INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 3, Pages 657–672



# MIXED FOURIER-GENERALIZED JACOBI RATIONAL SPECTRAL METHOD FOR TWO-DIMENSIONAL EXTERIOR PROBLEMS

JINGXIA WU AND ZHONGQING WANG*

**Abstract.** In this paper, we develop a mixed Fourier-generalized Jacobi rational spectral method for two-dimensional exterior problems. Some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations are established. Two model problems are considered. The convergence for the linear problem is proved. Numerical results demonstrate its spectral accuracy and efficiency.

**Key words.** Mixed Fourier-generalized Jacobi rational orthogonal approximations, spectral method, exterior problems.

## 1. Introduction

In the past several decades, spectral method has become increasingly popular in scientific computing and engineering applications (cf. [2, 5, 6, 7, 11, 20] and the references therein). Recently, more and more attentions were paid to its applications to numerical solutions of exterior problems (cf. [9, 10, 16, 17, 22, 25, 26, 28, 29]). Most existing literature of spectral method concerning exterior problems is based on Laguerre polynomial/function approximations. For instance, Guo, Shen and Xu [16] and Zhang and Guo [29] developed the mixed spectral methods for two-/three-dimensional exterior problems, by taking Laguerre polynomials as the basis functions. While Zhang, Wang and Guo [28] and Wang, Guo and Zhang [26] studied the mixed spectral methods for two-/three-dimensional exterior problems, by taking Laguerre functions as the basis functions. Besides, some authors also considered the pseudospectral method for symmetric solutions of certain specific exterior problems, which are reduced to one-dimensional problems on the half line, see [17, 25].

On the other hand, spectral methods based on rational approximations are developed rapidly, which are also very effective for simulating numerically various partial differential equations (PDEs) on unbounded domains, see [3, 4, 8, 14, 15]. By using this approach, we could also approximate differential equations on unbounded domains directly, without any artificial boundary and variable transformation. However, the existing rational functions are usually induced by the Legendre or Chebyshev polynomials. Accordingly, the weight functions of the corresponding orthogonal systems are fixed, which might not be the most appropriate in many cases. This drawback limits the applications of rational spectral method seriously.

Received by the editors July 24, 2010 and, in revised form, September 4, 2012.

2000 *Mathematics Subject Classification.* 65M70, 41A20, 35K55.

*Corresponding author. This work was partially supported by NSF of China, N.11171225, Innovation Program of Shanghai Municipal Education Commission, N.12ZZ131, Shuguang Project of Shanghai Education Commission, N.08SG45, and the Fund for E-institute of Shanghai Universities, N.E03004.

A natural idea is to construct an orthogonal system of rational functions induced by the Jacobi polynomials, so that the related rational spectral method is available for more practical problems, see [24]. But the orthogonal system given in [24] is induced by the standard Jacobi polynomials. Hence its application is still limited. Recently, some authors introduced a family of generalized Jacobi orthogonal polynomials/functions, see [12, 13, 21]. Meanwhile, Guo and Yi [18], and Yi and Guo [27] investigated the generalized Jacobi rational orthogonal approximations on unbounded domains, which enlarges applications of rational spectral method. The previous statements motivate us further study and applications of the generalized Jacobi rational spectral method for exterior problems.

This paper is devoted to the mixed Fourier-generalized Jacobi rational spectral method for two-dimensional exterior problems. We shall establish some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations. As examples, we design the mixed spectral schemes for two model problems and analyze the numerical error of the linear problem. Especially, taking suitable base functions, the resultant linear discrete systems are symmetric and sparse. Thereby, we can resolve them efficiently. The suggested method also provides accurate numerical solutions with the spectral accuracy.

The paper is organized as follows. In Section 2, we establish some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations. In Section 3, we propose the mixed spectral method for a linear model problem and analyze its numerical error. In Section 4, we present some numerical results for two model problems. The final section is for some concluding remarks.

## 2. Mixed orthogonal approximations

In this section, we derive some results on the mixed Fourier-generalized Jacobi rational orthogonal approximations.

**2.1. Generalized Jacobi rational orthogonal approximations.** Let $\omega^{\alpha,\beta}(y) = (1-y)^\alpha(1+y)^\beta$. Denote the standard Jacobi polynomials by $J_n^{\alpha,\beta}(y),\ \alpha,\ \beta > -1,\ n \geq 0$. Let $\Gamma(y)$ be the Gamma function. For $\alpha,\ \beta > -1$, the set of Jacobi polynomials forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, i.e.,

$$\int_{-1}^{1} J_m^{\alpha,\beta}(y) J_n^{\alpha,\beta}(y)\omega^{\alpha,\beta}(y)dy = \gamma_n^{\alpha,\beta}\delta_{m,n}, \tag{2.1}$$

where $\delta_{m,n}$ is the Kronecker symbol, and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

The Jacobi polynomials fulfill the recurrence relation (cf. [23]):

$$\begin{aligned} &2n(n+\alpha+\beta)(2n+\alpha+\beta-2)J_n^{\alpha,\beta}(y) \\ &= (2n+\alpha+\beta-1)\big[(2n+\alpha+\beta)(2n+\alpha+\beta-2)y + \alpha^2 - \beta^2\big] J_{n-1}^{\alpha,\beta}(y) \\ &-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)J_{n-2}^{\alpha,\beta}(y). \end{aligned} \tag{2.2}$$

For convenience of statements, we denote the set of real numbers by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and the set of negative integers by $\mathbb{N}^-$. For any $\alpha, \beta \in \mathbb{R}$, we set

$$\hat{\alpha} := \begin{cases} -\alpha, & \alpha \le -1, \\ 0, & \alpha > -1, \end{cases} \qquad \bar{\alpha} := \begin{cases} -\alpha, & \alpha \le -1, \\ \alpha, & \alpha > -1, \end{cases}$$

(likewise for $\hat{\beta}$ and $\bar{\beta}$). The symbol $[\alpha]$ represents the largest integer $\le \alpha$, and

$$n_0 := n_0^{\alpha,\beta} := [\hat{\alpha}] + [\hat{\beta}], \qquad n_1 := n_1^{\alpha,\beta} := n - n_0^{\alpha,\beta}.$$

The generalized Jacobi polynomials/functions with $\alpha, \beta \in \mathbb{R}$ are defined by (cf. [12, 13, 21])

$$j_n^{\alpha,\beta}(y) = \omega^{\hat{\alpha},\hat{\beta}}(y) J_{n_1}^{\bar{\alpha},\bar{\beta}}(y), \qquad n \ge n_0^{\alpha,\beta}. \tag{2.3}$$

We can rewrite (2.3) in a more explicit form:

$$j_n^{\alpha,\beta}(y) = \begin{cases} J_n^{\alpha,\beta}(y), & \alpha,\ \beta > -1, \\ (1+y)^{-\beta} J_{n_1}^{\alpha,-\beta}(y), & \alpha > -1,\ \beta \le -1,\ n_1 = n - [-\beta], \\ (1-y)^{-\alpha} J_{n_1}^{-\alpha,\beta}(y), & \alpha \le -1,\ \beta > -1,\ n_1 = n - [-\alpha], \\ (1-y)^{-\alpha}(1+y)^{-\beta} J_{n_1}^{-\alpha,-\beta}(y), & \alpha,\ \beta \le -1,\ n_1 = n - [-\alpha] - [-\beta]. \end{cases} \tag{2.4}$$

We next turn to the generalized Jacobi rational functions. To this end, let $\Lambda = (0, \infty)$ and $\chi(x)$ be a certain weight function. For any integer $r \ge 0$, we define the weighted Sobolev space $H_\chi^r(\Lambda)$ in the usual way, and denote its inner product, semi-norm and norm by $(u,v)_{r,\chi,\Lambda}$, $|v|_{r,\chi,\Lambda}$ and $\|v\|_{r,\chi,\Lambda}$, respectively. In particular, $L_\chi^2(\Lambda) = H_\chi^0(\Lambda)$, $(u,v)_{\chi,\Lambda} = (u,v)_{0,\chi,\Lambda}$ and $\|v\|_{\chi,\Lambda} = \|v\|_{0,\chi,\Lambda}$. For any $r > 0$, we define $H_\chi^r(\Lambda)$ and its norm by space interpolation. When $\chi(x) \equiv 1$, we omit the subscript $\chi$ in the notations. We also denote by $\omega_R^{\alpha,\beta}(x) = x^\beta (1+x)^{-\alpha-\beta-2}$. Obviously,

$$\omega_R^{\alpha,\beta}(x) \cong x^{-\alpha-2}, \text{ as } x \to \infty, \qquad \omega_R^{\alpha,\beta}(x) \cong x^\beta, \text{ as } x \to 0. \tag{2.5}$$

