INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 3, Pages 603–621

AN ALMOST FOURTH ORDER PARAMETER-ROBUST NUMERICAL METHOD FOR A LINEAR SYSTEM OF $(M \geq 2)$ COUPLED SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

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Abstract. We present a high order parameter-robust finite difference method for a linear system of $(M \geq 2)$ coupled singularly perturbed reaction-diffusion two point boundary value problems. The problem is discretized using a suitable combination of the fourth order compact difference scheme and the central difference scheme on a generalized Shishkin mesh. A high order decomposition of the exact solution into its regular and singular parts is constructed. The error analysis is given and the method is proved to have almost fourth order parameter robust convergence, in the maximum norm. Numerical experiments are conducted to demonstrate the theoretical results.

Key words. Parameter-robust convergence, System of coupled reaction-diffusion problem, Generalized-Shishkin mesh, Fourth order compact difference scheme, Central difference scheme.

1. Introduction

Consider the following system of $(M \geq 2)$ coupled singularly perturbed linear reaction-diffusion equations

$$\boldsymbol{T}\boldsymbol{u} := -\boldsymbol{E}\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f},\ x \in\ \Omega = (0,1) \tag{1a}$$

subject to the boundary conditions

$$\boldsymbol{u}(0) = \mathbf{p},\ \boldsymbol{u}(1) = \mathbf{q}, \tag{1b}$$

where $\boldsymbol{E} = \text{diag}(\varepsilon, \dots, \varepsilon)$, with small parameter $0 < \varepsilon \ll 1$. Suppose that the matrix $\boldsymbol{A} : \overline{\Omega} \to \mathbb{R}^{M,M}$ and the vector valued function $\boldsymbol{f} : \overline{\Omega} \to \mathbb{R}^M$ are four times continuously differentiable on $\overline{\Omega}$. We assume that the coupling matrix $\boldsymbol{A} = (\, a_{ij}(x)\,)_{M\times M}$ satisfy the following positivity conditions at each $x \in \overline{\Omega}$

$$a_{ij}(x) \leq 0,\ i \neq j, \tag{2}$$

$$a_{ii}(x) > 0,\ i = 1, \dots, M, \tag{3a}$$

$$\sum_{j=1, j\neq i}^{M} \left\| \frac{a_{ij}}{a_{ii}} \right\|_{\overline{\Omega}} < 1,\ i = 1, \dots, M, \tag{3b}$$

where $\|.\|_{\overline{\Omega}}$ denotes the continuous maximum norm on $\overline{\Omega}$. It is well known that under these assumptions the problem (1) possesses unique solution $\boldsymbol{u} \in C^6(\overline{\Omega})^M$ and exhibits two layers of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$ at both ends of the domain. These types of system of equations appear in the modeling of various physical phenomenon, such as the turbulent interaction of waves and currents [30], predator-prey population dynamics [8] and investigation of diffusion processes complicated by chemical reactions in electro analytic chemistry [29].

Received by the editors September 2, 2011 and, in revised form, May 23, 2012.
2000 *Mathematics Subject Classification.* 65L10, 65L11.

The use of classical numerical methods on uniform mesh for solving these problems may give rise to difficulties when the singular perturbation parameter ε is sufficiently small. This leads to the development of the numerical methods that are parameter-uniform/parameter-robust/uniformly convergent with respect to small parameter ε. There are two classes of parameter-robust numerical methods: fitted operator methods and fitted mesh methods. Such methods require the physical and the mathematical knowledge about the problem. In case of a fitted mesh method the accuracy is guaranteed for a fixed number of mesh points, irrespective of the magnitude of the perturbation parameter. To achieve this a class of non-equidistant meshes, dense in layers, are available in the literature, see [1],[26],[28],[31]. The construction of these meshes depends strongly on priori information of the solution and its derivatives. A wide class of parameter-robust numerical methods based on this approach are discussed in [5],[6],[13],[23], and the references therein.

The methods based on fitted meshes, particularly the Shishkin meshes [28] gained popularity because of their simplicity and applicability to more complicated problems in higher dimensions; see [5],[6],[13],[23]. Further, we refer readers to the review article [9] for the progress on the Shishkin meshes in the area of singular perturbation. The Shishkin mesh is preceded by the Bakhvalov mesh [1], which is somewhat more complicated. Shishkin meshes are piecewise equidistant and typically consist of two(three) equidistant parts on the basis of one(two) transition points: one(two) dense part(s) in the layer(s) and one coarse part outside the layer(s). Bakhvalov meshes are generated by a suitable mesh generating function which appropriately redistributes equidistantly spaced points, so that the mesh is dense in the layer(s) region(s). Combinations of these two meshes are developed in [13] and [27]. Bakhvalov meshes are generalized and simplified in [31] and some improvements of Shishkin meshes are considered in [32],[33]. One of the important modifications of Shishkin mesh called generalized Shishkin mesh is developed by Vulanović and used in establishing the high order parameter-robust convergence of numerical methods, see [32].

Although an extensive amount of literature is available for the numerical solution of (uncoupled) singularly perturbed reaction-diffusion problems, while only few papers deal with the numerical analysis of coupled system of singularly perturbed reaction-diffusion problems. Systems of singularly perturbed problems have been studied as back as Bakhvalov [1]. Shishkin [29] examined a system of two parabolic partial differential equations analogous to (1), posed on an infinite strip. For the system of two coupled reaction-diffusion equations, some parameter-robust numerical methods are designed and analyzed in [3],[4],[14],[15],[19],[22],[23].

It is natural to think about the parameter-robust numerical methods for systems of more than two singularly perturbed reaction-diffusion equations. Kellogg et al. [10] considered a system of singularly perturbed reaction-diffusion problems in two dimensions with the same perturbation parameter for all equations and proved that the standard finite difference method on piecewise-uniform Shishkin mesh is second-order accurate (up to logarithmic factor). Some parameter-robust numerical methods for solving problem (1) are analyzed in [7],[16], and the references therein. But in all the cases the order of convergence is atmost two. Nevertheless, first time in [3] a HODIE technique is used to derive a third order uniformly convergent numerical method for system of two reaction-diffusion equations. High order numerical methods are very convenient from numerical point of view; the reason is that these methods produce small errors with a low computational cost. The objective of the present paper is to construct an almost fourth order parameter-robust numerical method for a linear system of ($M \geq 2$) coupled singularly perturbed reaction-diffusion equations (1).

The paper is arranged as follows. In Section 2, a priori bounds on the solution of (1) and its derivatives are given; and a high order decomposition of the exact solution into its regular and layer parts is constructed. The generalized Shishkin mesh is used to disccretize the domain $\overline{\Omega}$ in Section 3. In Section 4, a high order finite difference scheme which is a suitable combination of the fourth order compact difference scheme and the standard central difference scheme is described on a generalized Shishkin mesh. The error analysis is given and the method is proved to have almost fourth order parameter-robust convergence, in the maximum norm in Section 5. In Section 6, numerical experiments are presented to validate the theoretical results. Finally the conclusions are included in Section 7.

Notations: In the remaining parts of the paper, C is a generic positive constant independent of the perturbation parameter ε and the discretization parameter N. Similarly, $\boldsymbol{C} = (C, C, ..., C)^T$ is a vector of identical constants with the same independencies. Define $\boldsymbol{v} \leq \boldsymbol{w}$ if $v_i \leq w_i,\ 1 \leq i \leq M$ and $|\boldsymbol{v}| = (|v_1|, \ldots, |v_M|)^T$. We consider the maximum norm and it is denoted by $||.||_D$, where D is a closed and bounded set. For a real valued function $v \in C(D)$ and for a vector valued function $\boldsymbol{v} = (v_1, \ldots, v_M)^T \in C(D)^M$, we define

$$||v||_D = \max_{x \in D} |v(x)| \text{ and } ||\boldsymbol{v}||_D = \max\{||v_1||_D, \ldots, ||v_M||_D\}.$$

If $D = \overline{\Omega}$, we drop D from the notation. The analogous discrete maximum norm on the mesh $\overline{\Omega}_N$ is denoted by $||.||_{\overline{\Omega}_N}$. For any function $g \in C(\overline{\Omega})$, g_i is used for $g(x_i)$; if $\mathbf{g} \in C(\overline{\Omega})^M$ then $\mathbf{g}_i = \mathbf{g}(x_i) = (g_{1,i}, \ldots, g_{M,i})^T$. $L(N)$ denotes the value of L with N intervals that solves (18) in Section 3. For simplicity, we use L_{N_0} to denote $L(N_0)$, where N_0 is a positive integer. If $N_0 = N$, we drop N as subscript from the notation and write L for $L(N)$.

2. Properties of the exact solution

2.1. Stability. Suppose the coupling matrix $\boldsymbol{A}$ satisfies (2). Under this assumption it has been proved that the vector valued differential operator $\boldsymbol{T}$ is maximum norm stable, see Linss [17]. The analysis is based on the following stability property for the scalar differential equations.

