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SOME ERROR ESTIMATES OF FINITE VOLUME ELEMENT APPROXIMATION FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, finite volume element method is applied to solve the distributed optimal control problems governed by an elliptic equation. We use the method of variational discretization concept to approximate the problems. The optimal order error estimates in L^2 and L^∞-norm are derived for the state, costate and control variables. The optimal H^1 and $W^{1,\infty}$-norm error estimates for the state and costate variables are also obtained. Numerical experiments are presented to test the theoretical results.

Key words. finite volume element method, variational discretization, optimal control problems, elliptic equation, distributed control.

1. Introduction

The finite volume element method is a discretization technique for partial differential equations. Due to its local conservative property and other attractive properties, such as the robustness with the unstructured meshes, the finite volume element method is widely used in computational fluid dynamics. In general, two different functional spaces (one for the trial space and one for the test space) are used in the finite volume element method. Owing to the two different spaces, the numerical analysis of the finite volume element method is more difficult than that of the finite element method and finite difference method. Since the method was proposed, there have been many results in the literature. Early work for the finite volume element method can be found in [2, 5, 7, 13, 15, 19]. In [2], Bank and Rose obtain the result that the finite volume approximation is comparable with the finite element approximation in H^1-norm. The optimal L^2-error estimate is obtained in [13, 19] under the assumption that $f \in H^1$. In [19], the authors also obtain the H^1-norm and maximum-norm error estimates. In [7], Chatzipantelidis proposes a nonconforming finite volume element method and obtains the L^2-norm and H^1-norm error estimates. Recently, Ye proposes a discontinuous finite volume element method. Unified error analysis for conforming, nonconforming and discontinuous finite volume element method is presented in [16]. High order finite volume element method can be found in, e.g., [8, 14]. For other recently development, we refer readers to see [6, 18, 21, 28] and the references therein.

The optimal control problems introduced in [23] are playing an increasingly important role in science and engineering. They have various application in the

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operation of physical, social, and economic processes. Finite element method is an important numerical method for the problems of partial differential equations and widely used in the numerical solution of optimal control problems. Only for the optimal control problems governed by linear elliptic equation, there have been many results in the literature. For instance, some a priori error estimates of the finite element approximation for the optimal control problems are established in [24]. A posteriori error estimates and adaptive finite element methods are studied in [22, 24]. Some superconvergence results are reported in, e.g. [24, 25]. The error estimates of mixed finite element approximation for optimal control problems are investigated in, for example, [11, 24]. Furthermore, some superconvergence results of the mixed finite element method are obtained in, e.g., [11, 24]. Other numerical methods for optimal control problems can be seen in [3, 12, 17, 29].

In most of these papers, the state and costate (adjoint state) variables are discretized by continuous linear elements and the control variable by piecewise constant or piecewise linear polynomials. The approximate order of the control variable is $O(h)$ or $O(h^{3/2})$ in the sense of L^2-norm or L^∞-norm (see, e.g., [26]). In [20], Hinze proposes a variational discretization concept for optimal control problems with control constraints. With the variational discretization concept, the control variable is not discretized directly, but discretized by a projection (defined later, see (3.7)) of the discrete costate variable. The convergent order of the control variable is $O(h^2)$.

There are two approaches to find the approximate solution of the optimal control problems governed by partial differential equation. One is of the *optimize-then-discretize* type. One first applies the Lagrange multiplier methods to obtain an optimal system, at the continuous level, consisting of the state equation, an adjoint equation and an optimal condition. Then one use some numerical method to discretize the resulting system. The other is of the *discretize-then-optimize* type. One first discretizes the optimal control problems by some means and then applies the Lagrange multiplier rule to the resulting discrete optimization problem. The two discrete systems, determined by the two approaches, are the same when finite element method is used. But in general, these discrete systems are not the same. In [17], the streamline upwind Galerkin method is applied to approximate the solution of elliptic optimal control problems using the *optimize-then-discretize* approach. In [29], the authors also use the *optimize-then-discretize* approach to solve the optimal control problem governed by convection dominated diffusion equation.

In engineering, there exist widely optimal control problems governed by fluid flow equation. And the finite volume element method is widely used in computational fluid dynamics. To our best knowledge, there is no published result in which the finite volume element method is applied to solve the optimal control problems. We want to use finite volume element method to solve fluid optimal control problems. But here we will use the *optimize-then-discretize* approach and the *finite volume element* method to find the approximation of elliptic optimal control problems.

In this paper, we consider the following optimal control problems: Find y, u such that

$$\min_{u\in U_{ad}} \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}, \tag{1.1}$$

$$-\nabla \cdot (A\nabla y) = Bu + f, \text{ in } \Omega, \tag{1.2}$$

$$y = 0, \text{ on } \Gamma, \tag{1.3}$$

where $\Omega \subset R^2$ is a bounded convex polygon domain and Γ is the boundary of Ω, α is a positive number, $f,\ y_d \in L^2(\Omega)$ or $H^1(\Omega)$, $A = (a_{i,j}(x))$ is a 2×2

symmetric, smooth enough and uniformly positive definite matrix in Ω, B is a bounded continuous linear operator, U_{ad} is denoted by

$$U_{ad} = \{u \in L^2(\Omega) : a \le u(x) \le b,\ a.e. \text{ in } \Omega,\ a, b \in \mathrm{R}\}.$$