The generalized Jacobi rational functions are defined by

$$R_n^{\alpha,\beta}(x) = j_n^{\alpha,\beta}\left(\frac{x-1}{x+1}\right), \qquad n \ge n_0, \quad x \in \Lambda, \tag{2.6}$$

which are the eigenfunctions of the following Sturm-Liouville problem (cf. [27]):

$$\partial_x(\omega_R^{\alpha-3,\beta+1}(x)\partial_x v(x)) + \lambda_n^{\alpha,\beta}\omega_R^{\alpha,\beta}(x)v(x) = 0, \qquad x \in \Lambda. \tag{2.7}$$

The corresponding eigenvalues are

$$\lambda_n^{\alpha,\beta} = \begin{cases} n_1(n_1+\alpha+\beta+1), & \alpha,\ \beta > -1, \\ n_1(n_1+\alpha-\beta+1) - \beta(\alpha+1), & \alpha > -1,\ \beta \le -1, \\ n_1(n_1-\alpha+\beta+1) - \alpha(\beta+1), & \alpha \le -1,\ \beta > -1, \\ (n_1+1)(n_1-\alpha-\beta), & \alpha,\ \beta \le -1. \end{cases}$$

According to [27], if $\alpha \le -2$ and $\beta = -1$, then

$$\partial_x R_n^{\alpha,\beta}(x) = -4(n - [-\alpha] - [-\beta] + 1)(x+1)^{-2} R_{n-1}^{\alpha+1,\beta+1}(x). \tag{2.8}$$

Moreover, the set of the generalized Jacobi rational functions forms a complete $L^2_{\omega_R^{\alpha,\beta}}(\Lambda)$ -orthogonal system, i.e.,

$$\int_\Lambda R_m^{\alpha,\beta}(x)R_n^{\alpha,\beta}(x)\omega_R^{\alpha,\beta}(x)dx = 2^{-\alpha-\beta-1}\eta_n^{\alpha,\beta}\delta_{m,n}, \tag{2.9}$$

where $\delta_{m,n}$ is the Kronecker symbol, and

$$\eta_n^{\alpha,\beta} = \frac{2^{\bar{\alpha}+\bar{\beta}+1}\Gamma(n_1+\bar{\alpha}+1)\Gamma(n_1+\bar{\beta}+1)}{(2n_1+\bar{\alpha}+\bar{\beta}+1)\Gamma(n_1+1)\Gamma(n_1+\bar{\alpha}+\bar{\beta}+1)}.$$

Thus, for any $v \in L^2_{\omega_R^{\alpha,\beta}}(\Lambda)$,

$$v(x) = \sum_{n=n_0}^{\infty} \hat{v}_n^{\alpha,\beta} R_n^{\alpha,\beta}(x), \qquad \hat{v}_n^{\alpha,\beta} = 2^{\alpha+\beta+1}(\eta_n^{\alpha,\beta})^{-1}\int_\Lambda v(x)R_n^{\alpha,\beta}(x)\omega_R^{\alpha,\beta}(x)dx. \tag{2.10}$$

Moreover, integrating (2.7) with $v = R_m^{\alpha,\beta}(x)$ by parts yields

$$\int_\Lambda \partial_x R_m^{\alpha,\beta}(x)\partial_x R_n^{\alpha,\beta}(x)\omega_R^{\alpha-3,\beta+1}(x)dx = 2^{-\alpha-\beta-1}\lambda_n^{\alpha,\beta}\eta_n^{\alpha,\beta}\delta_{m,n}. \tag{2.11}$$

For any $N \in \mathbb{N}$, we set

$$\mathcal{R}_N^{\alpha,\beta} = \text{span}\{R_l^{\alpha,\beta}(x),\ n_0 \le l \le N\}.$$

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega_R^{\alpha,\beta}}(\Lambda) \to \mathcal{R}_N^{\alpha,\beta}$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega_R^{\alpha,\beta},\Lambda} = 0, \qquad \forall \phi \in \mathcal{R}_N^{\alpha,\beta}.$$

For any $r \in \mathbb{N}$, we define the space

$$H^r_{\omega_R^{\alpha,\beta},A}(\Lambda) = \{\ v \mid v \text{ is measureable on } \Lambda \text{ and } \|v\|_{r,\omega_R^{\alpha,\beta},A} < \infty\},$$

equipped with the semi-norm and norm,

$$\begin{aligned}
&\|v\|_{0,\omega_R^{\alpha,\beta},A} = \|v\|_{\omega_R^{\alpha,\beta},\Lambda}, \qquad |v|_{1,\omega_R^{\alpha,\beta},A} = \|(x+1)^2\partial_x v\|_{\omega_R^{\alpha+1,\beta+1},\Lambda},\\
&|v|_{k,\omega_R^{\alpha,\beta},A} = |(x+1)^2\partial_x v|_{k-1,\omega_R^{\alpha+1,\beta+1},A}, \qquad k \ge 2,\\
&\|v\|_{r,\omega_R^{\alpha,\beta},A} = (\sum_{k=0}^{r} |v|^2_{k,\omega_R^{\alpha,\beta},A})^{\frac{1}{2}}.
\end{aligned} \tag{2.12}$$

For any real $r \ge 0$, we define the space $H^r_{\omega_R^{\alpha,\beta},A}(\Lambda)$ and its norm by space interpolation as in [1].

**Lemma 2.1.** *(cf. Theorem 3.1 of* [27]*). If one of the following conditions holds,*

(2.13)

(i) $\alpha,\ \beta > -1$, (ii) $\alpha > -1,\ \beta \le -r-1$ *or* $\beta \in \mathbb{N}^-$,

(iii) $\alpha \le -r-1$ *or* $\alpha \in \mathbb{N}^-,\ \beta > -1$, (iv) $\alpha,\ \beta \le -r-1$ *or* $\alpha,\ \beta \in \mathbb{N}^-$,

*then for any* $v \in H^r_{\omega_R^{\alpha,\beta},A}(\Lambda)$ *and integers* $0 \le \mu \le r \le N+1$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\omega_R^{\alpha,\beta},A} \le cN^{\mu-r}|v|_{r,\omega_R^{\alpha,\beta},A}. \tag{2.14}$$

We also introduce the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda),\ 0\le\mu\le 1$ with the norm $\|v\|_{\mu,\alpha,\beta,\gamma,\delta,\Lambda}$. For $\mu=0,\ H^0_{\alpha,\beta,\gamma,\delta}(\Lambda)=L^2_{\omega_R^{\gamma,\delta}}(\Lambda)$. For $\mu=1$,

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)=\{\ v\mid v\text{ is measurable and }\|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}<\infty\},$$

where $\|v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}=(|v|^2_{1,\omega_R^{\alpha,\beta},\Lambda}+\|v\|^2_{\omega_R^{\gamma,\delta},\Lambda})^{\frac12}$. For $0<\mu<1$, the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Lambda)$ and its norm are defined by space interpolation. Moreover, ${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)=\{v\mid v\in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\text{ and }v(0)=0\}$. Next let

$${}_0\mathcal{R}^{\gamma,\delta}_N=\{\ v\mid v\in\mathcal{R}^{\gamma,\delta}_N\text{ and }v(0)=0\}$$

and

$$a_{\alpha,\beta,\gamma,\delta}(u,v)=(\partial_x u,\partial_x v)_{\omega_R^{\alpha,\beta},\Lambda}+(u,v)_{\omega_R^{\gamma,\delta},\Lambda},\qquad\forall u,v\in{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

If $\beta,\ \delta>-1,\gamma\le-1$ and $\alpha-2\gamma>-3$, then by the definition of ${}_0\mathcal{R}^{\gamma,\delta}_N$, for any $\phi\in{}_0\mathcal{R}^{\gamma,\delta}_N$, $\phi(x)$ is a linear combination of the functions $(1+x)^{\gamma}J_n^{-\gamma,\delta}(\frac{x-1}{x+1}),\ 0\le n\le N-[-\gamma]$. Further, a direct calculation shows $\|\partial_x\phi\|_{\omega_R^{\alpha,\beta},\Lambda}<\infty$. Therefore, ${}_0\mathcal{R}^{\gamma,\delta}_N\subset{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$. Accordingly, in this case, we can define the orthogonal projection ${}_0P^1_{N,\alpha,\beta,\gamma,\delta}:{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\to{}_0\mathcal{R}^{\gamma,\delta}_N$ by

$$a_{\alpha,\beta,\gamma,\delta}({}_0P^1_{N,\alpha,\beta,\gamma,\delta}v-v,\phi)=0,\qquad\forall\phi\in{}_0\mathcal{R}^{\gamma,\delta}_N. \tag{2.15}$$