Lemma 2.1. *Consider the following scalar differential operator*

$$\mathcal{L}v := -\varepsilon v'' + bv' + av$$

with $\varepsilon > 0$, a, $b \in C[0,1]$ *and* $a > 0$ *on* $[0,1]$. *Then,*

$$||v|| \leq \max\left\{ \left\| \frac{\mathcal{L}v}{a} \right\|, |v(0)|, |v(1)| \right\}, \text{ for all } v \in C^2(0,1) \cap C[0,1].$$

If the matrix

$$\Upsilon := \begin{pmatrix} 1 & -\|a_{12}/a_{11}\| & \ldots & -\|a_{1M}/a_{11}\| \\ -\|a_{21}/a_{22}\| & 1 & \ldots & -\|a_{2M}/a_{22}\| \\ \vdots & \vdots & \ddots & \vdots \\ -\|a_{M1}/a_{MM}\| & -\|a_{M2}/a_{MM}\| & \ldots & 1 \end{pmatrix} \tag{4}$$

is inverse monotone, that is, all entries of Υ^{-1} are non-negative, then the following stability result holds.

Lemma 2.2. *(cf. Linss* [17]*) Let* $\boldsymbol{u}$ *be the solution of (1) and that the matrix* $\boldsymbol{A}$ *has strictly positive diagonal entries. Let* Υ *be inverse monotone. Then,*

$$\|u_i\| \leq \sum_{k=1}^{M} (\Upsilon^{-1})_{i\,k} \max \left\{ \left\| \frac{f_k}{a_{kk}} \right\|, |p_k|, |q_k| \right\}, \text{ for } i = 1, \ldots, M. \tag{5}$$

Lemma 2.2 conveys that the vector valued differential operator $\boldsymbol{T}$ is maximum norm stable although it does not in general satisfy the maximum principle. If the coupling matrix $\boldsymbol{A}$ satisfies (3) then Υ is a strictly diagonally dominant matrix with non-positive off-diagonal entries and the M-matrix criterion implies the inverse monotonicity of Υ. Moreover, if $\boldsymbol{f} \in C^4(\overline{\Omega})^M$ and $\boldsymbol{A} \in C^4(\overline{\Omega})^{M\times M}$, the stability of $\boldsymbol{T}$ along with the standard arguments from [11], ensures the existence of unique solution $\boldsymbol{u} \in C^6(\overline{\Omega})^M$.

The assumption (3) ensures the stability of $\boldsymbol{T}$. In addition, if (2) is also assumed, then $\boldsymbol{T}$ satisfies the following maximum principle.

Lemma 2.3. *(Maximum Principle) Assume that the coupling matrix* $\boldsymbol{A}$ *satisfies the positivity conditions (2)-(3). If* $\boldsymbol{u}(0) \geq 0$, $\boldsymbol{u}(1) \geq 0$ *with* $\boldsymbol{Tu} \geq 0$ *on* Ω*, then* $\boldsymbol{u} \geq 0$ *on* $\overline{\Omega}$.

Proof. The proof follows in the same way as that of Theorem 1 given in [21]. □

An immediate consequence of this lemma is the following comparison principle.

Lemma 2.4. *(Comparison Principle) Assume that the coupling matrix* $\boldsymbol{A}$ *satisfies the positivity conditions (2)-(3). If* $\boldsymbol{v}(0) \geq |\boldsymbol{u}(0)|, \boldsymbol{v}(1) \geq |\boldsymbol{u}(1)|$ *and* $\boldsymbol{Tv} \geq |\boldsymbol{Tu}|$ *on* Ω*, then* $\boldsymbol{v} \geq |\boldsymbol{u}|$ *for all* $x \in \overline{\Omega}$.

2.2. A priori bounds on the solution. The analysis in this subsection involves the frequent use of the following auxiliary result, see [1].

Lemma 2.5. *Let* $I := [a, a+\mu]$ *be an arbitrary interval with* $\mu > 0$*. Let* $g \in C^2(I)$*. Then*

$$||g'||_I \leq \frac{2}{\mu}||g||_I + \frac{\mu}{2}||g''||_I.$$

Lemma 2.6. *The solution* $\boldsymbol{u}$ *of (1) satisfies the bounds*

$$||\boldsymbol{u}^{(m)}|| \leq C\varepsilon^{-m/2}, \text{ for } m = 0, \ldots, 6. \tag{6}$$

Proof. The bound on $\boldsymbol{u}$, follows from the stability result (Lemma 2.2). The bound on the second derivative of $\boldsymbol{u}$ follows from (1) and the bound on $\boldsymbol{u}$. Applying Lemma 2.5 with $\mu = \varepsilon^{1/2}$ and $g = \boldsymbol{u}$, we obtain (6), for $m = 1$. The bounds on the higher derivatives of $\boldsymbol{u}$ can be obtained by differentiating $\boldsymbol{Tu} = \boldsymbol{f}$. □

We now derive sharper bounds on the derivatives of $\boldsymbol{u}$. Let ξ be an arbitrary number satisfying

$$\sum_{j=1, j\neq i}^{M} \left\| \frac{a_{ij}}{a_{ii}} \right\| < \xi < 1, \text{ for } i = 1, \ldots, M.$$

Because of (3b) such a number exists. Define $\alpha = \alpha(\xi) > 0$ by

$$\alpha := (1-\xi) \min_{i=1,\ldots,M} \min_{x\in[0,1]} a_{ii}(x). \tag{7}$$

Theorem 2.7. *Let $\boldsymbol{u}$ be the solution of (1). Let $\alpha^* \in (0, \alpha)$ be arbitrary but fixed. Then there exists a constant C, independent of ε, such that*

$$|\boldsymbol{u}^{(m)}(x)| \le C(1 + \varepsilon^{-m/2}(e^{-x\sqrt{\alpha^*/\varepsilon}} + e^{-(1-x)\sqrt{\alpha^*/\varepsilon}})) \tag{8}$$

for all $x \in \overline{\Omega}$ and $m = 0, \ldots, 4$.

Proof. The proof is by induction. Fix $\alpha^* \in (0, \alpha)$ and set $\widetilde{B}_m(x) = 1+\varepsilon^{-m/2}(e^{-x\sqrt{\alpha^*/\varepsilon}}+ e^{-(1-x)\sqrt{\alpha^*/\varepsilon}})$. The bound for $m = 0$ follows from Lemma 2.6. For $m = 1, \ldots, 4$, differentiating (1) by m-times we get

$$-\boldsymbol{E}\boldsymbol{u}^{(m+2)} + \boldsymbol{A}\boldsymbol{u}^{(m)} = \boldsymbol{f}^{(m)} - \sum_{l=0}^{m-1} \begin{pmatrix} m \\ l \end{pmatrix} \boldsymbol{A}^{(m-l)}\boldsymbol{u}^{(l)} := \Psi_m$$

where $\Psi_m = (\Psi_{m,1}, \ldots, \Psi_{m,M})^T$. Assume that (8) holds for all $0 \le j \le m-1$, that is,

$$|\boldsymbol{u}^{(j)}(x)| \le C(1 + \varepsilon^{-j/2}(e^{-x\sqrt{\alpha^*/\varepsilon}} + e^{-(1-x)\sqrt{\alpha^*/\varepsilon}})),\ 0 \le j \le m-1.$$

From this assumption it is clear that $|\Psi_{m,k}(x)| \le C\widetilde{B}_{m-1}(x),\ 1 \le k \le M$. Define $\tilde{\boldsymbol{u}}$ by $\boldsymbol{u}^{(m)}(x) = \widetilde{B}_m(x)\tilde{\boldsymbol{u}}(x)$. The kth-component of $\tilde{\boldsymbol{u}}$ satisfies

$$P_{k,\varepsilon} \ := \begin{cases} -\varepsilon\tilde{u}_k'' - 2\varepsilon\frac{\widetilde{B}_m'}{\widetilde{B}_m}\tilde{u}_k' + (a_{kk} - \varepsilon\frac{\widetilde{B}_m''}{\widetilde{B}_m})\tilde{u}_k = -\displaystyle\sum_{i=1, i\ne k}^{M} a_{ki}\tilde{u}_i + \frac{\Psi_{m,k}}{\widetilde{B}_m}, \\ \tilde{u}_k(0) = u_k^{(m)}(0)/\widetilde{B}_m(0),\ \tilde{u}_k(1) = u_k^{(m)}(1)/\widetilde{B}_m(1), \end{cases}$$

where $|\Psi_{m,k}/\widetilde{B}_m| \le C$.