We first apply Lagrange multiplier method to the problem (1.1)-(1.3) and obtain an optimal system. Then we use finite volume element method to discretize the state and adjoint equation of the system. For the optimal condition (variational inequality), we use the variational discretization concept to obtain the control. Assume that (y_h, p_h, u_h) is the numerical solution of the finite volume element method for the problem (1.1)-(1.3). Under some reasonable assumption, we mainly obtain the following results:

$$\begin{aligned} &||u-u_h||_{L^2(\Omega)} + ||y-y_h||_{L^2(\Omega)} + ||p-p_h||_{L^2(\Omega)} = O(h^2),\\ &||y-y_h||_{H^1(\Omega)} + ||p-p_h||_{H^1(\Omega)} = O(h),\\ &||u-u_h||_{L^\infty(\Omega)} + ||y-y_h||_{L^\infty(\Omega)} + ||p-p_h||_{L^\infty(\Omega)} = O(h^2|lnh|^{1/2}). \end{aligned}$$

The remainder of this paper is organized as follows. In Section 2, we present some notations and describe the finite volume element method briefly. In Section 3, we apply the piecewise linear finite volume element method and variational discretization concept to the problem (1.1)-(1.3) and obtain the discretized optimal system. In Section 4, we analyze the error estimates between the exact solution and the finite volume element approximation. And in Section 5, a numerical example is presented to test the theoretical results.

Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h.

2. Preliminaries

To begin with, we use the standard notations for Sobolev spaces $W^{m,p}(\Omega)$ with $1 \le p \le +\infty$ and their associated norms (see, e.g., [1, 4]). To simplify the notations, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the index $p = 2$ and Ω whenever possible, i.e., $||u||_{m,2,\Omega} = ||u||_{m,2} = ||u||_m, ||u||_0 = ||u||$. As usual, we also use $(\cdot,\cdot)$ to denote the $L^2(\Omega)$-inner product.

For the convex polygonal domain, we consider a quasi-uniform triangulation $\mathcal{T}_h$ consisting of closed triangle elements K such that $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} K$. We use N_h to denote the set of all nodes or vertices of $\mathcal{T}_h$. To define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$, we divide each $K \in \mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of K. The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as is shown in Figure 1. The dual partition $\mathcal{T}_h^*$ consists of the union of the control volume V_i. Let $h = max\{h_K\}$, where h_K is the diameter of the triangle K. As is shown in [19], the dual partition $\mathcal{T}_h^*$ is also quasi-uniform, i.e., there exists a positive constant C such that

$$C^{-1}h^2 \le meas(V_i) \le Ch^2,\ \forall\ V_i \in \mathcal{T}_h^*.$$

We define a finite dimensional space V_h (i.e. trial space) associated with $\mathcal{T}_h$ for the trial functions by

$$V_h = \{v :\ v \in C(\Omega),\ v|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h,\ v|_\Gamma = 0\}$$

and define a finite dimensional space Q_h (i.e. test space) associated with the dual partition $\mathcal{T}_h^*$ for the test functions by

$$Q_h = \{q \in L^2(\Omega) :\ q|_V \in P_0(V),\ \forall\ V \in \mathcal{T}_h^*;\ q|_{V_z} = 0,\ z \in \Gamma\},$$

where $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V.

To connect the trial space and test space, we define a transfer operator $I_h : V_h \to Q_h$ as follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i)\chi_i, \quad I_h v_h|_{V_i} = v_h(z_i), \ \forall \ V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i. For the operator I_h, it is well known that there exists a positive constant C such that for all $v \in V_h$

$$||v - I_h v|| \leq Ch||v||_1. \tag{2.1}$$

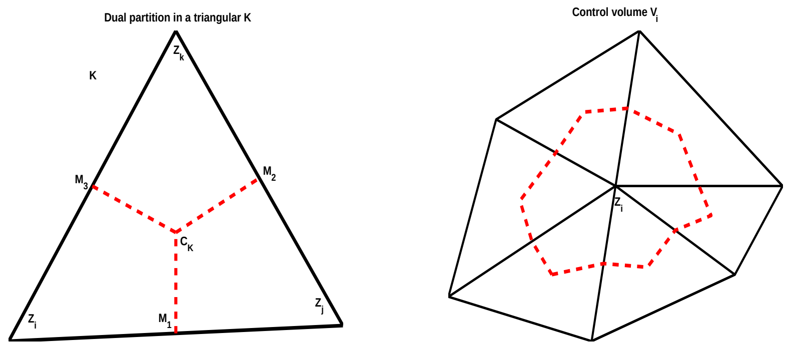

Figure 1. The dual partition of a triangular K on the left hand side and a control volume V_i on the right hand side.

To address the finite volume element method clearly, we consider the following problem

$$-\nabla \cdot (A\nabla\phi) = f, \text{ in } \Omega, \tag{2.2}$$
$$\phi = 0, \text{ on } \Gamma \tag{2.3}$$

where A, Ω, Γ are the same as in (1.2)-(1.3), $f \in L^2(\Omega)$ or $H^1(\Omega)$.

The finite volume element approximation ϕ_h of (2.2)-(2.3) is defined as the solution of the problem: Find $\phi_h \in V_h$ such that

$$a(\phi_h, I_h v_h) = (f, I_h v_h), \ \forall \ v_h \in V_h, \tag{2.4}$$

where the bilinear form $a(\phi_h, I_h v_h)$ is defined by

$$a(\phi, I_h v) = -\sum_{z_i \in N_h} v(z_i) \int_{\partial V_i} A\nabla\phi \cdot \mathbb{n} ds, \quad \phi, v \in H_0^1(\Omega),$$

where $\mathbb{n}$ is the unit outward normal vector to ∂V_i.