**Theorem 2.1.** *Let* $\gamma\le-1,\ \delta>-1,\ \alpha-2\gamma>-3,\ \gamma-\alpha-\sigma+2\in\mathbb{N}$, *and* $\sigma\le4,\ \theta\le0$. *If one of the following conditions holds,*

$$\begin{aligned}&\text{(i) }\alpha+\sigma\le-r-1\ \textit{or}\ \alpha+\sigma-1\in\mathbb{N}^-,\ 0\le\beta+\theta\le\delta+2,\\&\text{(ii) }\alpha+\sigma-1\in\mathbb{N}^-,\ \beta+\theta-1\in\mathbb{N}^-,\ \beta>-1,\end{aligned} \tag{2.16}$$

*then for any* $v\in{}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)\bigcap H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda)$ *and integers* $1\le r\le N+1$,

$$\|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v-v\|_{1,\alpha,\beta,\gamma,\delta,\Lambda}\le cN^{1-r}|v|_{r,\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}. \tag{2.17}$$

We can prove Theorem 2.1 by a similar argument as in the proof of Theorem 3.3 of [27].

**2.2. Fourier orthogonal approximation.** Let $I=(0,2\pi)$ and $H^r(I)$ be the Sobolev space with the norm $\|\cdot\|_{r,I}$ and semi-norm $|\cdot|_{r,I}$. For any non-negative integer $m$, $H^m_p(I)$ denotes the subspace of $H^m(I)$, consisting of all functions whose derivatives of order up to $m-1$ have the period $2\pi$. For any real $r>0$, the space $H^r_p(I)$ is defined by space interpolation as in [1].

For any positive integer $M$, we denote by $\widetilde{V}_M(I)=\text{span}\{\ e^{il\theta}\mid|l|\le M\}$, and $V_M(I)$ stands for the subset of $\widetilde{V}_M(I)$ consisting of all real-valued functions. The orthogonal projection $P_M:L^2(I)\to V_M(I)$ is defined by

$$\int_I(P_Mv(\theta)-v(\theta))\phi(\theta)d\theta=0,\qquad\forall\phi\in V_M(I).$$

It was shown in Theorem 2.3 of [11] that for any $v\in H^r_p(I)$, $r\ge0$ and $\mu\le r$,

$$\|P_Mv-v\|_{\mu,I}\le cM^{\mu-r}|v|_{r,I}. \tag{2.18}$$

**2.3. Fourier-generalized Jacobi rational orthogonal approximations.** We are now in position to establish the main results on the mixed Fourier-generalized Jacobi rational orthogonal approximations. For this purpose, let $\Omega = I \times \Lambda$ and introduce the space

$$
\begin{aligned}
{}_0H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega) = \quad & \{\ v \mid v \ \text{ is measurable on } \ \Omega,\ v(x,\theta+2\pi) = v(x,\theta), \\
& v(0,\theta) = 0 \text{ and } \|v\|_{1,\alpha,\beta,\gamma,\delta,\Omega} < \infty\},
\end{aligned}
$$

equipped with the following semi-norm and norm,

$$
\begin{aligned}
|v|_{1,\alpha,\beta,\Omega} &= (\|\partial_x v\|^2_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,L^2(I))} + \|\partial_\theta v\|^2_{L^2_\eta(\Lambda,L^2(I))})^{\frac{1}{2}}, \\
\|v\|_{1,\alpha,\beta,\gamma,\delta,\Omega} &= (|v|^2_{1,\alpha,\beta,\Omega} + \|v\|^2_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,L^2(I))})^{\frac{1}{2}},
\end{aligned}
$$

where $\eta = \frac{1}{x+1}$. Besides,

$$
(u,v)_{\chi,\Omega} = \int_\Omega u(x,\theta)v(x,\theta)\chi(x)dxd\theta, \qquad \|v\|_{\chi,\Omega} = (v,v)^{\frac{1}{2}}_{\chi,\Omega}.
$$

For $\chi(x) \equiv 1$, we drop the subscript $\chi$ in the notations.

Next let $V_{M,N,\alpha,\beta}(\Omega) = V_M(I) \otimes \mathcal{R}^{\alpha,\beta}_N(\Lambda)$ and ${}_0V_{M,N,\alpha,\beta}(\Omega) = V_M(I) \otimes {}_0\mathcal{R}^{\alpha,\beta}_N(\Lambda)$. The orthogonal projection $P_{M,N,\alpha,\beta} : L^2_{\omega_R^{\alpha,\beta}}(\Omega) \to V_{M,N,\alpha,\beta}(\Omega)$ is defined by

$$
(P_{M,N,\alpha,\beta}v - v, \phi)_{\omega_R^{\alpha,\beta},\Omega} = 0, \qquad \forall \phi \in V_{M,N,\alpha,\beta}(\Omega).
$$

**Theorem 2.2.** *If one of the conditions in* (2.13) *holds, then for any* $v \in H^r_{\omega_R^{\alpha,\beta},A}(\Lambda, L^2(I)) \cap L^2_{\omega_R^{\alpha,\beta}}(\Lambda, H^s_p(I))$, $s \geq 0$ *and integers* $0 \leq r \leq N+1$,

$$
\|P_{M,N,\alpha,\beta}v - v\|_{\omega_R^{\alpha,\beta},\Omega} \leq cN^{-r}|v|_{H^r_{\omega_R^{\alpha,\beta},A}(\Lambda,L^2(I))} + cM^{-s}|v|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,H^s(I))}. \tag{2.19}
$$

*Proof.* Clearly, by (2.14), (2.18) and the projection theorem,

$$
\begin{aligned}
\|P_{M,N,\alpha,\beta}v - v\|_{\omega_R^{\alpha,\beta},\Omega} &\leq \|P_{N,\alpha,\beta}P_M v - v\|_{\omega_R^{\alpha,\beta},\Omega} \\
&\leq \|P_{N,\alpha,\beta}P_M v - P_M v\|_{\omega_R^{\alpha,\beta},\Omega} + \|P_M v - v\|_{\omega_R^{\alpha,\beta},\Omega} \\
&\leq cN^{-r}|P_M v|_{H^r_{\omega_R^{\alpha,\beta},A}(\Lambda,L^2(I))} + cM^{-s}|v|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,H^s(I))} \\
&\leq cN^{-r}|v|_{H^r_{\omega_R^{\alpha,\beta},A}(\Lambda,L^2(I))} + cM^{-s}|v|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,H^s(I))}.
\end{aligned}
$$

□

We now assume that $\beta,\ \delta > -1$, $\gamma \leq -1$ and $\alpha - 2\gamma > -3$. Then by a similar argument as before, we can verify readily that ${}_0V_{M,N,\gamma,\delta}(\Omega) \subset {}_0H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)$. Therefore, in this case, we can define the orthogonal projection ${}_0P^1_{M,N,\mu,\alpha,\beta,\gamma,\delta}$ : ${}_0H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega) \to {}_0V_{M,N,\gamma,\delta}(\Omega)$ as

(2.20)

$$
\begin{aligned}
&(\partial_x({}_0P^1_{M,N,\mu,\alpha,\beta,\gamma,\delta}v - v), \partial_x\phi)_{\omega_R^{\alpha,\beta},\Omega} + (\partial_\theta({}_0P^1_{M,N,\mu,\alpha,\beta,\gamma,\delta}v - v), \partial_\theta\phi)_{\eta,\Omega} \\
&+\mu({}_0P^1_{M,N,\mu,\alpha,\beta,\gamma,\delta}v - v, \phi)_{\omega_R^{\gamma,\delta},\Omega} = 0, \qquad \forall \phi \in {}_0V_{M,N,\gamma,\delta}(\Omega), \quad \mu > 0.
\end{aligned}
$$