Since $|u_k^{(m)}(0)| \le C\varepsilon^{-m/2}$ and $|u_k^{(m)}(1)| \le C\varepsilon^{-m/2}$, we have

$$|\tilde{u}_k(0)| \le C,\ |\tilde{u}_k(1)| \le C.$$

By the definition of α and from the inequality $\widetilde{B}_m''(x) \le \varepsilon^{-1}\alpha^*\widetilde{B}_m(x)$, we have

$$a_{kk} - \varepsilon\frac{\widetilde{B}_m''}{\widetilde{B}_m} \ge a_{kk} - \alpha^* > 0 \text{ on } \overline{\Omega}.$$

On applying Lemma 2.1 to $P_{k,\varepsilon}$, we obtain

$$\|\tilde{u}_k\| - \sum_{i=1, i\ne k}^{M} \left\| \frac{a_{ki}}{a_{kk} - \alpha^*} \right\| \|\tilde{u}_i\| \le C \text{ for } k = 1, \ldots, M. \tag{9}$$

The choice of ξ and $\alpha^* \in (0, \alpha)$ implies

$$a_{kk}(x) - \alpha^* \ge \xi a_{kk}(x) \text{ for all } x \in [0, 1],\ k = 1, \ldots, M.$$

Let $x^* \in \overline{\Omega}$ be such that

$$\left\| \frac{a_{ki}}{a_{kk} - \alpha^*} \right\| = \frac{|a_{ki}(x^*)|}{a_{kk}(x^*) - \alpha^*} \le \frac{|a_{ki}(x^*)|}{\xi a_{kk}(x^*)} \le \frac{1}{\xi} \left\| \frac{a_{ki}}{a_{kk}} \right\|. \tag{10}$$

Summing (10) for $i = 1, \ldots, M$, $i \ne k$, we get

$$\sum_{i=1, i\ne k}^{M} \left\| \frac{a_{ki}}{a_{kk} - \alpha^*} \right\| \le \sum_{i=1, i\ne k}^{M} \frac{1}{\xi} \left\| \frac{a_{ki}}{a_{kk}} \right\| < 1 \text{ for } k = 1, \ldots, M. \tag{11}$$

Thus, the M-matrix criterion and (9),(11) give

$$\|\tilde{\boldsymbol{u}}\| \le C.$$

Then, recall the definition of $\tilde{\boldsymbol{u}}$, we obtain

$$|\boldsymbol{u}^{(m)}(x)| \le C(1+\varepsilon^{-m/2}(e^{-x\sqrt{\alpha^*/\varepsilon}} + e^{-(1-x)\sqrt{\alpha^*/\varepsilon}})), \text{ for all } x \in \overline{\Omega}.$$

This proves the lemma. □

Remarks: (i) If $\boldsymbol{f} \in C^6(\overline{\Omega})^M$ and $\boldsymbol{A} \in C^6(\overline{\Omega})^{M\times M}$, then the above result (8) can be extend for $m = 5, 6$. Thus we obtain

$$|\boldsymbol{u}^{(m)}(x)| \le C(1+\varepsilon^{-m/2}(e^{-x\sqrt{\alpha^*/\varepsilon}} + e^{-(1-x)\sqrt{\alpha^*/\varepsilon}})) \tag{12}$$

for all $x \in \overline{\Omega}$ and $m = 0, 1, \ldots, 6$.

(ii) Note that the bounds on the solution of (1) and its derivatives given in (8) or (12) are obtained without constructing any decomposition of $\boldsymbol{u}$.

2.3. Solution decomposition. For the analysis of the numerical method, it is necessary to have precise knowledge about the behavior of the exact solution $\boldsymbol{u}$ of (1) and its derivatives. Moreover, we require a special decomposition of the exact solution into its regular and layer parts. Suppose, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v}$ is a regular part and $\boldsymbol{w}$ is a layer part. This splitting is often called a Shishkin type decomposition; see [23]. We use the minimum regularity on the data $\boldsymbol{A}$ and $\boldsymbol{f}$ in deriving this decomposition. This is motivated from Linss [18], where the minimum regularity on the data is used to construct a high order decomposition of a solution in the case of a scalar singularly perturbed reaction-diffusion problem.

Theorem 2.8. *Let $\boldsymbol{f} \in C^4(\overline{\Omega})^M$, and let $\boldsymbol{A} \in C^4(\overline{\Omega})^{M\times M}$ satisfying assumptions (2)-(3). Then (1) possesses unique solution $\boldsymbol{u} \in C^6(\overline{\Omega})^M$ that can be decomposed as*

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \tag{13}$$

where the regular part $\boldsymbol{v}$ satisfies

$$||\boldsymbol{v}^{(m)}|| \le C(1+\varepsilon^{2-m/2}), \tag{14}$$

and the layer part $\boldsymbol{w}$ satisfies

$$|w_k^{(m)}(x)| \le C\varepsilon^{-m/2}B_\varepsilon(x), \tag{15}$$

for k=1,…,M and m=0,…,6, where $B_\varepsilon(x) = e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}}$.

Proof. Let $\Omega^* := [-1, 2]$ be the extension of the domain $\overline{\Omega}$. The functions $\boldsymbol{A}$ and $\boldsymbol{f}$ can be smoothly extended to functions $\boldsymbol{A}^* \in C^4(\Omega^*)^{M\times M}$ and $\boldsymbol{f}^* \in C^4(\Omega^*)^M$ with $\boldsymbol{A}^*|_{[0,1]} = \boldsymbol{A}$ and $\boldsymbol{f}^*|_{[0,1]} = \boldsymbol{f}$, in such a way that (3) remain valid for the extended functions (perhaps α be replaced by a smaller positive constant α^* and ξ be replaced by a slightly larger positive constant ξ^* that is still smaller than 1). Let $\boldsymbol{T}^*$ be the extended differential operator of $\boldsymbol{T}$. Let $\boldsymbol{v}^*$ be the solution of

$$\boldsymbol{T}^*\boldsymbol{v}^* = \boldsymbol{f}^*, \text{ in } (-1, 2), \ \boldsymbol{v}^*(-1) = \boldsymbol{v}^*(2) = 0 \text{ and set } \boldsymbol{v}^*|_{[0,1]} = \boldsymbol{v}.$$

The function $\boldsymbol{v}$ represents the regular part of the solution exhibiting no layers. An affine transformation and (8) gives $||\boldsymbol{v}^{(m)}|| \le C$ for $m = 0, \ldots, 4$. Using $\boldsymbol{T}\boldsymbol{v} = \boldsymbol{f}$, we immediately get $||v_k^{(6)}|| \le C\varepsilon^{-1}$, $k = 1, \ldots, M$. Finally, on applying Lemma 2.5 for $g = v_k^{(4)}$ with an interval $I \subseteq [0, 1]$ of length $\mu = \varepsilon^{1/2}$, we get

$$||v_k^{(5)}|| \le C\varepsilon^{-1/2}, \ k = 1, \ldots, M.$$

This proves the bounds for the regular part $\boldsymbol{v}$.

To obtain the bounds on the derivative of the layer part $\boldsymbol{w}$, we use the following property of the boundary layer function $B_\varepsilon(x)$

$$\max_{x\in I} B_\varepsilon(x) \leq 2e^{\delta\sqrt{\alpha/\varepsilon}} \min_{x\in I} B_\varepsilon(x) \text{ for any interval } I=[a,a+\delta]\subseteq[0,1]. \tag{16}$$

The layer part $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{v}$ is the solution of

$$\boldsymbol{T}\boldsymbol{w}=0, \text{ in } (0,1),\ \boldsymbol{w}(0)=\boldsymbol{u}(0)-\boldsymbol{v}(0),\ \boldsymbol{w}(1)=\boldsymbol{u}(1)-\boldsymbol{v}(1).$$

Under the assumptions (2) and (3) the operator $\boldsymbol{T}$ satisfies the comparison principle and its application to $\boldsymbol{w}$ yields (15) for $m=0$. The bounds on the second derivative of $\boldsymbol{w}$ follows from $\boldsymbol{T}\boldsymbol{w}=0$ and $|w_k(x)|\leq CB_\varepsilon(x)$. In order to get the bound on $\boldsymbol{w}'$, we apply Lemma 2.5 for $g=w_k$ with an interval $I\subseteq[0,1]$ of length $\mu=\varepsilon^{1/2}$ and use (16) with $\delta=\varepsilon$. This gives

$$|w_k'(x)|\leq C\varepsilon^{-1/2}B_\varepsilon(x),\ \text{for } x\in\overline{\Omega},\ k=1,\dots,M.$$

Further, the bounds on the higher-order derivatives on $\boldsymbol{w}$ can be obtained by differentiating $\boldsymbol{T}\boldsymbol{w}=0$. □

Remarks: (iii) The decomposition $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{w}$ described in Theorem 2.8 satisfies $\boldsymbol{T}\boldsymbol{u}=\boldsymbol{f}$ and $\boldsymbol{T}\boldsymbol{w}=0$; this decomposition is known as Shishkin decomposition. These additional properties played a key role in the analysis of a number of finite difference and finite element methods on Shishkin meshes and other layer adapted meshes, see [5],[6],[23],[25].