The bilinear form $a(\cdot,\cdot)$ is not symmetric though the problem is self-adjoint. It has the following property (see, e.g., [15, Lemma 2.4]). For all $w_h, v_h \in V_h$, there exist positive constants C and $h_0 \geq 0$ such that for all $0 < h < h_0$

$$|a(w_h, I_h v_h) - a(v_h, I_h w_h)| \leq Ch||w_h||_1 \, ||v_h||_1. \tag{2.5}$$

3. Finite volume element method for the optimal control problem

As is seen in [23], the necessary and sufficient optimal condition (system) consists of the state equation, a costate equation and a variational inequality, i.e., find $(y, p, u) \in H_0^1(\Omega) \times H_0^1(\Omega) \times U_{ad}$ such that

$$\begin{cases} (A\nabla y, \nabla w) = (Bu + f, w), \ \forall\ w \in H_0^1(\Omega), \\ (A\nabla p, \nabla q) = (y - y_d, q), \ \forall\ q \in H_0^1(\Omega), \\ (\alpha u + B^* p, v - u) \geq 0, \ \forall\ v \in U_{ad}. \end{cases} \tag{3.1}$$

If $y \in H_0^1(\Omega) \cap C^2(\Omega)$ and $p \in H_0^1(\Omega) \cap C^2(\Omega)$, then optimal system (3.1) can be written by

$$\begin{cases} -\nabla \cdot (A\nabla y) = Bu + f, \quad \text{in } \Omega, \quad y = 0, \ \text{on } \Gamma, \\ -\nabla \cdot (A\nabla p) = y - y_d, \ \text{in } \Omega, \quad p = 0, \ \text{on } \Gamma, \\ (\alpha u + B^* p, v - u) \geq 0, \ \forall\ v \in U_{ad}. \end{cases} \tag{3.2}$$

We use finite volume element method to discretized the state and costate equation directly. Then the continuous optimal system (3.2) can be approximated by: Find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ such that

$$a(y_h, I_h w_h) = (Bu_h + f, I_h w_h), \forall\ w_h \in V_h, \tag{3.3}$$

$$a(p_h, I_h q_h) = (y_h - y_d, I_h q_h), \forall\ q_h \in V_h, \tag{3.4}$$

$$(\alpha u_h + B^* p_h, v - u_h) \geq 0, \ \forall\ v \in U_{ad}. \tag{3.5}$$

Introducing a projection (see, e.g., [24, 20])

$$P_{[a,b]}(f(x)) = \max(a, \min(b, f(x))),$$

we can denote the variational inequality in system (3.2) by

$$u(x) = P_{[a,b]}(-\frac{B^* p}{\alpha}). \tag{3.6}$$

And the variational inequality (3.5) is equivalent to

$$u_h(x) = P_{[a,b]}(-\frac{B^* p_h}{\alpha}). \tag{3.7}$$

Then the discrete optimal system can be rewritten by: Find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ such that

$$\begin{cases} a(y_h, I_h w_h) = (Bu_h + f, I_h w_h), \forall\ w_h \in V_h, \\ a(p_h, I_h q_h) = (y_h - y_d, I_h q_h), \forall\ q_h \in V_h, \\ u_h(x) = P_{[a,b]}(-\frac{B^* p_h}{\alpha}). \end{cases} \tag{3.8}$$

The system (3.8) or (3.3)-(3.5) is an approximation of (3.1). But the existence and the uniqueness of the solution for (3.8) are not clear. To prove them, we first present an auxiliary lemma (Lemma 3.1) and then verify the existence and the uniqueness.

Let $y_h(u)$ be the solution of

$$a(y_h(u), I_h w_h) = (Bu + f, I_h w_h), \forall w_h \in V_h \tag{3.9}$$

and $p_h(y)$ be the solution of

$$a(p_h(y), I_h q_h) = (y - y_d, I_h q_h), \forall q_h \in V_h. \tag{3.10}$$

For $y_h(u)$, $p_h(y)$, noting that $y_h = y_h(u_h)$, $p_h = p_h(y_h)$, we have the following results.

Lemma 3.1. *Assume that $y_h(u), p_h(y)$ are the solutions of (3.9) and (3.10), respectively. Then the following results hold:*

$$||p_h(y) - p_h||_1 \leq C||y - y_h||, \tag{3.11}$$

$$||y_h(u) - y_h||_1 \leq C||u - u_h||. \tag{3.12}$$

Proof. Subtracting (3.4) from (3.10), we have

$$a(p_h(y) - p_h, I_h q_h) = (y - y_h, I_h q_h), \forall q_h \in V_h.$$

Let $q_h = p_h(y) - p_h$. Then (3.11) can easily follows from [19, Lemma 2.2] and the Cauchy-Schwarz inequality. In the same way, (3.12) can be verified easily. □

Lemma 3.2. *The system (3.3)-(3.5) admits a unique solution for sufficiently small h.*

Proof. We first introduce a projection $P_k : L^2(\Omega) \to U_{ad}$ which is defined by

$$||z - P_k(z)|| = \min_{z_h \in U_{ad}} ||z - z_h||. \tag{3.13}$$

The projection P_k has the property of

$$||P_k(z') - P_k(z'')|| \leq ||z' - z''||, \ \forall z', z'' \in L^2(\Omega). \tag{3.14}$$