For description of approximation result, we introduce the non-isotropic space

$$
\begin{aligned}
&B^{r,s}_{\alpha,\beta,\gamma,\delta,\sigma,\theta}(\Omega) \\
= \quad &H^1_{\omega_R^{\alpha,\beta}}(\Lambda, H^{s-1}_p(I)) \cap L^2_\eta(\Lambda, H^s_p(I)) \cap L^2_{\omega_R^{\gamma,\delta}}(\Lambda, H^{s-1}_p(I)) \cap H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda, H^1_p(I)),
\end{aligned}
$$

equipped with the norm

$$\|v\|_{B^{r,s}_{\alpha,\beta,\gamma,\delta,\sigma,\theta}(\Omega)} = \begin{aligned}[t] &(\|v\|^2_{H^1_{\omega_R^{\alpha,\beta}}(\Lambda,H^{s-1}(I))} + \|v\|^2_{L^2_\eta(\Lambda,H^s(I))} \\ &+\|v\|^2_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,H^{s-1}(I))} + \|v\|^2_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,H^1(I))})^{\frac{1}{2}}.\end{aligned}$$

**Theorem 2.3.** *If $\beta > -1$, $\delta \le 0$, and the conditions for (2.17) hold, then for any $s \ge 1$, integers $1 \le r \le N+1$ and $v \in B^{r,s}_{\alpha,\beta,\gamma,\delta,\sigma,\theta}(\Omega) \cap {}_0H^1_{p,\alpha,\beta,\gamma,\delta}(\Omega)$,*

$$\|v - {}_0P^1_{M,N,\mu,\alpha,\beta,\gamma,\delta}v\|_{1,\alpha,\beta,\gamma,\delta,\Omega} \le c(M^{1-s} + N^{1-r}), \tag{2.21}$$

*where $c$ depends only on $|v|_{H^1_{\omega_R^{\alpha,\beta}}(\Lambda,H^{s-1}(I))}$, $|v|_{L^2_\eta(\Lambda,H^s(I))}$, $|v|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,H^{s-1}(I))}$, $|v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,L^2(I))}$ and $|v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,H^1(I))}$.*

*Proof.* By the projection theorem,

$$\|v - {}_0P^1_{M,N,\mu,\alpha,\beta,\gamma,\delta}v\|_{1,\alpha,\beta,\gamma,\delta,\Omega} \le c\|v-\phi\|_{1,\alpha,\beta,\gamma,\delta,\Omega}, \qquad \forall \phi \in {}_0V_{M,N,\gamma,\delta}(\Omega).$$

We take $\phi = {}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v$. With the aid of (2.17) and (2.18), we deduce that

$$\begin{aligned} &\|\partial_x(v - {}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v)\|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,L^2(I))} \\ &\le \|\partial_x v - P_M\partial_x v\|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,L^2(I))} + \|\partial_x(P_M v - {}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v)\|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,L^2(I))} \\ &\le cM^{1-s}|\partial_x v|_{L^2_{\omega_R^{\alpha,\beta}}(\Lambda,H^{s-1}(I))} + cN^{1-r}|P_M v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,L^2(I))} \\ &\le cM^{1-s}|v|_{H^1_{\omega_R^{\alpha,\beta}}(\Lambda,H^{s-1}(I))} + cN^{1-r}|v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,L^2(I))}. \end{aligned} \tag{2.22}$$

Next, due to $\gamma \le -1$ and $\delta \le 0$, we have $(1+x)^{-1} \le \omega_R^{\gamma,\delta}(x)$ for $x \in \Lambda$. Moreover, $\partial_\theta P_M v = P_M \partial_\theta v$. Therefore, we use (2.17) and (2.18) again to obtain

$$\begin{aligned} &\|\partial_\theta(v - {}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v)\|_{L^2_\eta(\Lambda,L^2(I))} \\ &\le \|\partial_\theta v - P_M\partial_\theta v\|_{L^2_\eta(\Lambda,L^2(I))} + \|\partial_\theta(P_M v - {}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v)\|_{L^2_\eta(\Lambda,L^2(I))} \\ &\le \|\partial_\theta v - P_M\partial_\theta v\|_{L^2_\eta(\Lambda,L^2(I))} + c\|\partial_\theta P_M v - {}_0P^1_{N,\alpha,\beta,\gamma,\delta}\partial_\theta P_M v\|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,L^2(I))} \\ &\le cM^{1-s}|v|_{L^2_\eta(\Lambda,H^s(I))} + cN^{1-r}|\partial_\theta P_M v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,L^2(I))} \\ &\le cM^{1-s}|v|_{L^2_\eta(\Lambda,H^s(I))} + cN^{1-r}|v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,H^1(I))}. \end{aligned} \tag{2.23}$$

In the same manner, we verify that

$$\begin{aligned} &\|v - {}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v\|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,L^2(I))} \\ &\le \|v - P_M v\|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,L^2(I))} + \|{}_0P^1_{N,\alpha,\beta,\gamma,\delta}P_M v - P_M v\|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,L^2(I))} \\ &\le cM^{1-s}|v|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,H^{s-1}(I))} + cN^{1-r}|P_M v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,L^2(I))} \\ &\le cM^{1-s}|v|_{L^2_{\omega_R^{\gamma,\delta}}(\Lambda,H^{s-1}(I))} + cN^{1-r}|v|_{H^r_{\omega_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\Lambda,L^2(I))}. \end{aligned} \tag{2.24}$$

Finally, the desired result comes immediately from a combination of (2.22)-(2.24). $\square$

## 3. Spectral method for exterior problems

In this section, we propose the mixed Fourier-generalized Jacobi rational spectral method for exterior problems. We consider the following linear model problem:

$$
(3.1)\quad \begin{cases}
-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}W(\rho,\theta)) - \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}W(\rho,\theta) + \mu W(\rho,\theta) & \\
= F(\rho,\theta), & \mu > 0, \rho > 1, \theta \in \overline{I}, \\
W(\rho,\theta+2\pi) = W(\rho,\theta), & \rho > 1, \theta \in \overline{I}, \\
W(1,\theta) = 0, & \theta \in \overline{I}, \\
\lim_{\rho\to\infty} \rho W(\rho,\theta) = \lim_{\rho\to\infty} \rho\partial\rho W(\rho,\theta) = 0, & \theta \in \overline{I}.
\end{cases}
$$

We make the variable transformation

$$
\rho = x+1, \qquad U(x,\theta) = W(\rho,\theta), \qquad f(x,\theta) = F(\rho,\theta).
$$

Then (3.1) is changed to

$$
(3.2)\quad \begin{cases}
-\partial_x((x+1)\partial_x U(x,\theta)) - \frac{1}{x+1}\partial_\theta^2 U(x,\theta) + \mu(x+1)U(x,\theta) & \\
= (x+1)f(x,\theta), & x > 0, \theta \in \overline{I}, \\
U(x,\theta+2\pi) = U(x,\theta), & x > 0, \theta \in \overline{I}, \\
U(0,\theta) = 0, & \theta \in \overline{I}, \\
\lim_{x\to\infty} xU(x,\theta) = \lim_{x\to\infty} x\partial_x U(x,\theta) = 0, & \theta \in \overline{I}.
\end{cases}
$$

In order to derive a proper weak formulation of (3.2), we introduce the bilinear form with $\mu > 0$,

$$
\begin{aligned}
A_\mu(u,v) = \; & \int_\Omega (x+1)\partial_x u(x,\theta)\partial_x v(x,\theta)dxd\theta + \int_\Omega \frac{1}{x+1}\partial_\theta u(x,\theta)\partial_\theta v(x,\theta)dxd\theta \\
& +\mu \int_\Omega (x+1)u(x,\theta)v(x,\theta)dxd\theta.
\end{aligned}
$$

Obviously

$$
(3.3)\qquad A_\mu(u,v) \le (\mu+1)\|u\|_{1,-3,0,-3,0,\Omega}\|v\|_{1,-3,0,-3,0,\Omega},
$$

$$
(3.4)\qquad A_\mu(v,v) \ge \min(\mu,1)\|v\|^2_{1,-3,0,-3,0,\Omega}.
$$

We now look for the solution of (3.2) in the space ${}_0H^1_{p,-3,0,-3,0}(\Omega)$. In this case, $\lim_{x\to\infty} xU(x,\theta) = \lim_{x\to\infty} x\partial_x U(x,\theta) = 0$ for $\theta \in \overline{I}$. Accordingly, for any $v \in {}_0H^1_{p,-3,0,-3,0}(\Omega)$, $\lim_{x\to\infty}(x+1)\partial_x U(x,\theta)v(x,\theta) = 0$, Therefore, by multiplying (3.2) by $v \in {}_0H^1_{p,-3,0,-3,0}(\Omega)$ and integrating the resulting equation by parts over $\Omega$, we obtain a weak formulation of (3.2). It is to find $U \in {}_0H^1_{p,-3,0,-3,0}(\Omega)$ such that

$$
(3.5)\qquad A_\mu(U,v) = (f,v)_{\omega_R^{-3,0},\Omega} \qquad \forall v \in {}_0H^1_{p,-3,0,-3,0}(\Omega).
$$

