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FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY LINEAR QUASI-PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the mathematical formulation for a quadratic optimal control problem governed by a linear quasi-parabolic integro-differential equation is studied, the optimality conditions are derived, and then the a priori error estimate for its finite element approximation is given. Furthermore some numerical tests are performed to verify the theoretical results.

Key words. optimal control, linear quasi-parabolic integro-differential equations, optimality conditions, finite element methods, a priori error estimate.

1. Introduction

Linear quasi-parabolic integro-differential equations and their control appear in many scientific problems and engineering applications such as biology mechanics, nuclear reaction dynamics, heat conduction in materials with memory, and visco-elasticity, etc.. The existence and uniqueness of the solution of the linear quasi-parabolic integro-differential equations have been studied by Wheeler M. F. in [17]. Furthermore the finite element methods for linear quasi-parabolic integro-differential equations with a smooth kernel have been discussed in, e.g., X. Cui [2]. However there exists little research on optimal control problems governed by quasi-parabolic integro-differential equations, in spite of the fact that such control problems are often encountered in practical engineering applications and scientific computations. Furthermore the finite element methods of the optimal control problem governed by such equations have not been studied although there has existed much research on the finite element approximations of quasi-parabolic integro-differential equations.

Finite element approximations of optimal control problems governed by various partial differential equations have been extensively studied in the literature. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems, for examples, see [1, 4, 5, 10, 11, 12, 13, 14, 15, 16]. For optimal control problems governed by linear PDEs, the optimality conditions and their finite element approximation and the a prior error estimates were established long ago, for example, see [4, 7]. The purpose of this paper is to study the mathematical formulation and its finite element approximation of the optimal control problem governed by a linear quasi-parabolic integro-differential equation. In particular we establish the optimality conditions and analyze the a priori error estimates for these constrained optimal control problems. We also present some numerical tests to verify the theoretical analysis.

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The outline of the paper is as follows. In Section 2, we present the weak formulation and analyze the existence of the solution for the optimal control problem. In Section 3, we give the optimality conditions and the finite element approximation of the optimal control problems. In Section 4, we derive the a priori error estimates for the finite element approximation of the control problem. In the last section, we perform some numerical tests, which illustrate the theoretical results.

2. Model problem and its weak formulation

Let Ω, with Lipschitz boundary $\partial\Omega$, and Ω_U be bounded open sets in $\mathbb{R}^d$, $1 \leq d \leq 3$, and $T > 0$. Introduce the objective functional

$$J(u,y) = \Big\{\frac{1}{2}\int_0^T \int_\Omega |y - z_d|^2 + \frac{\alpha}{2}\int_0^T \int_{\Omega_U} |u|^2\Big\},$$

where α is a positive regularity constant. We investigate the optimal control problem governed by a quasi-parabolic integro-differential equation as follows:

$$\min_{u \in U_{ad}} J(u, y(u)) \tag{1}$$

subject to

$$\begin{cases} y_t - \nabla \cdot \big(A\nabla y_t + D\nabla y + \displaystyle\int_0^t C(t,\tau)\nabla y(\tau)d\tau\big) & \\ = f + Bu & \text{in } \ \Omega \times (0,T], \\ y = 0 & \text{on } \ \partial\Omega \times [0,T], \\ y|_{t=0} = y^0 & \text{in } \ \Omega, \end{cases} \tag{2}$$

where u is the control, y is the state, z_d is the observation, U_{ad} is a closed convex subset with respect to the control, f, z_d and y^0 are some given functions to be specified later, and

$$A = (a_{ij}(\mathbf{x}))_{d\times d}, \ D = (d_{ij}(\mathbf{x}))_{d\times d}, \ \ C = (c_{ij}(\mathbf{x},t,\tau))_{d\times d},$$

B is a bounded operator independent of t from $L^2(0,T;L^2(\Omega_U))$ to $L^2(0,T;L^2(\Omega))$.

