} = 0.2i$ , i = 1, 2, ..., 10, we define the error term

$$\epsilon^N = \max |x(t_i) - x^N(t_i)|.$$

For this definition of error we compare our scheme with the two schemes in [20]. We denote by  $\epsilon_S^N$  and  $\epsilon_H^N$  the error reported in [20] for cubic splines and cubic Hermite splines, respectively. Our linear spline scheme demonstrates suitable convergence behavior.

TABLE 2. Approximation error for Example 2

Ν	4	8	16	32	64
$\epsilon^N$	.025634	.008336	.002617	.001340	.000765

Example 2. Consider the neutral equation with two delays

$$\dot{x}(t) + \frac{1}{4}\dot{x}(t - \frac{1}{2}) = -x(t) + \frac{1}{4}x(t - 1), \quad t \ge 0$$
  
$$x(t) = -t \quad \text{for } -1 \le t \le 0.$$

The true solution is given by

$$\begin{aligned} x(t) &= h(t) = \frac{1}{4}(3 - t - 3e^{-t}), & t \in [0, 1/2], \\ x(t) &= h(t) - \frac{3}{16} + \frac{1}{32}(9 - 6t)e^{-(t - \frac{1}{2})}, & t \in [1/2, 1]. \end{aligned}$$

We must choose weight functions  $g_1$ ,  $g_2$  so that conditions (15)-(16) are satisfied. For the numerical results we chose

$$g_1(\theta) = a_1 s + 1, \qquad -1/2 \le \theta \le 0, g_2(\theta) = a_2(s + 1/2) + 1, \qquad -1 \le \theta \le -1/2,$$

with  $a_1 = -4$  and  $a_2 = -1$ . In this example we calculate the error  $\epsilon^N$  over the interval  $0 \le t \le 1$  for  $t_i = 0.1i$ , i = 1, 2, ..., 10 and list the result in Table 2.

This approximation scheme has been tested on several problems, and the examples here are illustrative of the qualitative behavior of the scheme. We also refer to [9] where this scheme was used on an LQR control problem for a scalar neutral delay equation with excellent results.

# 4. Conclusion

We have provided a general variational framework (Theorem 1) which allows us to construct a new semidiscrete approximation scheme for neutral delay equations. This extends to neutral equations a scheme which was originally constructed for retarded delay equations in [21]. One advantage of the scheme is the general variational framework, which can accomodate various types of basis functions (we use linear splines). Another advantage of this framework is the possibility of establishing adjoint semigroup convergence, especially if one can choose the Hilbert space V to contain both dom  $\mathcal{A}$  and dom  $\mathcal{A}^*$ . This is a subject for further investigation, although for the scalar single delay case the adjoint semigroup convergence has been established in [10].

# 5. Appendix

We provide a proof of Theorem 1. The theorem and proof are clearly based on ideas in the proof of Cea's Lemma [6], [24] as well as some of the variational versions of the Trotter-Kato semigroup convergence theorem in [17]. We include the result since it is particularly suitable for the approximation framework we construct.

Proof of Theorem 1: Since  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semigroup, it follows that dom  $\mathcal{A}^s$  is dense in H. For any  $x \in X$ ,  $v \in \text{dom } \mathcal{A}^s$ , and  $v^N \in X^N$ , we

have

(21) 
$$||P^N x - x||_X \le ||v^N - x||_X \le c_V ||v^N - v||_V + ||v - x||_X,$$

so by (12) and the denseness of  $\operatorname{dom} \mathcal{A}^s$  we get

(22) 
$$\lim_{N \to \infty} \|P^N x - x\|_X = 0 \quad \text{for all } x \in X.$$

Fix  $\lambda > \omega$ . By (9) it follows that  $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}^N)$  for all N. We claim that

(23) 
$$\lim_{N \to \infty} \| (\lambda I - \mathcal{A}^N)^{-1} P^N w - (\lambda I - \mathcal{A})^{-1} w \|_X = 0 \quad \text{for all } w \in X.$$

To show this claim, we shall first verify (23) on the dense subset dom $A^{s-1}$ , and then use denseness together with a uniform bound on the resolvent to argue that (23) holds on all of X. To proceed, let  $w \in \text{dom}A^{s-1}$ , set

$$y = (\lambda I - \mathcal{A})^{-1} w,$$

and for each N set

$$y^N = (\lambda I - \mathcal{A}^N)^{-1} P^N w.$$

Thus  $y \in \text{dom}A^s$ , and by hypothesis there exists a sequence of vectors  $\{\tilde{y}^N\}_{N=1}^{\infty}$ ,  $\tilde{y}^N \in V^N$ , with the properties (11) and (12). Observe that

(24) 
$$\lambda \langle y, \Omega \rangle_X - \sigma(y, \Omega) = \langle w, \Omega \rangle_X \quad \forall \Omega \in V,$$

and for each N,

$$\lambda \langle y^N, \Omega \rangle_X - \sigma(y^N, \Omega) = \langle w, \Omega \rangle_X \quad \forall \Omega \in V^N$$

Of course, (24) also holds for all  $\Omega \in V^N$ . Thus we may subtract the above equations and for each N we have

.