(iv) Suppose $x^*=4\sqrt{\varepsilon/\alpha^*}\ln(1/\sqrt{\varepsilon})$. For each $k\in\{1,\dots,M\}$ and $x\in\overline{\Omega}$, we set $v_k(x)=u_k(x)$ for $x\in[x^*,1-x^*]$ and $\boldsymbol{v}$ extends to a smooth function defined on $[0,1]$. Further, for each $k\in\{1,\dots,M\}$ and $x\in\overline{\Omega}$, we define $w_k(x)=u_k(x)-v_k(x)$. Then the result (12) and the choice of x^* implies that (see [12])

$$||\boldsymbol{v}^{(m)}||\leq C(1+\varepsilon^{2-m/2}),$$

$$|w_k^{(m)}(x)|\leq C\varepsilon^{-m/2}(e^{-x\sqrt{\alpha^*/\varepsilon}}+e^{-(1-x)\sqrt{\alpha^*/\varepsilon}}),$$

for $k=1,\dots,M$ and $m=0,\dots,6$. Here $\boldsymbol{u}$ is decomposed into a sum of the regular part $\boldsymbol{v}=(v_1,\dots,v_m)^T$ and the layer part $\boldsymbol{w}=(w_1,\dots,w_m)^T$. This Shishkin type decompisition $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{w}$ does not in general satisfy $\boldsymbol{T}\boldsymbol{u}=\boldsymbol{f}$ and $\boldsymbol{T}\boldsymbol{w}=0$, see [12].

3. The Mesh

In this section, we construct a generalized Shishkin mesh using a suitable mesh generating function $\mathcal{K}$ as given in [32]. Let $\overline{\Omega}=[0,1]$ be the given interval. Let $\overline{\Omega}_N$ be a partitioning of $\overline{\Omega}$ defined by

$$\overline{\Omega}_N:=0=x_0<x_1<x_2<\dots<x_{N-1}<x_N=1,$$

with mesh spacing $h_i=x_i-x_{i-1},\ i=1,2,\dots,N$. For simplicity, we assume that $N\geq 4$ is an even integer and that $x_{N-i}=1-x_i,\ i=0,1,2,...,N$. It then suffices to describe the mesh on the interval $[0,1/2]$.

Define the transition point

$$\tau=\min\{\ q,\ m\sqrt{\varepsilon}L\ \},$$

where $L=L(N)$ is the value of L with N intervals that solves (18), $m\geq a/\sqrt{\alpha}$ with a is a positive constant and α is defined by (7). Assume that qN is an integer. Divide the interval $[0,\tau]$ into qN subintervals and $[\tau,1/2]$ into $N/2-qN$ subintervals. The Shishkin type mesh $S(L)$ is defined by

$$h_i = \begin{cases} h = \dfrac{\tau}{Nq}, \text{ for } i = 1,\dots,qN; \\ H = \dfrac{1-2\tau}{(1-2q)N}, \text{ for } i = (qN+1),(qN+2),\dots,N/2, \end{cases} \tag{17}$$

where h and H are respectively the fine and the coarse mesh widths, $q \in (0,1/2)$. The standard Shishkin mesh uses $L = \ln N$ in defining τ and it is denoted by $S(\ln N)$. Now $S(\ln N)$ is used to discretize the domain $\overline{\Omega}$ and the Shishkin discterized domain $\overline{\Omega}_N^S$ is given by

$$\overline{\Omega}_N^S := 0 = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = 1,$$

where $h_i = x_i - x_{i-1},\ i = 1,2,\dots,N$ using (17).

The use of $L < \ln N$ enables meshes with a greater density in the layers, which improves accuracy of the numerical results. This is of practical importance only since theoretically any L behaves like $\ln N$ as $N \to \infty$, see [32].

Now we define the generalized Shishkin mesh, denoted by $\widetilde{S}(L)$, that changes smoothly in the transition points $x = \tau$ from the fine part to the coarse part, where $L = L(N)$ satisfying $\ln(\ln N) < L \le \ln N$ and

$$e^{-L} \le \frac{L}{N}. \tag{18}$$

Such a generalized Shishkin mesh $\widetilde{S}(L)$ can be defined by $x_i = \mathcal{K}(i/N),\ i = 0,1,\dots,N/2$, where $\mathcal{K} \in C^2[0,1/2]$ is a mesh generating function

$$\mathcal{K}(t) = \begin{cases} \dfrac{\tau}{q}t, \text{ for } t \in [0,q]; \\ p(t-q)^3 + \dfrac{\tau}{q}(t-q) + \tau, \text{ for } t \in [q,1/2]. \end{cases} \tag{19}$$

The coefficient p is determined from $\mathcal{K}(1/2) = 1/2$.

The above defined generalized Shishkin mesh $\widetilde{S}(L)$ is used to discretize the domain $\overline{\Omega}$ and the generalized Shishkin discterized domain $\overline{\Omega}_N^{\tilde{S}}$ is given by

$$\overline{\Omega}_N^{\tilde{S}} := 0 = x_0 < \dots < x_{N/4} = \tau < \dots < x_{N/2} = 1/2 < \dots < x_{3N/4} = 1-\tau < \dots < x_N = 1,$$

where we choose $q = 1/4$, and $a \ge 4$ (as taken in Section 5). Assuming $h_{\max} = \max_{\forall i}\ h_i$, where $h_i = x_i - x_{i-1},\ i = 1,2,\dots,N$; it can be easily verified that $h_{\max} = h_{N/2} = h_{N/2+1}$.

Observe that the fine parts of $S(L)$ and $\widetilde{S}(L)$ are identical, but the coarse part of $\widetilde{S}(L)$ is a smooth continuation of the fine mesh and is no longer equidistant. In the case of generalized Shishkin mesh $\widetilde{S}(L)$, the mesh width h_i, for $i = \{N/4, N/4+1,\dots,3N/4\}$, satisfies the following.

(A) For some $\eta_i \in (i/N,(i+1)/N)$,

$$h_{i+1} = \mathcal{K}((i+1)/N) - \mathcal{K}(i/N) = N^{-1}\mathcal{K}'(\eta_i) \le CN^{-1}. \tag{20a}$$

(B) For some $\phi_i \in ((i-1)/N,(i+1)/N)$,

$$\begin{aligned} |h_{i+1} - h_i| &= |\mathcal{K}((i+1)/N) - 2\mathcal{K}(i/N) + \mathcal{K}((i-1)/N)| \\ &= N^{-2}|\mathcal{K}''(\phi_i)| \le CN^{-2}. \end{aligned} \tag{20b}$$

4. The Discretization

In this section, we introduce a hybrid scheme to discretize a linear system of ($M \geq 2$) coupled singular perturbed reaction-diffusion equations (1) on the generalized Shishkin discretized domain $\overline{\Omega}_N^{\tilde{S}}$. The hybrid scheme is a combination of the fourth order compact difference scheme (where the coefficients $q_i's$ and $r_i's$ of the scheme are determined so that the scheme is exact for polynomials up to degree four and satisfy the normalization condition $q_i^{k,-}+q_i^{k,c}+q_i^{k,+}=1,\ i=1,\dots,N-1,$ $k=1,\dots,M$) and the central difference scheme, is given by

$$[\boldsymbol{T}^N\boldsymbol{U}]_i=\boldsymbol{f}_i \tag{21a}$$

with

$$\boldsymbol{U}_0=\boldsymbol{u}(0),\ \ \boldsymbol{U}_N=\boldsymbol{u}(1), \tag{21b}$$

where

$$[\boldsymbol{T}^N\boldsymbol{U}]_i:=\begin{pmatrix}[T_1^N\boldsymbol{U}]_i\\ [T_2^N\boldsymbol{U}]_i\\ \cdot\\ \cdot\\ [T_M^N\boldsymbol{U}]_i\end{pmatrix}=\begin{pmatrix}[R(U_1)]_i\ +\ [Q(a_{12}U_2)]_i+\cdots+\ [Q(a_{1M}U_M)]_i\\ [R(U_2)]_i\ +\ [Q(a_{21}U_1)]_i+\cdots+\ [Q(a_{2M}U_M)]_i\\ \cdot\\ \cdot\\ [R(U_M)]_i+[Q(a_{M1}U_1)]_i+\cdots+[Q(a_{MM-1}U_{M-1})]_i\end{pmatrix},$$

$$\boldsymbol{f}_i:=\begin{pmatrix}[f_1]_i\\ [f_2]_i\\ \cdot\\ \cdot\\ [f_M]_i\end{pmatrix}=\begin{pmatrix}[Q(f_1)]_i\\ [Q(f_2)]_i\\ \cdot\\ \cdot\\ [Q(f_M)]_i\end{pmatrix},$$

and

$[R(V_k)]_i=r_i^{k,-}V_{k,i-1}+r_i^{k,c}V_{k,i}+r_i^{k,+}V_{k,i+1},\ [Q(V_k)]_i=q_i^{k,-}V_{k,i-1}+q_i^{k,c}V_{k,i}+q_i^{k,+}V_{k,i+1}.$

The coefficients $r_i^{k,*}, i=1,\dots N-1, k=1,\dots,M,\ *=-,c,+$ are given by

$$\begin{cases} r_i^{k,-}\ =\frac{-2\varepsilon}{h_i(h_i+h_{i+1})}+q_i^{k,-}a_{kk,i-1};\\ r_i^{k,c}\ =\frac{2\varepsilon}{h_ih_{i+1}}+q_i^{k,c}a_{kk,i};\\ r_i^{k,+}\ =\frac{-2\varepsilon}{h_{i+1}(h_i+h_{i+1})}+q_i^{k,+}a_{kk,i+1}.\end{cases} \tag{22a}$$

The coefficients $q_i^{k,*}, i=1,\dots,N-1,\ k=1,\dots,M,\ *=-,c,+$ are defined in two different ways.