For a given $v_h \in L^2(\Omega)$, Let $(y_h(v_h), p_h(v_h))$ be the solution of the following auxiliary problem: Find $(y_h(v_h), p_h(v_h)) \in V_h \times \ V_h$ such that

$$a(y_h(v_h), I_h w_h) = (f + Bv_h, I_h w_h), \forall w_h \in V_h, \tag{3.15}$$

$$a(p_h(v_h), I_h q_h) = (y_h(v_h) - y_d, I_h q_h), \forall q_h \in V_h. \tag{3.16}$$

Define a mapping $\Phi : L^2(\Omega) \to L^2(\Omega)$ by

$$\Phi(z_h) = z_h - \rho(\alpha z_h + B^* p_h(z_h)), \forall z_h \in L^2(\Omega), \rho > 0. \tag{3.17}$$

Let $T(z_h) = P_k \Phi(z_h)$. Then the proof of the existence and uniqueness of (3.3)-(3.5) is to show that $T(z_h)$ is a contractive mapping. It follows from (3.14) that for all $z'_h, z''_h \in L^2(\Omega)$

$$\begin{aligned}
&||T(z'_h) - T(z''_h)||^2 = ||P_k(\Phi(z'_h)) - P_k(\Phi(z''_h))||^2 \\
&\leq ||\Phi(z'_h) - \Phi(z''_h)||^2 = (\Phi(z'_h) - \Phi(z''_h), \Phi(z'_h) - \Phi(z''_h)).
\end{aligned}$$

Note that

$$\begin{aligned}
& (\Phi(z'_h) - \Phi(z''_h), \Phi(z'_h) - \Phi(z''_h)) \\
= \ & (1 - 2\rho\alpha)(z'_h - z''_h, z'_h - z''_h) \\
& -2\rho(B(z'_h - z''_h), p_h(z'_h) - p_h(z''_h)) \\
& +\rho^2||\alpha(z'_h - z''_h) + B^* p_h(z'_h) - B^* p_h(z''_h)||^2.
\end{aligned}$$

We have

$$\begin{aligned}
& ||T(z'_h) - T(z''_h)||^2 \\
\leq \ & (1 - 2\rho\alpha)(z'_h - z''_h, z'_h - z''_h) \\
& -2\rho(B(z'_h - z''_h), p_h(z'_h) - p_h(z''_h)) \\
& +\rho^2||\alpha(z'_h - z''_h) + B^* p_h(z'_h) - B^* p_h(z''_h)||^2.
\end{aligned} \tag{3.18}$$

For $z'_h, z''_h \in L^2(\Omega)$, it follows from (3.15)-(3.16) that

$$a(y_h(z'_h) - y_h(z''_h), I_h w_h) = (B(z'_h - z''_h), I_h w_h), \forall w_h \in V_h,$$
$$a(p_h(z'_h) - p_h(z''_h), I_h q_h) = (y_h(z'_h) - y_h(z''_h), I_h q_h), \forall q_h \in V_h.$$

Let $w_h = p_h(z_h') - p_h(z_h'')$ and $q_h = y_h(z_h') - y_h(z_h'')$. We have

$$\begin{aligned}
& (B(z_h' - z_h''), p_h(z_h') - p_h(z_h'')) \\
= \ & (y_h(z_h') - y_h(z_h''), I_h(y_h(z_h') - y_h(z_h''))) \\
& + \{a(y_h(z_h') - y_h(z_h''), I_h(p_h(z_h') - p_h(z_h''))) \\
& - a(p_h(z_h') - p_h(z_h''), I_h(y_h(z_h') - y_h(z_h'')))\} \\
& + (B(z_h' - z_h''), (p_h(z_h') - p_h(z_h'')) - I_h(p_h(z_h') - p_h(z_h''))) \\
\geq \ & \{a(y_h(z_h') - y_h(z_h''), I_h(p_h(z_h') - p_h(z_h''))) \\
& - a(p_h(z_h') - p_h(z_h''), I_h(y_h(z_h') - y_h(z_h'')))\} \\
& + (B(z_h' - z_h''), (p_h(z_h') - p_h(z_h'')) - I_h(p_h(z_h') - p_h(z_h''))),
\end{aligned}$$

where we have use the fact that $(v_h, I_h v_h) \geq 0$ (see, e.g., [21, Lemma 3.2]). Using [15, Lemma 2.4] and Lemma 3.1, we have

$$\begin{aligned}
& \{a(y_h(z_h') - y_h(z_h''), I_h(p_h(z_h') - p_h(z_h''))) \\
& - a(p_h(z_h') - p_h(z_h''), I_h(y_h(z_h') - y_h(z_h'')))\} \\
& \geq -C_1 h ||p_h(z_h') - p_h(z_h'')||_1 \, ||y_h(z_h') - y_h(z_h'')||_1 \\
& \geq -C_1 C_2 h ||z_h' - z_h''||^2. \qquad (3.19)
\end{aligned}$$

Note that (2.1) and Lemma 3.1. We have

$$\begin{aligned}
& (B(z_h' - z_h''), (p_h(z_h') - p_h(z_h'')) - I_h(p_h(z_h') - p_h(z_h''))) \\
& \geq -C_3 h ||p_h(z_h') - p_h(z_h'')||_1 ||| z_h' - z_h''|| \\
& \geq -C_3 C_4 h ||z_h' - z_h''||^2. \qquad (3.20)
\end{aligned}$$

Combining (3.19)-(3.20), we deduce that

$$(B(z_h' - z_h''), p_h(z_h') - p_h(z_h'')) \geq -(C_1C_2 + C_3C_4)h \; ||z_h' - z_h''||^2. \qquad (3.21)$$