(25) 
$$\lambda \langle y - y^N, \Omega \rangle_X - \sigma(y - y^N, \Omega) = 0 \quad \forall \Omega \in V^N$$

Hence for each N and all  $\Omega \in V^N$  we have

$$\begin{split} \lambda \langle y - y^N, y - y^N \rangle_X &- \sigma (y - y^N, y - y^N) \\ &= \lambda \langle y - y^N, y - \Omega \rangle_X - \sigma (y - y^N, y - \Omega) \\ &+ \lambda \langle y - y^N, \Omega - y^N \rangle_X - \sigma (y - y^N, \Omega - y^N) \\ &= \lambda \langle y - y^N, y - \Omega \rangle_X - \sigma (y - y^N, y - \Omega) \end{split}$$

by (25) since  $\Omega - y^N \in V^N$ . Thus for each N

$$\begin{split} \lambda \|y - y^N\|_X^2 &= \lambda \langle y - y^N, y - \Omega \rangle_X + \sigma(y - y^N, y - y^N) - \sigma(y - y^N, y - \Omega) \\ &= \lambda \langle y - y^N, y - \Omega \rangle_X - \sigma(y - y^N, y^N - \Omega) \\ \end{split}$$
(26) 
$$\begin{split} &= \lambda \operatorname{Re} \langle y - y^N, y - \Omega \rangle_X - \operatorname{Re} \sigma(y - y^N, y^N - \Omega), \end{split}$$

for all  $\Omega \in V^N$  since  $\lambda$  is real. It follows from (9) and (26) that for each N and all  $\Omega \in V^N$ ,

$$\begin{aligned} (\lambda - \gamma) \|y - y^{N}\|_{X}^{2} &= \lambda \|y - y^{N}\|_{X}^{2} - \gamma \|y - y^{N}\|_{X}^{2} \\ &\leq \lambda \|y - y^{N}\|_{X}^{2} - \operatorname{Re} \sigma(y - y^{N}, y - y^{N}) \\ &= \lambda \operatorname{Re} \langle y - y^{N}, y - \Omega \rangle_{X} - \operatorname{Re} \sigma(y - y^{N}, y^{N} - \Omega) \\ &- \operatorname{Re} \sigma(y - y^{N}, y - y^{N}) \\ &= \lambda \operatorname{Re} \langle y - y^{N}, y - \Omega \rangle_{X} - \operatorname{Re} \sigma(y - y^{N}, y - \Omega) \\ &\leq |\lambda| \|y - y^{N}\|_{X} \|y - \Omega\|_{X} + |\sigma(y - y^{N}, y - \Omega)|. \end{aligned}$$

$$(27)$$

Now for each N we choose  $\Omega = \tilde{y}^N$ , where  $\{\tilde{y}^N\}_{N=1}^{\infty}$  is the sequence noted above with the properties (11) and (12). It follows that for each N

$$(\lambda - \gamma) \|y - y^N\|_X^2 \le |\lambda| \|y - y^N\|_X \|y - \tilde{y}^N\|_X + L \|y - y^N\|_X \|y - \tilde{y}^N\|_V,$$

and hence

(28) 
$$(\lambda - \gamma) \|y - y^N\|_X \le |\lambda| \|y - \tilde{y}^N\|_X + L \|y - \tilde{y}^N\|_V.$$

Thus

(29) 
$$\lim_{N \to \infty} \|(\lambda I - \mathcal{A}^N)^{-1} P^N w - (\lambda I - \mathcal{A})^{-1} w\|_X = 0$$

for all  $w \in \text{dom}A^{s-1}$ . Since  $\text{dom}A^{s-1}$  is dense in X and we have the uniform bound  $\|(\lambda I - A^N)^{-1}P_X^N\|_X \leq \frac{1}{\lambda - \gamma}$  for all N, it follows that (23) is true for all  $w \in X$ , and the claim (23) is proved. The result now follows from Theorem 3.1 in [15].  $\Box$ 

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