(i) For the mesh points located in $(0,\tau)\cup(1-\tau,1)$; the coefficients $q_i^{k,*}, i=\{1,\dots,N/4-1\}\cup\{3N/4+1,\dots,N-1\},\ *=-,c,+,$ are given by

$$\begin{cases} q_i^{k,-}\ =\frac{1}{6}-\frac{h_{i+1}^2}{6h_i(h_i+h_{i+1})};\\ q_i^{k,c}\ =\frac{h_i^2+h_{i+1}^2+3h_ih_{i+1}}{6h_ih_{i+1}};\\ q_i^{k,+}\ =\frac{1}{6}-\frac{h_i^2}{6h_{i+1}(h_i+h_{i+1})}.\end{cases} \tag{22b}$$

(ii) For the mesh points located in $[\tau,1-\tau]$, depending on the relation between $h_{\max}$ and ε, the coefficients $q_i^{k,*}$, where $*=-,c,+$, are defined in two different cases.

In the first case, when $\gamma h_{\max}^2 ||a_{kk}||_\infty \leq \varepsilon, k = 1, \ldots, M$, where γ is a positive constant independent of N and ε, the coefficients $q_i^{k,*}, i = N/4, \ldots, 3N/4,\ k = 1, \ldots, M,\ * = -, c, +$, are defined again by ($22b$).

While in the second case, when $\gamma h_{\max}^2 ||a_{kk}||_\infty > \varepsilon, k = 1, \ldots, M$, where γ is a positive constant independent of N and ε, the coefficients $q_i^{k,*}, i = N/4, \ldots, 3N/4,\ k = 1, \ldots, M,\ * = -, c, +$, are given by

$$q_i^{k,-} = 0,\ q_i^{k,c} = 1,\ q_i^{k,+} = 0. \tag{22c}$$

The above definition of the coefficients q_i^k's and r_i^k's shows that the present scheme (21) is defined by the fourth order compact difference scheme (or new HODIE scheme as derived in [2] for scalar singularly perturbed reaction diffusion equations) within the boundary layer region $(0, \tau) \cup (1-\tau, 1)$. While in the regular region $[\tau, 1-\tau]$, the present scheme (21) is defined by the fourth order compact difference scheme when $\gamma h_{\max}^2 ||a_{kk}||_\infty \leq \varepsilon$ and is defined by the central difference scheme when $\gamma h_{\max}^2 ||a_{kk}||_\infty > \varepsilon$. This means high-order approximation is used only when the local mesh width is small enough to give non-positive off-diagonal entries while at all other mesh points the central difference scheme is used. This combination leads to the following lemma.

Lemma 4.1. *Let $\gamma = 1/6$ and N_0 be the smallest positive integer such that*

$$4m^2 ||a_{kk}||_\infty / 3 < N_0^2 / L_{N_0}^2,$$

where $L_{N_0} = L(N_0)$ as defined in Section 2. Then, for any $N \geq N_0$, the discrete operator $\boldsymbol{T}^N$ defined in (21) *is of positive type.*

Proof. First, for $x_i \in (0, \tau) \cup (1-\tau, 1)$, the fourth order compact differnce scheme is considered. The condition $4m^2 ||a_{kk}||_\infty / 3 < N_0^2 / L_{N_0}^2$ for any $N \geq N_0$, with the the coefficients $q_i^{k,*},\ * = -, c, +,\ k = 1, \ldots, M$, defined by (22b), and the assumptions (2)-(3) concludes the lemma.

Secondly, for $x_i \in [\tau, 1-\tau]$ when $\gamma h_{\max}^2 ||a_{kk}||_\infty > \varepsilon$, the central difference scheme is considered. Hence the proof is trivial.

While in the opposite case, when $\gamma h_{\max}^2 ||a_{kk}||_\infty \leq \varepsilon$, the fourth order compact difference scheme is considered in $[\tau, 1-\tau]$. On $\overline{\Omega}_N^{\tilde{S}}$, $(h_i^2 - h_{i+1}^2 + h_i h_{i+1}) \geq 0$, for $N/4 \leq i \leq N/2$. The assertion is trivially true for $i = N/2$. For $N/4 \leq i \leq N/2 - 1$,

$$(h_i^2 + h_i h_{i+1}) \geqslant h_{i+1}^2$$

follows if

$$h_{i+1} \leqslant \sqrt{2} h_i,$$

if

$$\mathcal{K}'((i+1)/N) \leqslant \sqrt{2}\mathcal{K}'((i-1)/N),$$

that is, if

$$\widetilde{w}(z) = 3pz^2 - 28.971pz - 28.971p + 4\tau N^2 \geqslant 0,$$

where $z = i - 1 - N/4 \geqslant -1$ and p is determined by $\mathcal{K}(1/2) = 1/2$, see Section 3. It is easy to verify that $\widetilde{w}$ is non-negative if $\tau N^2 \geqslant 24.728p$. Since $\gamma h_{\max}^2 ||a_{kk}|| \leqslant \varepsilon$ implies $\tau N^2 \geqslant 24.728p$, it follows that $\widetilde{w}(z) \geqslant 0$ for all z.

Similarly $(h_{i+1}^2 - h_i^2 + h_i h_{i+1}) \geq 0$ for $N/2 \leq i \leq 3N/4$. Thus the condition $\gamma h_{\max}^2 ||a_{kk}||_\infty \leq \varepsilon$ with the the coefficients $q_i^{k,*},\ * = -, c, +,\ k = 1, \ldots, M$, defined by (22b), and the assumptions (2)-(3) concludes the lemma. □

Under the assumptions of the above lemma, the discretized operator $\boldsymbol{T}^N$ is of positive type and it satisfies the following discrete comparison principle

Lemma 4.2. *(Discrete Comparison Principle) Let* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *be two mesh functions and satisfy* $\boldsymbol{V}_0 \geq \boldsymbol{W}_0$, $\boldsymbol{V}_N \geq \boldsymbol{W}_N$ *and* $[\boldsymbol{T}^N \boldsymbol{V}]_i \geq [\boldsymbol{T}^N \boldsymbol{W}]_i$, *for* $i = 1, \ldots, N-1$, *then* $\boldsymbol{V}_i \geq \boldsymbol{W}_i$ *for* $i = 0, \ldots, N$.

An immediate consequence of the discrete comparison principle is the following stability estimate.

Lemma 4.3. *(Stability Estimate) Let* $\boldsymbol{V}$ *be any mesh function such that* $\boldsymbol{V}_0 = \boldsymbol{0}$ *and* $\boldsymbol{V}_N = \boldsymbol{0}$, *then*

$$||\,\boldsymbol{V}||_{\overline{\Omega}_N^{\tilde{S}}} \leq C||\,\boldsymbol{T}^N \boldsymbol{V}||_{\overline{\Omega}_N^{\tilde{S}}},$$

where C *is independent of* N *and* ε.