Moreover, it is easy to see that

$$||\alpha(z_h' - z_h'') + B^* p_h(z_h') - B^* p_h(z_h'')||^2 \leq C_5 ||z_h' - z_h''||^2. \qquad (3.22)$$

Then it follows from (3.18), (3.21) and (3.22) that

$$||T(z_h') - T(z_h'')||^2 \leq C^* ||z_h' - z_h''||^2 \qquad (3.23)$$

where $C^* = (1 - \rho(2\alpha - 2(C_1C_2 + C_3C_4)h) + \rho^2 C_5)$. For sufficiently small h we have $\alpha - (C_1C_2 + C_3C_4)h > 0$. Then $0 < C^* < 1$ if $0 < \rho < (2\alpha - 2(C_1C_2 + C_3C_4)h)/C_5$. Therefore $T(z_h)$ is a contractive mapping and hence (3.3)-(3.5) admits a unique solution. □

4. Error estimates

In this section, we analyze the error estimates of the finite volume element approximation. We first estimate the error between the exact solution and the FVEM approximation in L^2-norm. Then we estimate H^1-norm error. At the end of this section we present some maximum-norm error estimates.

4.1. L^2 error estimates. In this subsection, we analyze the L^2-error estimates. Owing to the property of the variational inequality, we first estimate the error of the approximate control in L^2-norm. Using the properties of the control, we then estimate the error of the numerical solutions for the state and the costate.

Theorem 4.1. *Assume that* $A \in W^{1,\infty}(\Omega)$ *and* $u, f, y_d \in L^2(\Omega)$. *Let* $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ *and* $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ *be the solutions of (3.1) and (3.3)-(3.5), respectively. Then there exists an* $h_0 > 0$ *such that for all* $0 < h \le h_0$

$$||u - u_h|| \le Ch. \tag{4.1}$$

Moreover, if $A \in W^{2,\infty}(\Omega)$ *and* $u, f, y_d \in H^1(\Omega)$, *then there exists an* $h_0 > 0$ *such that for all* $0 < h \le h_0$

$$||u - u_h|| \le Ch^2. \tag{4.2}$$

Proof. Let $v = u$ in (3.5) and $v = u_h$ in the variational inequality of (3.2). Then we have

$$\begin{aligned}
\alpha(u - u_h, u - u_h) \ &\le \ (B^*(p - p_h), u_h - u) = (p - p_h, B(u_h - u)) \\
&= \ (p - p_h(y), B(u_h - u)) + (p_h(y) - p_h, B(u_h - u)) \\
&= \ (p - p_h(y), B(u_h - u)) + (I_h(p_h(y) - p_h), B(u_h - u)) \\
&\quad +((p_h(y) - p_h) - I_h(p_h(y) - p_h), B(u_h - u)) \\
&= \ (p - p_h(y), B(u_h - u)) + a(y_h - y_h(u), I_h(p_h(y) - p_h)) \\
&\quad +((p_h(y) - p_h) - I_h(p_h(y) - p_h), B(u_h - u))
\end{aligned}$$

The second term can be written by

$$\begin{aligned}
&\quad a(y_h - y_h(u), I_h(p_h(y) - p_h)) \\
&= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&\quad +a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&\quad +(y - y_h, I_h(y_h - y_h(u))) \\
&= a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&\quad +(y - y_h(u), I_h(y_h - y_h(u))) - (y_h - y_h(u), I_h(y_h - y_h(u))) \\
&\le a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&\quad +(y - y_h(u), I_h(y_h - y_h(u))),
\end{aligned}$$

where we have used the fact that $(y_h - y_h(u), I_h(y_h - y_h(u))) \ge 0$. Connecting the previous two inequalities, we have that

$$\begin{aligned}
&\alpha(u - u_h, u - u_h) \\
&\le (p - p_h(y), B(u_h - u)) + (y - y_h(u), I_h(y_h - y_h(u))) \\
&\quad +((p_h(y) - p_h) - I_h(p_h(y) - p_h), B(u_h - u)) \\
&\quad +a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.3}$$

(i) We first consider the case that $A \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. In this case, we can easily obtain

$$\begin{aligned}
I_1 \ &= \ (p - p_h(y), B(u_h - u)) \\
&\le \ ||p - p_h(y)|| \ ||B(u_h - u)|| \\
&\le \ ||p - p_h(y)|| \ ||u_h - u|| \le Ch||u_h - u||,
\end{aligned} \tag{4.4}$$

where we have used the estimate of [19, Theorem 3.5]. Using Lemma 3.1, and noticing the fact that $(I_h(y_h - y_h(u)), I_h(y_h - y_h(u)))$ is equivalent to $(y_h - y_h(u), y_h -$

$y_h(u)$) (see, e.g., [19]), we have that

$$
\begin{aligned}
I_2 &= (y - y_h(u), I_h(y_h - y_h(u))) \\
&\leq ||y - y_h(u)||\ ||y_h - y_h(u)|| \\
&\leq ||y - y_h(u)||\ ||u_h - u|| \leq Ch||u_h - u||. \qquad (4.5)
\end{aligned}
$$

Lemma 3.1 and (2.1) imply that

$$
\begin{aligned}
I_3 &= ((p_h(y) - p_h) - I_h(p_h(y) - p_h), B(u_h - u)) \\
&\leq Ch||p_h(y) - p_h||_1\ ||u_h - u|| \\
&\leq Ch||y - y_h||\ ||u_h - u|| \\
&\leq Ch(Ch||y||_2 + ||u_h - u||)\ ||u_h - u|| \\
&\leq Ch||u_h - u||^2. \qquad (4.6)
\end{aligned}
$$

Using (2.5) and Lemma 3.1, we have that

$$
\begin{aligned}
I_4 &= (a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&\leq Ch||y_h - y_h(u)||_1\ ||p_h(y) - p_h||_1 \\
&\leq Ch||u_h - u||\ ||y - y_h|| \\
&\leq Ch(Ch||y||_2 + ||u_h - u||)\ ||u_h - u|| \\
&\leq Ch||u_h - u||^2. \qquad (4.7)
\end{aligned}
$$

Inequalities of (4.3)-(4.7) imply that there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$ (4.1) holds.