The mixed spectral scheme for (3.5) is to seek $u_{M,N} \in {}_0V_{M,N,-3,0}(\Omega)$ such that

$$
(3.6)\qquad A_\mu(u_{M,N},\phi) = (f,\phi)_{\omega_R^{-3,0},\Omega}, \quad \forall \phi \in {}_0V_{M,N,-3,0}(\Omega).
$$

**Theorem 3.1.** *Let $U$ and $u_{M,N}$ be the solutions of* (3.5) *and* (3.6), *respectively. If $\mu > 0$ and $U \in B^{r,s}_{-3,0,-3,0,1,0}(\Omega) \cap {}_0H^1_{p,-3,0,-3,0}(\Omega)$ with $s \geq 1$ and integers $1 \leq r \leq N+1$, then*

$$\|U - u_{M,N}\|_{1,-3,0,-3,0,\Omega} \leq c(M^{1-s} + N^{1-r}). \tag{3.7}$$

*Proof.* Let $U_{M,N} = {}_0P^1_{M,N,\mu,-3,0,-3,0}U$. Then by (2.20) and (3.5),

$$A_\mu(U_{M,N}, \phi) = A_\mu(U, \phi) = (f, \phi)_{\omega_R^{-3,0},\Omega}. \tag{3.8}$$

Subtracting (3.8) from (3.6) yields

$$A_\mu(u_{M,N} - U_{M,N}, \phi) = 0. \tag{3.9}$$

Therefore $u_{M,N} = U_{M,N}$. Finally, we use (2.21) with $\alpha = \gamma = -3$, $\beta = \delta = \theta = 0$ and $\sigma = 1$ to obtain the desired result. $\square$

## 4. Numerical results

In this section, we describe the numerical implementations and present some numerical results.

Let $\psi_k(x) = R_k^{-3,-1}(x)$, $k \geq 4$. Clearly, $\psi_k(0) = 0$ and $\{\psi_k(x)\}_{k=4}^N$ spans the space ${}_0\mathcal{R}_N^{-3,0}(\Lambda)$. The basis functions are chosen as

$$\phi^1_{k,m}(x,\theta) = \frac{1}{\sqrt{2\pi}}\psi_k(x)\sin(m\theta), \quad 4 \leq k \leq N, \quad 1 \leq m \leq M,$$

$$\phi^2_{k,m}(x,\theta) = \frac{1}{\sqrt{2\pi}}\psi_k(x)\cos(m\theta), \quad 4 \leq k \leq N, \quad 0 \leq m \leq M.$$

**4.1. Linear problem.** We begin with the linear problem (3.1). Under the previous basis functions, we write the numerical solution as

$$u_{M,N}(x,\theta) = \sum_{4\leq k\leq N}\sum_{1\leq m\leq M} u^1_{k,m}\phi^1_{k,m}(x,\theta) + \sum_{4\leq k\leq N}\sum_{0\leq m\leq M} u^2_{k,m}\phi^2_{k,m}(x,\theta).$$

Take $\phi = \phi^q_{j,l}$, $q = 1,2$ in (3.6). Then by the orthogonality of trigonometric functions, we obtain a sequence of one-dimensional problems:

$$\begin{aligned}&\sum_{k=4}^{N}\Big(\int_\Lambda (x+1)\partial_x\psi_k(x)\partial_x\psi_j(x)dx + l^2\int_\Lambda \frac{1}{x+1}\psi_k(x)\psi_j(x)dx\\ &+\mu\int_\Lambda (x+1)\psi_k(x)\psi_j(x)dx\Big)u^q_{k,l} = d(l,q)g^q_{j,l}, \quad 4 \leq j \leq N, \quad q = 1,2,\end{aligned} \tag{4.1}$$

where $d(0,2) = 1$, $d(l,q) = 2$ otherwise, and $g^q_{j,l} = (f, \phi^q_{j,l})_{\omega_R^{-3,0},\Omega}$.

For deriving a compact matrix form of (4.1), we introduce the matrices $\mathbb{A} = (a_{j,k})$, $\mathbb{B} = (b_{j,k})$ and $\mathbb{C} = (c_{j,k})$ with the following entries:

$$\begin{aligned}&a_{j,k} = \int_\Lambda (x+1)\partial_x\psi_k(x)\partial_x\psi_j(x)dx, \qquad b_{j,k} = \int_\Lambda \frac{1}{x+1}\psi_k(x)\psi_j(x)dx,\\ &c_{j,k} = \int_\Lambda (x+1)\psi_k(x)\psi_j(x)dx.\end{aligned}$$

With the aid of (2.1), (2.2) and (2.8), a direct calculation shows

$$a_{j,k}=\begin{cases}-\frac{32j^2(j+2)(j-3)(j-2)}{(2j+3)(2j+1)(2j-3)(2j-1)}, & j=k-3,\\ \frac{96(j^3-2j^2+1)(j-3)}{(2j+1)(2j-3)(2j-1)}, & j=k-2,\\ -\frac{480(j^4-5j^3+7j^2-j-2)(j-3)}{(2j+1)(2j-5)(2j-3)(2j-1)}, & j=k-1,\\ \frac{320(j^2-3j+2)(j-3)^2}{(2j-5)(2j-3)(2j-1)}, & j=k,\\ -\frac{480(j^4-9j^3+28j^2-34j+12)(j-4)}{(2j-7)(2j-5)(2j-3)(2j-1)}, & j=k+1,\\ \frac{96(j^3-8j^2+20j-15)(j-5)}{(2j-7)(2j-5)(2j-3)}, & j=k+2,\\ -\frac{32(j-6)(j-5)(j-3)^2(j-1)}{(2j-9)(2j-7)(2j-5)(2j-3)}, & j=k+3,\\ 0, & \text{otherwise},\end{cases}$$

$$b_{j,k}=\begin{cases}\frac{32(j+2)(j-2)(j-3)}{(2j+3)(2j+1)(2j-3)(2j-1)}, & j=k-3,\\ -\frac{32(j^2-j+4)(j-3)}{j(2j+1)(2j-3)(2j-1)}, & j=k-2,\\ -\frac{32(j^4-4j^3+j^2+6j-40)(j-3)}{j(2j+1)(2j-5)(2j-3)(2j-1)(j-1)}, & j=k-1,\\ \frac{64(j^4-6j^3+15j^2-18j+20)(j-3)}{j(2j-5)(2j-3)(2j-1)(j-2)(j-1)}, & j=k,\\ -\frac{32(j^4-8j^3+19j^2-12j-40)(j-4)}{(2j-7)(2j-5)(2j-3)(2j-1)(j-2)(j-1)}, & j=k+1,\\ -\frac{32(j^2-5j+10)(j-5)}{(2j-7)(2j-5)(2j-3)(j-2)}, & j=k+2,\\ \frac{32(j-6)(j-5)(j-1)}{(2j-9)(2j-7)(2j-5)(2j-3)}, & j=k+3,\\ 0, & \text{otherwise},\end{cases}$$

$$c_{j,k}=\begin{cases}\frac{128(j-3)(j-2)}{(2j-3)(2j-1)(j-1)}, & j=k-1,\\ \frac{128(j-3)^2}{(2j-3)(j-2)(j-1)}, & j=k,\\ \frac{128(j-4)(j-3)}{(2j-5)(2j-3)(j-2)}, & j=k+1,\\ 0, & \text{otherwise}.\end{cases}$$

Furthermore, we set

$$\vec{\mathbf{X}}_l^q=(u_{4,l}^q,u_{5,l}^q,\cdots,u_{N,l}^q),\qquad \vec{\mathbf{G}}_l^q=(g_{4,l}^q,g_{5,l}^q,\cdots,g_{N,l}^q),\qquad q=1,2.$$

Then, we have from (4.1) that

$$[\mathbb{A}+l^2\mathbb{B}+\mu\mathbb{C}]\vec{\mathbf{X}}_l^q=d(l,q)\vec{\mathbf{G}}_l^q,\qquad q=1,2.$$

To examine the accuracy of the above scheme, we test the scheme on several examples.