5. Error Analysis

In this section, high order parameter-robust convergence of the proposed scheme (21) on the generalized Shishkin discretized domain $\overline{\Omega}_N^{\tilde{S}}$ is established. First, we investigate the truncation error estimate. Let $\boldsymbol{U} = (U_1, \ldots, U_M)^T$, a vector mesh function on $\overline{\Omega}_N^{\tilde{S}}$, be the solution to the discrete problem (21). Recall the decomposition (13) of the continuous problem (1). We begin the analysis by defining an analogous decomposition of the discrete solution as $\boldsymbol{U} = \boldsymbol{V} + \boldsymbol{W}$, where $\boldsymbol{V}$ is the solution of

$$[\boldsymbol{T}^N \boldsymbol{V}]_i = \boldsymbol{f}_i,\ i = 1, \ldots, N-1,\ \ \boldsymbol{V}_0 = \boldsymbol{v}(0),\ \ \boldsymbol{V}_N = \boldsymbol{v}(1),$$

and $\boldsymbol{W}$ is the solution of

$$[\boldsymbol{T}^N \boldsymbol{W}]_i = \boldsymbol{0},\ i = 1, \ldots, N-1,\ \ \boldsymbol{W}_0 = \boldsymbol{w}(0),\ \ \boldsymbol{W}_N = \boldsymbol{w}(1).$$

The error $\boldsymbol{u} - \boldsymbol{U}$ satisfies

$$||\boldsymbol{u} - \boldsymbol{U}||_{\overline{\Omega}_N^{\tilde{S}}} \leq ||\boldsymbol{v} - \boldsymbol{V}||_{\overline{\Omega}_N^{\tilde{S}}} + ||\boldsymbol{w} - \boldsymbol{W}||_{\overline{\Omega}_N^{\tilde{S}}}.$$

To obtain the bound on $||\boldsymbol{u} - \boldsymbol{U}||_{\overline{\Omega}_N^{\tilde{S}}}$, the regular and the layer parts of the error can be estimated separately.

Note that if $\tau = 1/4$, then $\overline{\Omega}_N^{\tilde{S}}$ is uniform, N^{-1} is very small respect to ε and therefore a classical analysis could be made to prove the convergence of the present scheme. So, in the analysis we only consider the case $\tau = m\sqrt{\varepsilon}L$.

Let $[\boldsymbol{T}^N(\boldsymbol{u} - \boldsymbol{U})]_i$ be the truncation error denoted by $[\Gamma(\boldsymbol{u})]_i$. The truncation error estimate on $\overline{\Omega}_N^{\tilde{S}}$ is discussed in the following cases.

Case I: When $x_i \in (0, \tau) \cup (1-\tau, 1)$, we have $h_i = h_{i+1} = 4m\sqrt{\varepsilon}N^{-1}L$. By Taylor's expansion we obtain

$$|\,[\Gamma_k(\boldsymbol{u})]_i\,| \leq C\varepsilon h_i^4 ||u_k^{(6)}||_{[x_{i-1}, x_{i+1}]},\ k = 1, \ldots, M.$$

Using $h_i = 4m\sqrt{\varepsilon}N^{-1}L$ and (6), it follows that

$$|[\Gamma_k(\boldsymbol{u})]_i| \leq C(L/N)^4,\ k = 1, \ldots, M. \tag{23}$$

Case II: When $x_i \in [\tau, 1-\tau\,]$, according to the decomposition of $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, we split the truncation error into two parts as

$$|\,[\Gamma_k(\boldsymbol{u})]_i\,| \leq |\,[\Gamma_k(\boldsymbol{v})]_i\,| + |\,[\Gamma_k(\boldsymbol{w})]_i|,\ k = 1, \ldots, M. \tag{24}$$

Here the present scheme (21) is defined by the fourth order compact difference scheme when $\gamma h_{\max}^2 ||a_{kk}||_\infty \leq \varepsilon,\ k = 1, \ldots, M$, and is defined by the central scheme when $\gamma h_{\max}^2 ||a_{kk}||_\infty > \varepsilon,\ k = 1, \ldots, M$. The error analysis for both the cases is given as follows.

(i) For the case $\gamma h_{\max}^2||a_{kk}||_\infty \le \varepsilon,\ \ k=1,\dots,M$, suppose $\mathbf{g}\in C^6[0,1]^M$, then by Taylor's expansion we obtain

$$|[\Gamma_k(\mathbf{g})]_i| \le C\varepsilon(Q_{k,i}+R_{k,i}),\ k=1,\dots,M, \tag{25}$$

where

$$Q_{k,i} = |h_{i+1}-h_i|(h_{i+1}+h_i)^2\|g_k^{(5)}\|_{[x_{i-1},x_{i+1}]},$$
$$R_{k,i} = (h_i^4+h_{i+1}^4)\|g_k^{(6)}\|_{[x_{i-1},x_{i+1}]}.$$

Using (14),(20) in (25), we obtain the bound on the truncation error with respect to the regular part $\boldsymbol{v}$

$$|\,[\Gamma_k(\boldsymbol{v})]_i\,| \le CN^{-4},\ k=1,\dots,M. \tag{26}$$

Again, using (15),(20) in (25), we obtain the bound on the truncation error with respect to the boundary layer part $\boldsymbol{w}$

$$|\,[\Gamma_k(\boldsymbol{w})]_i\,| \le C\varepsilon^{-2}N^{-4}||B_\varepsilon||_{[x_{i-1},x_{i+1}]},\ k=1,\dots,M. \tag{27}$$

For $x_i\in[\tau,1-\tau]$, we have

$$||B_\varepsilon||_{[x_{i-1},x_{i+1}]} \le e^{(-x_{N/4-1}\sqrt{\alpha/\varepsilon})} + e^{(-(1-x_{3N/4+1})\sqrt{\alpha/\varepsilon})}.$$

To obtain the bound on the term $||B_\varepsilon||_{[x_{i-1},x_{i+1}]}$, we split the interval $[\tau,1-\tau]$ into two subintervals $[\tau,\frac{1}{2}]$ and $[\frac{1}{2},1-\tau]$, and then we look at each case separately. When $x_i\in[\tau,\frac{1}{2}]$

$$||B_\varepsilon||_{[x_{i-1},x_{i+1}]} \le 2e^{(-x_{N/4-1}\sqrt{\beta/\varepsilon})} = 2e^{(-\tau\sqrt{\beta/\varepsilon})}e^{(h_{N/4}\sqrt{\beta/\varepsilon})} \le Ce^{-aL},$$
$$\text{using } \tau = m\sqrt{\varepsilon}L \text{ and } \ m\ge a/\sqrt{\beta}.$$

Similarly, when $x_i\in[\frac{1}{2},1-\tau]$

$$||B_\varepsilon||_{[x_{i-1},x_{i+1}]} \le 2e^{(-(1-x_{3N/4+1})\sqrt{\beta/\varepsilon})} \le Ce^{-aL},\ \text{using } \tau=m\sqrt{\varepsilon}L,\ m\ge a/\sqrt{\beta}.$$

With $e^{-L}\le L/N$, choose $a\ge 4$; this leads to

$$||B_\varepsilon||_{[x_{i-1},x_{i+1}]} \le C(L/N)^4. \tag{28}$$

Using (28) in (27), we obtain

$$|\,[\Gamma_k(\boldsymbol{w})]_i\,| \le C(L/N)^4,\ k=1,\dots,M. \tag{29}$$

On combining (26),(29) with (24), we obtain

$$|\,[\Gamma_k(\boldsymbol{u})]_i\,| \le C(L/N)^4,\ \text{for } \gamma h_{\max}^2||a_{kk}||_\infty \le \varepsilon,\ k=1,\dots,M. \tag{30}$$

(ii) When $\gamma h_{\max}^2||a_{kk}||_\infty > \varepsilon,\ k=1,\dots,M$, the discrete scheme (21) is defined by central scheme (see (22c)). In that case suppose $\mathbf{g}\in C^4[0,1]^M$ then by Taylor's expansion we obtain

$$|[\Gamma_k(\mathbf{g})]_i| \le C\varepsilon(Y_{k,i}+Z_{k,i}),\ k=1,\dots,M. \tag{31}$$

where

$$Y_{k,i} = |h_{i+1}-h_i|||g_k^{(3)}||_{[x_i,1-x_i]},\ Z_{k,i} = h_{i+1}^2||g_k^{(4)}||_{[x_i,1-x_i]}.$$

Using (14),(20) in (31), we obtain the bound on the truncation error with respect to the regular part $\boldsymbol{v}$

$$|\,[\Gamma_k(\boldsymbol{v})]_i\,| \le C\varepsilon N^{-2}.$$

Under the condition $\gamma h_{\max}^2||a_{kk}||_\infty > \varepsilon$, we obtain

$$|\,[\Gamma_k(\boldsymbol{v})]_i\,| \le CN^{-4},\ k=1,\dots,M. \tag{32}$$

To estimate the error with respect to the layer part $\boldsymbol{w}$. Suppose $\mathbf{g} \in C^2[0,1]^M$, and

$$|\,[\Gamma_k(\boldsymbol{g})]_i\,| \le C\varepsilon \|g_k''\|_{[x_{i-1},x_{i+1}]}.$$

Using (15) and (28), we obtain

$$|\,[\Gamma_k(\boldsymbol{w})]_i\,| \le C(L/N)^4. \tag{33}$$

On combining (32),(33) with (24), we get

$$|\,[\Gamma_k(\boldsymbol{u})]_i\,| \le C(L/N)^4, \text{ for } \gamma h_{\max}^2 ||a_{kk}||_\infty > \varepsilon. \tag{34}$$

Combining Cases **I** and **II**, we obtain the truncation error estimate for the discrete scheme (21) on $\overline{\Omega}_N^{\tilde{S}}$ and it is given by

$$|\,[\Gamma(\boldsymbol{u})]_i\,| \le C(L/N)^4. \tag{35}$$

The truncation error estimate (35) and the uniform stability result given in Lemma 4.3 conclude the main result of this section.