(ii) We then consider the case that $A \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$. In this case, we can easily obtain

$$
\begin{aligned}
I_1 &= (p - p_h(y), B(u_h - u)) \\
&\leq ||p - p_h(y)||\ ||B(u_h - u)|| \\
&\leq ||p - p_h(y)||\ ||u_h - u|| \leq Ch^2||u_h - u||, \qquad (4.8)
\end{aligned}
$$

where we have used the estimate of [19, Theorem 3.5]. Using Lemma 3.1 and [19, Theorem 3.5], and noticing the fact that $(y_h - y_h(u), I_h(y_h - y_h(u)))$ is equivalent to $(y_h - y_h(u), y_h - y_h(u))$ (see, e.g., [19]), we have that

$$
\begin{aligned}
I_2 &= (y - y_h(u), I_h(y_h - y_h(u))) \\
&\leq ||y - y_h(u)||\ ||y_h - y_h(u)|| \\
&\leq ||y - y_h(u)||\ ||u_h - u|| \leq Ch^2||u_h - u||. \qquad (4.9)
\end{aligned}
$$

Using (4.3) and (4.6)-(4.9), we have that there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$ (4.2) holds. $\square$

Theorem 4.2. *Assume that $A \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (3.1) and (3.3)-(3.5), respectively. Then there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$
||y - y_h|| + ||p - p_h|| \leq Ch. \qquad (4.10)
$$

Moreover, if $A \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$, then there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$

$$
||y - y_h|| + ||p - p_h|| \leq Ch^2. \qquad (4.11)
$$

Proof. Using the triangle inequality, we have that

$$||y - y_h|| \le ||y - y_h(u)|| + ||y_h(u) - y_h||,$$
$$||p - p_h|| \le ||p - p_h(y)|| + ||p_h(y) - p_h||.$$

Lemma 3.1 implies that

$$||y - y_h|| \le ||y - y_h(u)|| + C||u - u_h||, \tag{4.12}$$
$$||p - p_h|| \le ||p - p_h(y)|| + C||y - y_h||. \tag{4.13}$$

(i) We first consider the case that $A \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. In this case, noticing [19, Theorem 3.5], we can easily obtain

$$||y - y_h(u)|| \le Ch, \quad \text{and} \quad ||u - u_h|| \le Ch. \tag{4.14}$$

From (4.12) and (4.14) we have that

$$||y - y_h|| \le Ch. \tag{4.15}$$

Using (4.13), (4.15), and noticing that $||p - p_h(y)|| \le Ch$, we have

$$||p - p_h|| \le Ch. \tag{4.16}$$

From (4.14)-(4.15) we can immediately obtain (4.10).

(ii) We then consider the case that $A \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$. In this case, noticing [19, Theorem 3.5], we can easily obtain

$$||y - y_h(u)|| \le Ch^2, \quad \text{and} \quad ||p - p_h(y)|| \le Ch^2. \tag{4.17}$$

Using (4.12), (4.13), (4.2) and (4.17), we can immediately obtain (4.11). □

4.2. H^1 error estimates.

In this subsection, we estimate the error of the numerical solutions of the state and costate in H^1-norm.

Theorem 4.3. *Assume that $A \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ are the solutions of (3.1) and (3.3)-(3.5), respectively. Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$*

$$||y - y_h||_1 + ||p - p_h||_1 \le Ch. \tag{4.18}$$

Proof. Using the triangle inequality, we have that

$$||y - y_h||_1 \le ||y - y_h(u)||_1 + ||y_h(u) - y_h||_1,$$
$$||p - p_h||_1 \le ||p - p_h(y)||_1 + ||p_h(y) - p_h||_1.$$

Lemma 3.1 implies that

$$||y - y_h||_1 \le ||y - y_h(u)||_1 + C||u - u_h||, \tag{4.19}$$
$$||p - p_h||_1 \le ||p - p_h(y)||_1 + C||y - y_h||. \tag{4.20}$$

From [19, Theorem 3.3] we can obtain

$$||y - y_h(u)||_1 \le Ch, \quad ||p - p_h(y)||_1 \le Ch. \tag{4.21}$$

From Theorem 4.2 and (4.19)-(4.21) we can easily obtain (4.18) □

4.3. Maximum-norm error estimates. In this subsection, we estimate the error of the numerical solutions of control, state and costate in $L^\infty(\Omega)$-norm. Then we estimate $W^{1,\infty}$-error for the state and costate.