**Example 1.** We take $\mu=1$ in (3.1), and test the exact solution $U(x,\theta)=\frac{x}{x+1}\sin(1+\theta)e^{-\frac{x}{2}}$, which decays exponentially as $x$ increases with oscillation. In Figures 4.1 and 4.2, we plot the discrete $L^2-$ and $L^\infty-$errors against various $M$ with $N=2M$, which indicates an exponential convergence rate.

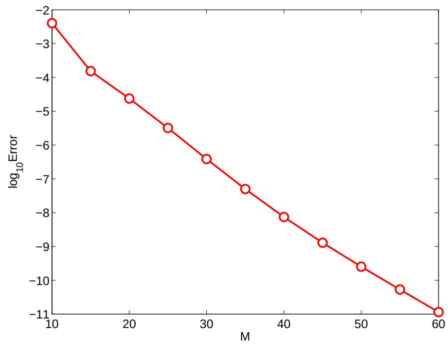


FIGURE 4.1. $L^2-$errors of Example 1.

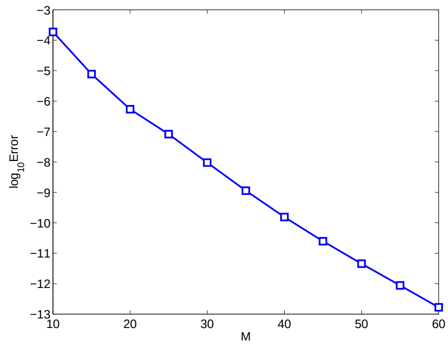


FIGURE 4.2. $L^\infty-$errors of Example 1.

**Example 2.** We take $\mu = 1$ in (3.1), and test the exact solution $U(x, \theta) = \frac{x\sin(1+\theta)}{(x+1)^{k+\frac{1}{2}}}$, $k > \frac{3}{2}$, which decays algebraically at infinity with oscillation. In Figures 4.3 and 4.4, we plot the discrete $L^2-$ and $L^\infty-$errors against various $M$ with $N = 2M$ and $k = 3, 4, 5$, respectively. It is clear that in all cases, the errors decay at certain algebraic rate. It also shows that the smoother the exact solution is, the smaller the numerical errors will be.

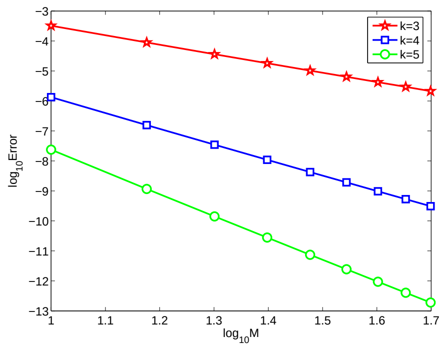


FIGURE 4.3. $L^2-$errors of Example 2.

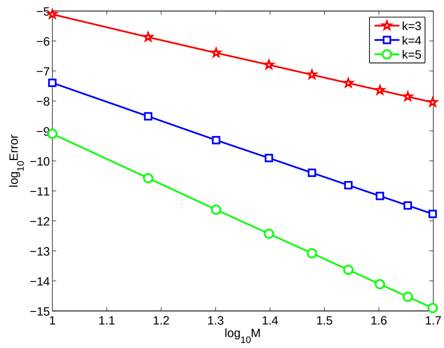


FIGURE 4.4. $L^\infty-$errors of Example 2.

**4.2. Nonlinear problem.** The proposed method is also useful for some nonlinear problems. For instance, we consider the following nonlinear problem (see [19]):

(4.2)

$$\begin{cases} \partial_t W(\rho,\theta,t) - \Delta W(\rho,\theta,t) + W^2(\rho,\theta,t) = F(\rho,\theta,t), & \rho > 1, \theta \in \bar{I}, t \in (0,T], \\ W(\rho,\theta+2\pi,t) = W(\rho,\theta,t), & \rho > 1, \theta \in \bar{I}, t \in [0,T], \\ W(\rho,\theta,0) = W_0(\rho,\theta), & \rho > 1, \theta \in \bar{I}, \\ W(1,\theta,t) = 0, \lim\limits_{\rho\to\infty} \rho W(\rho,\theta,t) = \lim\limits_{\rho\to\infty} \rho\partial_\rho W(\rho,\theta,t) = 0, & \theta \in \bar{I}, t \in [0,T], \end{cases}$$

where the Laplacian:

$$\Delta W(\rho,\theta,t) = \frac{\partial^2 W(\rho,\theta,t)}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial W(\rho,\theta,t)}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 W(\rho,\theta,t)}{\partial\theta^2}.$$

We make the variable transformation

$$\rho = x+1,\ U(x,\theta,t) = W(\rho,\theta,t),\ U_0(x,\theta) = W_0(\rho,\theta),\ f(x,\theta,t) = F(\rho,\theta,t).$$

Then (4.2) is changed to

(4.3)

$$\begin{cases} (x+1)\partial_t U(x,\theta,t) - (x+1)\partial_x^2 U(x,\theta,t) - \partial_x U(x,\theta,t) - & \dfrac{1}{x+1}\partial_\theta^2 U(x,\theta,t) \\ \quad +(x+1)U^2(x,\theta,t) = (x+1)f(x,\theta,t), & x>0, \theta\in\bar{I}, t\in(0,T], \\ U(x,\theta+2\pi,t) = U(x,\theta,t), & x>0, \theta\in\bar{I}, t\in[0,T], \\ U(x,\theta,0) = U_0(x,\theta), & x>0, \theta\in\bar{I}, \\ U(0,\theta,t) = 0,\ \lim\limits_{x\to\infty} xU(x,\theta,t) = \lim\limits_{x\to\infty} x\partial_x U(x,\theta,t) = 0, & \theta\in\bar{I}, t\in[0,T]. \end{cases}$$

Multiplying (4.3) by $v(x,\theta,t)$ and integrating the resulting equation by parts, we derive a weak formulation. It is to find $U(t)\in {}_0H^1_{p,-3,0,-3,0}(\Omega)$ such that

(4.4)

$$\begin{cases} (\partial_t U(t),v)_{\omega_R^{-3,0},\Omega} + (\partial_x U(t),\partial_x v)_{\omega_R^{-3,0},\Omega} + (\partial_\theta U(t),\partial_\theta v)_{\eta,\Omega} + (U^2(t),v)_{\omega_R^{-3,0},\Omega} \\ = (f(t),v)_{\omega_R^{-3,0},\Omega}, \\ U(x,\theta,0) = U_0(x,\theta). \end{cases}$$

The mixed spectral scheme for (4.4) is to seek $u_{M,N}(x,\theta,t)\in {}_0V_{M,N,-3,0}(\Omega)$ such that

(4.5)

$$\begin{cases} (\partial_t u_{M,N}(t),\phi)_{\omega_R^{-3,0},\Omega} + (\partial_x u_{M,N}(t),\partial_x\phi)_{\omega_R^{-3,0},\Omega} + (\partial_\theta u_{M,N}(t),\partial_\theta\phi)_{\eta,\Omega} \\ \quad +(u^2_{M,N}(t),\phi)_{\omega_R^{-3,0},\Omega} = (f(t),\phi)_{\omega_R^{-3,0},\Omega}, \qquad \forall\phi\in {}_0V_{M,N,-3,0}(\Omega), \\ u_{M,N}(x,\theta,0) = P_{M,N,-3,0}U_0(x,\theta). \end{cases}$$

We next write the numerical solution as

$$u_{M,N}(x,\theta,t) = \sum_{4\le k\le N}\sum_{1\le m\le M} u^1_{k,m}(t)\phi^1_{k,m}(x,\theta) + \sum_{4\le k\le N}\sum_{0\le m\le M} u^2_{k,m}(t)\phi^2_{k,m}(x,\theta).$$

Denote by $\tau$ the mesh size in time $t$. The fully discrete scheme for (4.5) is as follows,