Theorem 5.1. *(Parameter-Robust Convergence) Let $\boldsymbol{u} \in C^6(\overline{\Omega})^M$ be the exact solution of (1). Let $\boldsymbol{U}$ be the approximate solution to discrete problem (21) on the generalized Shishkin discretized domain $\overline{\Omega}_N^{\tilde{S}}$. Under the assumptions of Lemma 4.1, there exist a positive constant C such that*

$$||\boldsymbol{u} - \boldsymbol{U}||_{\overline{\Omega}_N^{\tilde{S}}} \le C(L/N)^4,$$

where C is independent of N and ε.

For the proof of Theorem 5.1, the choice of $a(\ge 4)$ in the definition of the transition parameter of generalized Shishkin mesh is crucial. In practice we observe that if we take $a > 4$, the uniform errors are larger but the orders of uniform convergence is preserved. In mesh construction the small value of a shows more number of nodal points within the boundary layer region(s). Nevertheless, if we take $a < 4$, then we cannot achieve the same order of uniform convergence. So in the present scheme we choose $a = 4$.

6. Numerical Results

In this section numerical results are presented which confirm the theoretical error estimates established in the previous section. We first construct the generalized Shishkin discretized domain $\overline{\Omega}_N^{\tilde{S}}$ by considering the optimal value of L instead of $\ln N$ in $\widetilde{S}(L)$ from (18) based on the fact that $L < \ln N$. This provides higher density of the mesh points in the layers. Such an optimal value of L is chosen to be $L^* = L^*(N)$ which satisfies

$$e^{-L^*} = L^*/N.$$

The discrete scheme (21) is then implemented on $\overline{\Omega}_N^{\tilde{S}}$ to solve three test problems. The numerical results of the present scheme are compared with hybrid finite difference scheme of HODIE type [3] and central difference (CD) scheme [15].

Example 1. Consider the following system of (two) coupled reaction-diffusion problem (see [3])

$$\begin{cases} -\varepsilon u_1'' + u_1 - 0.5u_2 = f_1, \\ -\varepsilon u_2'' - 2u_1 + 4u_2 = f_2, \end{cases}$$

the right hand side and the boundary conditions are such that the exact solution of the problem $\boldsymbol{u} = (u_1, u_2)^T$ is given by

$$u_1 = g_1(x)/k_1 + g_2(x)/k_2 - x + x^2 + \cos^2(\pi x),\ u_2 = g_1(x)/k_1 - g_2(x)/k_2 + \sin(\pi x),$$

where

$$g_1(x) = \exp\big(-x/\sqrt{\varepsilon}\big) + \exp\big(-(1-x)/\sqrt{\varepsilon}\big),$$
$$g_2(x) = \exp\big(-2x/\sqrt{\varepsilon}\big) + \exp\big(-2(1-x)/\sqrt{\varepsilon}\big),$$

with

$$k_1 = 1 + \exp\big(-1/\sqrt{\varepsilon}\big),\ k_2 = 1 + \exp\big(-2/\sqrt{\varepsilon}\big).$$

Example 2. Consider the following system of (two) coupled reaction-diffusion problem (see [22])

$$\begin{cases} -\varepsilon u_1'' + 2(x+1)^2 u_1 - (1+x^3)u_2 = 2\exp(x),\ u_1(0) = u_1(1) = 0; \\ -\varepsilon u_2'' - 2\cos(\pi x/4)u_1 + (1+\sqrt{2})\exp(1-x)u_2 = 10x + 1,\ u_2(0) = u_2(1) = 0. \end{cases}$$

Example 3. Consider the following system of (three) coupled reaction-diffusion problem

$$\begin{cases} -\varepsilon u_1'' + 3u_1 - (1-x)u_2 - (1-x)u_3 = \exp(x),\ u_1(0) = u_1(1) = 0; \\ -\varepsilon u_2'' - 2u_1 + (4+x)u_2 - u_3 = \cos(x),\ u_2(0) = u_2(1) = 0; \\ -\varepsilon u_3'' - 2u_1 - 3u_2 + (6+x)u_3 = 1 + x^2,\ u_3(0) = u_3(1) = 0. \end{cases}$$

Examples 1, 2, and 3 satisfy all the assumptions (2)-(3) as given in Section 1, for any $\alpha \in (0,1)$. Therefore we assume $\alpha = 0.99 < 1$ in the construction of $\widetilde{S}(L)$.

Let $\boldsymbol{U}^N$ be the approximate solution to (21) on $\overline{\Omega}_N^{\tilde{S}} = \{x_i\}_{i=0}^N$ with N intervals. As the exact solution $\boldsymbol{u}$ is known for Example 1, for fixed value of N and ε, the maximum pointwise errors are calculated by

$$\boldsymbol{E}_\varepsilon^N = \max_{0 \le i \le N} |\boldsymbol{U}^N(x_i) - \boldsymbol{u}(x_i)|,\ E_\varepsilon^N = \max_{k=1,\dots,M} E_{\varepsilon,k}^N.$$

The parameter-robust error is computed by

$$E^N = \max_\varepsilon E_\varepsilon^N.$$

When the exact solution $\boldsymbol{u}$ is not known as in the case of Examples 2 and 3, we use a variant of the double mesh principle (see [5] for a justification of the method) to estimate the error. For this, we compute not only the solution $\boldsymbol{U}^N$, but also another approximate solution $\widetilde{\boldsymbol{U}}^{2N}$ to the problems on the domain $\widetilde{\overline{\Omega}}_N^{\tilde{S}} = \{\widehat{x}_i\}_{i=0}^{2N}$ with $2N$ intervals that contain the mesh points of the original domain $\overline{\Omega}_N^{\tilde{S}}$ and their midpoints, i.e., the mesh points are defined by

$$\widehat{x}_{2i} = x_i,\ i = 0,\dots,N,\ \widehat{x}_{2i+1} = (x_i + x_{i+1})/2,\ i = 0,\dots,N-1.$$

Then, for fixed values of N and ε, the maximum pointwise errors are calculated by

$$\boldsymbol{E}_\varepsilon^N = \max_{0 \le i \le N} |\boldsymbol{U}^N(x_i) - \widetilde{\boldsymbol{U}}^{2N}(\widehat{x}_{2i})|,\ E_\varepsilon^N = \max_{k=1,\dots,M} E_{\varepsilon,k}^N.$$

The parameter-robust error is computed by

$$E^N = \max_\varepsilon E_\varepsilon^N.$$

From these estimates for the maximum errors we obtain the corresponding classical convergence rate in standard way, for each fixed ε, by

$$\rho_\varepsilon^N = \frac{\ln(E_\varepsilon^N/E_\varepsilon^{2N})}{\ln(2)},$$

and the parameter-robust convergence rate ρ^N by

$$\rho^N = \frac{\ln(E^N/E^{2N})}{\ln(2)}.$$

TABLE 1. Maximum errors and numerical rate of convergence of the present scheme to solve the Example 1.

ε		$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-4}	E_ε^N	5.43e-07	3.41e-08	2.15e-09	1.34e-10	8.37e-12
	ρ_ε^N	3.99	3.99	4.00	4.00	
2^{-8}	E_ε^N	8.82e-05	5.55e-06	3.47e-07	2.16e-08	1.47e-09
	ρ_ε^N	3.99	4.00	4.00	3.88	
2^{-12}	E_ε^N	7.20e-03	8.81e-04	8.82e-05	5.55e-06	3.47e-07
	ρ_ε^N	3.03	3.32	3.99	4.00	
2^{-16}	E_ε^N	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
	ρ_ε^N	3.03	3.12	3.24	3.34	
2^{-20}	E_ε^N	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
	ρ_ε^N	3.03	3.12	3.24	3.34	
2^{-24}	E_ε^N	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
	ρ_ε^N	3.03	3.12	3.24	3.34	
2^{-28}	E_ε^N	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
	ρ_ε^N	3.03	3.12	3.24	3.34	
2^{-32}	E_ε^N	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
	ρ_ε^N	3.03	3.12	3.24	3.34	
	E^N	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
	ρ^N	3.03	3.12	3.24	3.34	

TABLE 2. Comparison of the maximum errors of the present scheme with the hybrid HODIE scheme [3] and central difference(CD) scheme [15] to solve the Example 1 for $\varepsilon = 2^{-24}$.