Theorem 4.4. *Assume that* $A \in W^{2,\infty}(\Omega)$ *and* $u, f, y_d \in H^1(\Omega)$. *Let* $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ *and* $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ *be the solutions of (3.1) and (3.3)-(3.5), respectively. Then there exists an* $h_0 > 0$ *such that for all* $0 < h \le h_0$

$$||u - u_h||_\infty \le C||p - p_h||_\infty \le Ch^2(|\ln h|)^{1/2}, \tag{4.22}$$

$$||y - y_h||_\infty \le Ch^2(|\ln h|)^{1/2}. \tag{4.23}$$

Proof. Using the definition of $P_{[a,b]}(\cdot)$ and (3.6)-(3.7), we have that

$$\begin{aligned} ||u - u_h||_\infty &\le C||p - p_h||_\infty \\ &\le C(||p - p_h(y)||_\infty + ||p_h(y) - p_h||_\infty) \\ &\le C||p - p_h(y)||_\infty + C(|\ln h|)^{1/2}||p_h(y) - p_h||_1 \\ &\le C||p - p_h(y)||_\infty + C(|\ln h|)^{1/2}||y - y_h|| \\ &\le Ch^2(|\ln h|)^{1/2}, \end{aligned}$$

where we have used the inverse inequality, [19, Theorem 3.11], Lemma 3.1, and Theorem 4.1. Here we complete the proof of (4.22). Analogous to (4.22), we have that

$$\begin{aligned} ||y - y_h||_\infty &\le ||y - y_h(u)||_\infty + ||y_h(u) - y_h||_\infty \\ &\le ||y - y_h(u)||_\infty + C(|\ln h|)^{1/2}||y_h(u) - y_h||_1 \\ &\le ||y - y_h(u)||_\infty + C(|\ln h|)^{1/2}||u - u_h|| \\ &\le Ch^2(|\ln h|)^{1/2}. \end{aligned}$$

Then we complete the proof of (4.23). □

Theorem 4.5. *Assume that* $A \in W^{2,\infty}(\Omega)$ *and* $u, f, y_d \in H^1(\Omega)$. *Let* $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ *and* $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ *be the solutions of (3.1) and (3.3)-(3.5), respectively. Then there exists an* $h_0 > 0$ *such that for all* $0 < h \le h_0$

$$||p - p_h||_{1,\infty} \le Ch\ |\ln h|, \tag{4.24}$$

$$||y - y_h||_{1,\infty} \le Ch\ |\ln h|. \tag{4.25}$$

Proof. Using the inverse inequality, and considering [19, Theorem 3.10] and Lemma 3.1 , we have that

$$\begin{aligned} ||\nabla(p - p_h)||_\infty &\le ||\nabla(p - p_h(y))||_\infty + ||\nabla(p_h(y) - p_h)||_\infty \\ &\le ||\nabla(p - p_h(y))||_\infty + Ch^{-1}||\nabla(p_h(y) - p_h)|| \\ &\le ||\nabla(p - p_h(y))||_\infty + Ch^{-1}||y - y_h|| \\ &\le Ch\ |\ln h| + Ch \le Ch\ |\ln h|. \end{aligned}$$

Here we complete the proof of (4.24). Analogous to (4.24), we have that

$$\begin{aligned} ||\nabla(y - y_h)||_\infty &\le ||\nabla(y - y_h(u))||_\infty + ||\nabla(y_h(u) - y_h)||_\infty \\ &\le ||\nabla(y - y_h(u))||_\infty + Ch^{-1}||y_h(u) - y_h|| \\ &\le ||\nabla(y - y_h(u))||_\infty + Ch^{-1}||u - u_h|| \\ &\le Ch\ |\ln h| + Ch \le Ch\ |\ln h|. \end{aligned}$$

Then we complete the proof of (4.25). □

5. Numerical example

In order to test the theory of the previous section, we present one numerical example to illustrate them. We solve the discrete problem (3.3)-(3.5) or (3.8) using the algorithm presented in [27].

Example 5.1. *We investigate a distributed optimal control problem with Dirichlet boundary value condition ([25]).*

$$\min_{u\in U_{ad}} \frac{1}{2}||y-y_d||^2_{L^2(\Omega)} + \frac{1}{2}||u||_{L^2(\Omega)}, \tag{5.1}$$

$$-\Delta y = u, \text{ in } \Omega, \quad y = 0, \text{ on } \Gamma, \tag{5.2}$$

where $\Omega = \{(x_1, x_2);\ 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$, Γ *denotes the boundary of* Ω. *The exact state* y *is* $sin(\pi x_1)sin(\pi x_2) - y_g$. *Where* y_g *is the solution of the problem*

$$\begin{aligned} -\Delta y_g &= g, \text{ in } \Omega, \\ y_g &= 0, \text{ on } \Gamma. \end{aligned}$$

The function g *is given by*

$$g(x_1,x_2) = \begin{cases} u_f(x_1,x_2) - a, & if\ u_f(x_1,x_2) < a, \\ 0, & if\ u_f(x_1,x_2) \in [a,b], \\ u_f(x_1,x_2) - b, & if\ u_f(x_1,x_2) > b, \end{cases}$$

with $u_f(x_1,x_2) = 2\pi^2 sin(\pi x_1)sin(\pi x_2)$. *And* $y_d = (4\pi^4+1)sin(\pi x_1)sin(\pi x_2) - y_g$, $p = -2\pi^2 sin(\pi x_1)sin(\pi x_2)$, $u = P_{[a,b]}(-p) = max(a, min(b,p))$, $a = 3$, $b = 15$. *(Choose* $u_h^{(0)}(x) = 8.0$*).*