(4.6)

$$\begin{aligned} &2(u_{M,N}(t+\tau),\phi)_{\omega_R^{-3,0},\Omega} + \tau(\partial_x u_{M,N}(t+\tau),\partial_x\phi)_{\omega_R^{-3,0},\Omega} + \tau(\partial_\theta u_{M,N}(t+\tau),\partial_\theta\phi)_{\eta,\Omega} \\ &+\tau(u^2_{M,N}(t+\tau),\phi)_{\omega_R^{-3,0},\Omega} = 2(u_{M,N}(t),\phi)_{\omega_R^{-3,0},\Omega} - \tau(\partial_x u_{M,N}(t),\partial_x\phi)_{\omega_R^{-3,0},\Omega} \\ &-\tau(\partial_\theta u_{M,N}(t),\partial_\theta\phi)_{\eta,\Omega} - \tau(u^2_{M,N}(t),\phi)_{\omega_R^{-3,0},\Omega} + \tau(f(t+\tau)+f(t),\phi)_{\omega_R^{-3,0},\Omega}. \end{aligned}$$

Take $\phi = \phi^q_{j,l},\ q = 1,\ 2$ in (4.6). By the orthogonality of trigonometric functions, we also obtain a sequence of one-dimensional problems:

(4.7)

$$\sum_{k=4}^{N}\Big(2\int_\Lambda (x+1)\psi_k\psi_j dx + \tau\int_\Lambda (x+1)\partial_x\psi_k\partial_x\psi_j dx + l^2\tau\int_\Lambda \frac{1}{x+1}\psi_k\psi_j dx\Big)u^q_{k,l}(t+\tau)$$
$$+\tau d(l,q)(u^2_{M,N}(t+\tau),\phi^q_{j,l})_{\omega_R^{-3,0},\Omega} = \sum_{k=4}^{N}\Big(2\int_\Lambda (x+1)\psi_k\psi_j dx - \tau\int_\Lambda (x+1)\partial_x\psi_k\partial_x\psi_j dx$$
$$-l^2\tau\int_\Lambda \frac{1}{x+1}\psi_k\psi_j dx\Big)u^q_{k,l}(t) + d(l,q)g^q_{j,l}(t),\qquad 4\le j\le N,\quad q=1,2,$$

where $d(0,2) = 1$ , $d(l,q) = 2$ otherwise, and

$$g^q_{j,l}(t) = \tau(f(t+\tau)+f(t),\phi^q_{j,l})_{\omega_R^{-3,0},\Omega} - \tau(u^2_{M,N}(t),\phi^q_{j,l})_{\omega_R^{-3,0},\Omega},\qquad q=1,2.$$

Next let $\mathbb{A} = (a_{j,k})$, $\mathbb{B} = (b_{j,k})$ and $\mathbb{C} = (c_{j,k})$ be the same as in the last subsection,

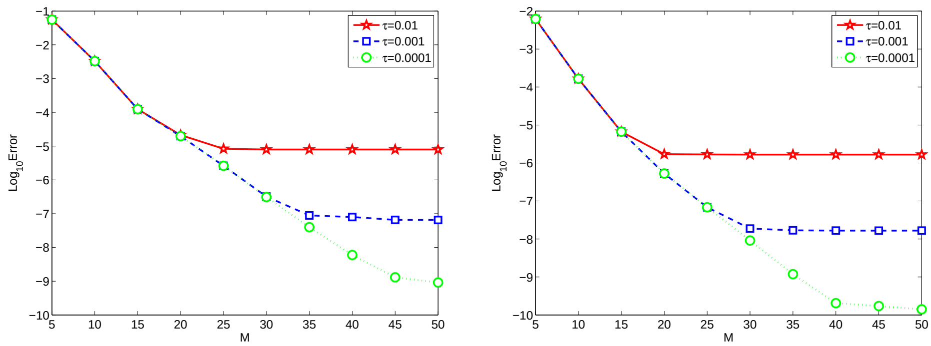


FIGURE 4.5. $L^2-$errors of Example 3.

FIGURE 4.6. $L^\infty-$errors of Example 3.

and set

$$\vec{\mathbf{X}}^q_l(t) = (u^q_{4,l}(t),u^q_{5,l}(t),\cdots,u^q_{N,l}(t))^T,\vec{\mathbf{G}}^q_l(t) = (g^q_{4,l}(t),g^q_{5,l}(t),\cdots,g^q_{N,l}(t))^T, q=1,2.$$

Then, we have from (4.7) that

$$\begin{aligned}&[\tau\mathbb{A}+l^2\tau\mathbb{B}+2\mathbb{C}]\vec{\mathbf{X}}^q_l(t+\tau)+\tau d(l,q)(u^2_{M,N}(t+\tau),\phi^q_{j,l})_{\omega_R^{-3,0},\Omega}\\ &=[-\tau\mathbb{A}-l^2\tau\mathbb{B}+2\mathbb{C}]\vec{\mathbf{X}}^q_l(t)+d(l,q)\vec{\mathbf{G}}^q_l(t).\end{aligned}\tag{4.8}$$

To examine the accuracy of the above scheme, we test the scheme on several examples.

**Example 3.** We test the exact solution $U(x,\theta,t) = \frac{x\sin(1+\theta)\sin(t)}{x+1}e^{-\frac{x}{2}}$, which decays exponentially as $x$ increases with oscillation. In Figures 4.5 and 4.6, we plot the discrete $L^2-$ and $L^\infty-$errors at $t=1$ against various $M$ with $N=2M$ and $\tau = 0.01,\ 0.001,\ 0.0001$, respectively. Clearly, the errors decay exponentially as $M$ and $N$ increase and $\tau$ decreases. It is also observed from Figure 4.5 that for fixed $\tau = 0.01$ and the mode $M\le 25$, the total numerical errors are dominated by the approximation errors in the space and so decay fast as $M$ increases. But for $M>25$, the total numerical errors are dominated by the approximation errors in time $t$. Thus, the numerical results keep the same accuracy, even if $M$ and $N$

increase again. A similar situation happens in other cases, see Figures 4.5-4.6 and 4.9-4.10.

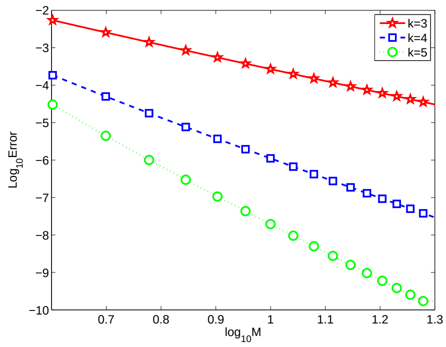


FIGURE 4.7. $L^2-$errors of Example 4.

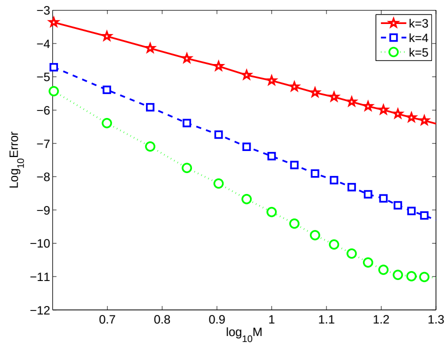


FIGURE 4.8. $L^\infty-$errors of Example 4.

**Example 4.** We test the exact solution $U(x,\theta,t) = \frac{x\sin(1+\theta)\sin(t)}{(x+1)^{k+\frac{1}{2}}}$, $k > \frac{3}{2}$, which decays algebraically at infinity with oscillation. In Figures 4.7 and 4.8, we plot the discrete $L^2-$ and $L^\infty-$errors at $t = 1$ against various $M$ with $N = 2M$, $\tau = 0.0001$ and $k = 3, 4, 5$, respectively. It is clear that in all cases, the errors decay at certain algebraic rate. We also observe that the numerical results with $k = 5$ are better than that with $k = 3, 4$.

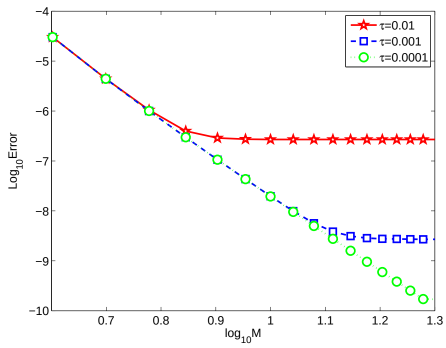


FIGURE 4.9. $L^2-$errors of Example 4.