	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
Present scheme	7.20e-03	8.81e-04	1.01e-04	1.07e-05	1.06e-06
HODIE scheme [3]	2.20e-02	2.95e-03	3.29e-04	3.33e-05	3.19e-06
CD scheme [15]	3.10e-02	1.14e-02	3.83e-03	1.22e-03	3.77e-04

The present scheme (21) on the standard Shishkin mesh $S(\ln N)$ can be viewed as an extension of the new HODIE scheme given in [2] to the system of ($M \geq 2$) coupled reaction-diffusion problem (1). The authors in [2] proved that the scheme is third order uniformly convergent for scalar singularly perturbed reaction-diffusion problems. But in general this extension of new HODIE scheme on standard Shishkin mesh is not possible in the case of system of coupled reaction diffusion problem (1). It can be seen that the coefficients (q's) given by (22b) are not always positive at the transition points, due to the fact that the standard Shishkin mesh is very

TABLE 3. Maximum errors and numerical rate of convergence of the present scheme to solve the Example 2.

ε		$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-4}	E_ε^N	1.57e-08	9.81e-10	6.07e-11	3.92e-12	2.45e-13
	ρ_ε^N	4.00	4.00	3.95	4.00	
2^{-8}	E_ε^N	5.62e-05	3.64e-06	2.29e-07	1.43e-08	8.95e-10
	ρ_ε^N	3.95	3.99	4.00	4.00	
2^{-12}	E_ε^N	4.55e-03	5.73e-04	5.55e-05	3.57e-06	2.24e-07
	ρ_ε^N	2.98	3.37	3.96	3.99	
2^{-16}	E_ε^N	4.70e-03	5.72e-04	6.39e-05	6.86e-06	6.86e-07
	ρ_ε^N	3.04	3.16	3.22	3.32	
2^{-20}	E_ε^N	4.74e-03	5.74e-04	6.45e-05	6.85e-06	6.85e-07
	ρ_ε^N	3.05	3.16	3.24	3.32	
2^{-24}	E_ε^N	4.75e-03	5.75e-04	6.46e-05	6.85e-06	6.85e-07
	ρ_ε^N	3.05	3.16	3.24	3.32	
2^{-28}	E_ε^N	4.75e-03	5.75e-04	6.46e-05	6.85e-06	6.85e-07
	ρ_ε^N	3.05	3.16	3.24	3.32	
2^{-32}	E_ε^N	4.75e-03	5.75e-04	6.46e-05	6.85e-06	6.85e-07
	ρ_ε^N	3.05	3.16	3.24	3.32	
	E^N	4.75e-03	5.75e-04	6.46e-05	6.85e-06	6.85e-07
	ρ^N	3.05	3.15	3.24	3.32	

TABLE 4. Comparison of the maximum errors of the present scheme with the hybrid HODIE scheme [3] and central difference(CD) scheme [15] to solve the Example 2 for $\varepsilon = 2^{-24}$.

	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
Present scheme	4.75e-03	5.75e-04	6.46e-05	6.85e-06	6.85e-07
HODIE scheme [3]	1.54e-02	1.86e-03	2.07e-04	2.13e-05	2.05e-06
CD scheme [15]	3.12e-02	1.14e-02	3.84e-03	1.22e-03	3.78e-04

anisotropic in nature. This shows that the operator in (21) is not a positive type on a standard Shishkin mesh. At the moment, when $N^{-1} < \sqrt{\varepsilon}$ it is hard to find a high order difference scheme of positive type on the standard Shishkin mesh for system of coupled reaction-diffusion problems. While in [3] the authors extended the idea of HODIE technique to system of coupled reaction-diffusion problems by choosing a particular value of coefficients (q's) in the scheme at the transition points of Shishkin mesh. But this results in a combination of three schemes and gives third order uniformly convergent result. In order to increase the order of convergence more than three and to maintain the positivity of the present discrete operator in (21), we consider the scheme (21) on the generalized Shishkin mesh $\widetilde{S}(L)$. Lemma 4.1 shows that the discrete operator in (21) on the generalized Shishkin mesh is of positive type and the analysis in Section 5 shows that the scheme (21) is almost fourth order uniformly convergent with respect to perturbation parameters on a generalized Shishkin mesh.

For different values of N and ε, Tables $1, 3$, and 5 represent the maximum errors E_ε^N and the classical rate of convergence ρ_ε^N of the present scheme (21) on $\overline{\Omega}_N^{\tilde{S}}$ to solve the Examples $1, 2$ and 3, respectively. The last two rows in each of the

TABLE 5. Maximum errors and numerical rate of convergence of the present scheme to solve the Example 3.

ε		$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-4}	E_ε^N	1.11e-07	6.96e-09	4.35e-10	2.62e-11	1.57e-12
	ρ_ε^N	4.00	4.00	4.05	4.06	
2^{-8}	E_ε^N	2.37e-05	1.50e-06	9.42e-08	5.89e-09	3.69e-10
	ρ_ε^N	3.98	3.99	4.00	4.00	
2^{-12}	E_ε^N	1.85e-03	2.30e-04	2.28e-05	1.44e-06	9.05e-08
	ρ_ε^N	3.01	3.33	3.98	3.99	
2^{-16}	E_ε^N	1.83e-03	2.28e-04	2.62e-05	2.77e-06	2.75e-07
	ρ_ε^N	3.00	3.12	3.24	3.33	
2^{-20}	E_ε^N	1.82e-03	2.27e-04	2.61e-05	2.76e-06	2.75e-07
	ρ_ε^N	3.00	3.12	3.24	3.33	
2^{-24}	E_ε^N	1.82e-03	2.27e-04	2.61e-05	2.76e-06	2.75e-07
	ρ_ε^N	3.00	3.12	3.24	3.33	
2^{-28}	E_ε^N	1.82e-03	2.27e-04	2.61e-05	2.76e-06	2.75e-07
	ρ_ε^N	3.00	3.12	3.24	3.33	
2^{-32}	E_ε^N	1.82e-03	2.27e-04	2.61e-05	2.76e-06	2.75e-07
	ρ_ε^N	3.00	3.12	3.24	3.33	
	E^N	1.82e-03	2.27e-04	2.61e-05	2.76e-06	2.75e-07
	ρ^N	3.00	3.12	3.24	3.33	

TABLE 6. Comparison of the maximum errors of the present scheme with the hybrid HODIE scheme [3] and central difference(CD) scheme [15] to solve the Example 3 for $\varepsilon = 2^{-24}$.

	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
Present scheme	1.82E-03	2.27E-04	2.61E-05	2.76E-06	2.75E-07
HODIE scheme [3]	5.20E-03	7.78E-04	8.35E-05	8.59E-06	8.20E-07
CD scheme [15]	7.76e-03	2.81e-03	9.34e-04	2.99e-04	9.29e-05

tables represent the parameter-robust errors E^N and parameter-robust convergence rate ρ^N. Numerical results reported in Tables $1, 3$ and 5 clearly indicate that the present scheme is almost fourth order uniformly convergent on the generalized Shishkin mesh, and this supports the theoretical result proved in Section 5. The small reduction in parameter-robust convergence rate, that is, slightly less than 4, is because of the fact that L presents in the error estimate. We also compare the maximum errors of the present scheme (21) on the generalized Shishkin mesh with two earlier reference methods of hybrid HODIE scheme on Shishkin mesh [3] and central difference (CD) scheme on Shishkin mesh [15]. Numerical results are reported in Tables $2, 4$, and 6 for the test Examples 1,2, and 3, respectively. It can be seen from these tables that the present scheme is more accurate than earlier reference methods of hybrid HODIE scheme [3] and CD scheme [15]. It can be noted that the numerical results of the hybrid HODIE scheme [3] are comparable with the present scheme. A reason for this is that both the schemes are the same in the layer regions, but considered on different types of meshes. This gives us a theoretical benefit to prove an almost fourth order parameter-robust convergence of present scheme (21) on a generalized Shishkin mesh while in [3] a hybrid HODIE scheme is considered on a general Shishkin mesh and a third order parameter-robust convergence is proved.

7. Conclusions

We presented a high order parameter-robust finite difference method for a linear system of ($M \geq 2$) coupled singularly perturbed reaction-diffusion problems with the same singular perturbation parameter in all the equations. A high order decomposition of the exact solution into its regular and layer parts is constructed. The problem is discretized using a suitable combination of the fourth order compact difference scheme and the central difference scheme on the generalized Shishkin mesh $\widetilde{S}(L)$. The essential idea in this method is to use the generalized Shishkin mesh $\widetilde{S}(L)$ in order to obtain a high order parameter-robust convergence. Observe that the fine parts of the standard Shishkin mesh $S(L)$ and the generalized Shishkin mesh $\widetilde{S}(L)$ are identical, but the coarse part of $\widetilde{S}(L)$ is a smooth continuation of the fine mesh and is no longer equidistant. Using this fact the present scheme is proved to be almost fourth-order uniformly convergent with respect to perturbation parameter. The numerical results validate the theoretical results.

Acknowledgements

The authors gratefully acknowledge the valuable comments and suggestions from the anonymous referees.

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