Table 1. Numerical results of L^2*-error with Delaunay mesh*

finite element method			finite volume element method			
$\|\|u-u_h\|\|$	$\|\|y-y_h\|\|$	$\|\|p-p_h\|\|$	$\|\|u-u_h\|\|$	$\|\|y-y_h\|\|$	$\|\|p-p_h\|\|$	Dof
0.1315	0.0173	0.2046	0.0719	0.0067	0.1320	126
0.0328	0.0040	0.0485	0.0190	0.0015	0.0299	444
0.0084	0.0011	0.0122	0.0048	0.0004	0.0073	1642
0.0021	0.0003	0.0031	0.0012	0.0001	0.0019	6193
0.0005	0.0001	0.0008	0.0003	0.0000	0.0005	23642

Table 2. Numerical results of L^∞*-error with Delaunay mesh*

FEM			FEVM		
$\|\|u-u_h\|\|_\infty$	$\|\|y-y_h\|\|_\infty$	$\|\|p-p_h\|\|_\infty$	$\|\|u-u_h\|\|_\infty$	$\|\|y-y_h\|\|_\infty$	$\|\|p-p_h\|\|_\infty$
0.1090	0.0206	0.1368	0.1741	0.0160	0.2980
0.0292	0.0049	0.0348	0.0594	0.0043	0.0878
0.0074	0.0013	0.0092	0.0186	0.0013	0.0262
0.0022	0.0004	0.0033	0.0052	0.0004	0.0075
0.0006	0.0001	0.0010	0.0014	0.0001	0.0022

To compare with the finite element method, we list the results of finite element approximation and FVEM approximation in the same table. In Table 1, we present the error in L^2-norm for the numerical solution of the triple (u, y, p). In Table 2, we present the error in L^∞-norm for the numerical solution of the triple (u, y, p). We present H^1-error and $W^{1,\infty}$-error in Table 3 and Table 4, respectively. The corresponding convergent rates of FEVM approximation are presented in Figure 2 and Figure 3.

Table 3. Numerical results of $H^1(\Omega)$-error with Delaunay mesh

FEM		FVEM		
$\|\|y-y_h\|\|_1$	$\|\|p-p_h\|\|_1$	$\|\|y-y_h\|\|_1$	$\|\|p-p_h\|\|_1$	Dof
0.1912	3.6573	0.1869	3.6698	126
0.0859	1.6807	0.0854	1.6823	444
0.0421	0.8279	0.0420	0.8282	1642
0.0211	0.4152	0.0210	0.4153	6193
0.0106	0.2100	0.0106	0.2100	23642

Seen from the numerical results listed in these tables, the finite volume element approximation and the finite element approximation have almost the same accuracy. The convergent rates listed in Figure 2 and Figure 3 match the theories derived in the previous section.

Table 4. Numerical results of $W^{1,\infty}(\Omega)$-error with Delaunay mesh

FEM		FVEM		
$\|\|y-y_h\|\|_{1,\infty}$	$\|\|p-p_h\|\|_{1,\infty}$	$\|\|y-y_h\|\|_{1,\infty}$	$\|\|p-p_h\|\|_{1,\infty}$	Dof
0.3266	6.4466	0.3266	6.4466	126
0.1866	3.6797	0.1871	3.6743	444
0.1036	2.0452	0.1037	2.0486	1642
0.0530	1.0474	0.0531	1.0487	6193
0.0279	0.5500	0.0279	0.5501	23642

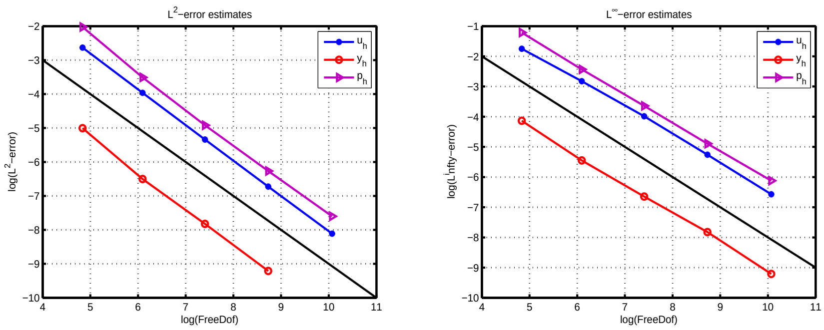

Figure 2. The convergent rates in the L^2-norm on the left hand side and in the L^∞-norm on the right hand side for the finite volume element approximation. (The ***slopes*** *of the solid lines are -1)*

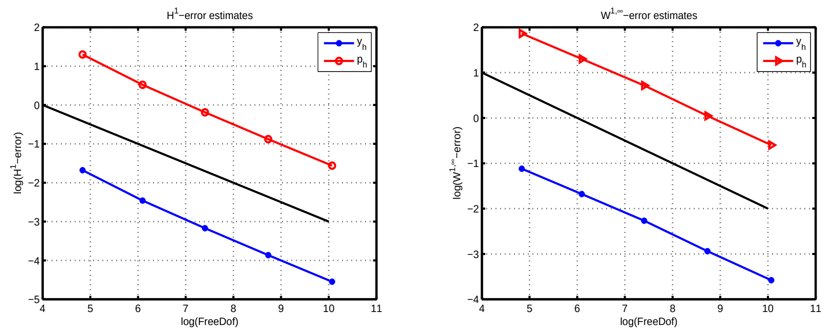

Figure 3. The convergent rates in the H^1-norm on the left hand side and in the $W^{1,\infty}$-norm on the right hand side for the finite volume element approximation. (The ***slopes*** *of the solid lines are $-1/2$)*

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