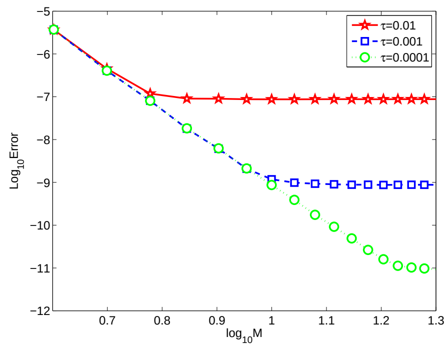


FIGURE 4.10. $L^\infty-$errors of Example 4.

In Figures 4.9 and 4.10, we plot the discrete $L^2-$ and $L^\infty-$errors at $t = 1$ against various $M$ with $N = 2M$, $k = 5$ and $\tau = 0.01,\ 0.001,\ 0.0001$, respectively, which shows that the numerical results with $\tau = 0.0001$ are better than that with $\tau = 0.01,\ 0.001$.

In Figures 4.11 and 4.12, we plot the discrete $L^2-$ and $L^\infty-$errors against various $t$ with $N = 2M = 30$, $k = 5$ and $\tau = 0.001$, which demonstrates the stability of scheme (4.6) for long time calculation.

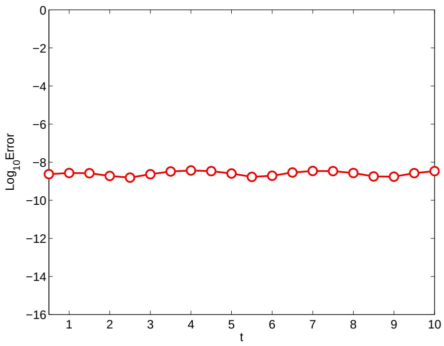


FIGURE 4.11. Stability of $L^2-$errors.

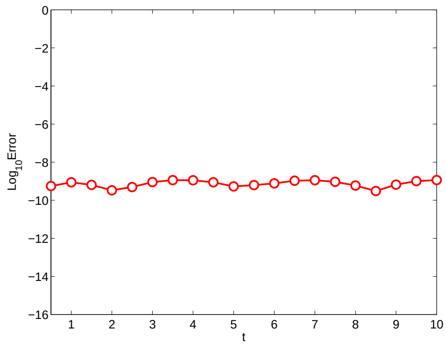


FIGURE 4.12. Stability of $L^\infty-$errors.

## 5. Concluding Remarks

In this paper, we proposed the mixed Fourier-generalized Jacobi rational spectral method for two-dimensional exterior problems, and established some basic results on the mixed Fourier-generalized Jacobi rational orthogonal approximations. These results form the mathematical foundation of the related spectral method for various two-dimensional problems on unbounded or exterior domains. To compare with the existing rational spectral method, the suggested method not only enlarges applications and simplifies numerical analysis, but also leads to very efficient numerical algorithms.

As examples of applications, we provided the mixed spectral schemes for two model exterior problems, and analyzed the convergence of the linear problem. In particular, by choosing suitable basis functions, we are able to design proper numerical algorithms, such that the resultant linear discrete systems are symmetric and sparse. Hence they can be solved efficiently. The numerical results demonstrate their high accuracy.

## References

[1] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction,* Spring-Verlag, Berlin, 1976.

[2] C. Bernardi and Y. Maday, *Spectral Methods, in Handbook of Numerical Analysis,* Vol.5, Techniques of Scientific Computing, 209-486, edited by P. G. Ciarlet and J. L. Lions, Elsevier, Amsterdam, 1997.

[3] J. P. Boyd, Spectral method using rational basis functions on an infinite interval, *J. Comp. Phys.,* **69** (1987), 112-142.

[4] J. P. Boyd, Orthogonal rational functions on a semi-infinite interval, *J. Comp. Phys.,* **70** (1987), 63-88.

[5] J. P. Boyd, *Chebyshev and Fourier Spectral Methods,* Dover Publications Inc., Mineola, NY, second edition, 2001.

[6] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains,* Springer, Berlin, 2006.

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods: Evolution to Complex Geometries and Applications to Fluid Dynamics,* Springer, Berlin, 2007.

[8] C. I. Christov, A complete orthogonal system of functions in space, *SIAM J. on Appl. Math.,* **42** (1982), 1337-1344.

[9] O. Coulaud, D. Funaro and O. Kavian, Laguerre spectral approximation of elliptic problems in exterior domains, *Comput. Meth. in Appl. Mech. and Engin.,* **80** (1990), 451-458.

[10] V. Girault, The Stokes problem and vector potential operator in three dimensional exterior domains, an approach in weighted Sobolev spaces, *Diff. Integ. Equ.,* **7** (1994), 535-570.

[11] Ben-yu Guo, *Spectral Methods and Their Applications,* World Scientific, Singapore, 1998.

[12] Ben-yu Guo, Jie Shen and Li-lian Wang, Optional spectral-Galerkin methods using generalizd Jacobi polynomials, *J. Sci. Comp.,* **27** (2006), 305-322.

[13] Ben-yu Guo, Jie Shen and Li-lian Wang, Generalized Jacobi polynomials/functions and their applications, *Appl. Numer. Math.,* **59** (2009), 1011-1028.

[14] Ben-yu Guo, Jie Shen and Zhong-qing Wang, A rational approximation and its applications to differential equations on the half line, *J. Sci. Comp.,* **15** (2000), 117-147.

[15] Ben-yu Guo, Jie Shen and Zhong-qing Wang, Chebyshev rational spectral and pseudospectral methods on a semi-infinite interval, *Inter. J. Numer. Meth. Engin.,* **53** (2002), 65-84.

[16] Ben-yu Guo, Jie Shen and Cheng-long Xu, Generalized Laguerre approximation and its applications to exterior problems, *J. Comp. Math.,* **23** (2005), 113-130.

[17] Ben-yu Guo, Li-lian Wang and Zhong-qing Wang, Generalized Laguerre interpolation and pseudospectral method for unbound domains, *SIAM J. Numer. Anal,* **43** (2006), 2567-2589.

[18] Ben-yu Guo and Yong-gang Yi, Generalized Jacobi rational spectral method and its applications, *J. Sci. Comp.*, **43** (2010), 201-238.

[19] L. A. Herraiz, A nonliner parabolic problem in an exterior domain, *J. Diff. Equ.,* **142** (1998), 371-412.

[20] G. E. Karniadakis and S. J. Sherwin, *Spectral/hp Element Methods for Computational Fluid Dynamics,* Oxford University Press, Oxford, 2005.

[21] A. B. J. Kuijlaars, A. Martinez-Finkelshtein and R. Orive, Orthogonality of Jacobi polynomials with general parameters, *Elec. Tran. on Numer. Anal.,* **19** (2005), 1-17.

[22] A. Márquez and S. Meddahi, A combination of spectral and finite elements for an exterior problem in the plane, *Appl. Numer. Math.,* **43** (2002), 275-295.

[23] G. Szegö, *Orthogonal Polynomials,* Amer. Math. Soc., 1959.

[24] Zhong-qing Wang and Ben-yu Guo, Jacobi rational approximation and spectral method for differential equations of degenerate type, *Math. Comp.,* **77** (2008), 883-907.

[25] Zhong-qing Wang, Ben-yu Guo and Yan-na Wu, Pseudospectral method using generalized Laguerre functions for singular problems on unbounded domains, *Disc. Cont. Dyn. Sys. B,* **11** (2009), 1019-1038.

[26] Zhong-qing Wang, Ben-yu Guo and Wei Zhang, Mixed spectral method for three-dimensional exterior problems using spherical harmonic and generalized Laguerre functions, *J. Comp. Appl. Math.,* **217** (2008), 277-298.

[27] Yong-gang Yi and Ben-yu Guo, Generalized Jacobi rational spectral method on the half line, *Adv. Comput. Math.,* **37** (2012), 1-37.

[28] Rong Zhang, Zhong-qing Wang and Ben-yu Guo, Mixed Fourier-Laguerre spectral and pseudospectral methods for exterior problems using generalized Laguerre functions, *J. Sci. Comp.,* **36** (2008), 263-283.

[29] Xiao-yong Zhang and Ben-yu Guo, Spherical harmonic-generalized Laguerre spectral method for exterior problems, *J. Sci. Comp.,* **27** (2006), 523-537.



Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China

Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China, Scientific Computing Key Laboratory of Shanghai Universities, Division of Computational Science of E-Institute of Shanghai Universities

*E-mail*: zqwang@shnu.edu.cn.