We give the weak formulation of the problem mentioned-above and study the existence and regularity of the solution. To this end, let us introduce some Sobolev spaces. Throughout the paper, we adopt the standard notations, such as $W^{m,s}(\Omega)$, for Sobolev spaces on Ω with norm $\|\cdot\|_{m,s,\Omega}$ and semi-norm $|\cdot|_{m,s,\Omega}$ for $m \geq 0$ and $1 \leq s \leq \infty$. Set $W_0^{m,s}(\Omega) = \big\{w \in W^{m,s}(\Omega) : w|_{\partial\Omega} = 0\big\}$. Also denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$), with norm $\|\cdot\|_{m,\Omega}$, and semi-norm $|\cdot|_{m,\Omega}$. Denote by $L^r(0,T;W^{m,s}(\Omega))$ the Banach space of all L^r integrable functions from $(0,T)$ into $W^{m,s}(\Omega)$ with norm $\| v \|_{L^r(0,T;W^{m,s}(\Omega))} = \big(\int_0^T \| v \|^r_{W^{m,s}(\Omega)} dt\big)^{\frac{1}{r}}$ for $1 \leq r \leq \infty$. Similarly, one can define the spaces $H^1(0,T;W^{m,s}(\Omega))$ and $C^k(0,T;W^{m,s}(\Omega))$. The details can be found in [8]. To fix idea, we shall take the state space $W = L^2(0,T;V)$ with V $= H_0^1(\Omega)$ and the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega_U)$. Let the observation space $Y = L^2(0,T;H)$ with $H = L^2(\Omega)$ and $U_{ad} \subseteq X$ a convex subset. In addition c or C denotes a general positive constant independent of unknowns and the meshes parameters introduced later. Introduce L^2-inner products:

$$(f_1, f_2) = \int_\Omega f_1 f_2 \ \ \forall\, f_1, \ f_2 \in H, \ \ (u,v)_U = \int_{\Omega_U} uv \ \ \forall\, u, \ v \in U$$

and bilinear forms:

$$a(z,w)=(A\nabla z,\nabla w),\quad d(z,w)=(D\nabla z,\nabla w),$$
$$c(t,\tau;z,w)=(C(t,\tau)\nabla z,\nabla w),\quad c'_t(t,\tau;z,w)=(C'_t(t,\tau)\nabla z,\nabla w)$$

for any z and w in V. In the case that $f_1\in V$ and $f_2\in V^*$, the dual pair (f_1,f_2) is understood as $\langle f_1,f_2\rangle_{V\times V^*}$. Therefore the control problem (1) - (2) can be restated as:

$$\min_{u\in U_{ad}} J(u,y(u))$$

$$(OCP)\qquad \begin{cases} (y_t,w)+a(y_t,w)+d(y,w)+\displaystyle\int_0^t c(t,\tau;y(\tau),w)d\tau \\ =(f+Bu,w)\quad \forall\ w\in V,\quad t\in(0,T], \\ y|_{t=0}=y^0. \end{cases}$$

Next, we will analyze the existence, uniqueness and the regularity of the solution of (OCP). Assume that there are positive constants c_0 and C_0 such that for all t and τ in $[0,T]$:

$$(3)\qquad \begin{aligned} &(a)\ \ a(z,z)\ge c_0\|z\|^2_{1,\Omega},\quad d(z,z)\ge c_0\|z\|^2_{1,\Omega},\\ &(b)\ \ |a(z,w)|\le C_0\|z\|_{1,\Omega}\|w\|_{1,\Omega},\quad |\ d(z,w)\ |\le C_0\|z\|_{1,\Omega}\|w\|_{1,\Omega},\\ &(c)\ \ |c(t,\tau;z,w)|\le C_0\|z\|_{1,\Omega}\|w\|_{1,\Omega},\quad |c'_t(t,\tau;z,w)|\le C_0\|z\|_{1,\Omega}\|w\|_{1,\Omega}\end{aligned}$$

for any z and w in V. The following theorem gives the existence and uniqueness of the solution of the system (OCP).

Theorem 2.1. *Assume that the condition (3) hold. There exists the unique solution (u,y) for the minimization problem (OCP) such that $u\in L^2(0,T;L^2(\Omega_U))$ and $y\in L^\infty(0,T;H^1(\Omega))$ and $y'_t\in L^2(0,T;H^1(\Omega))$.*

Proof. Let $\{(u^n,y^n)\}_{n=1}^\infty$ be a minimization sequence for the system (OCP), then it is clear that $\{u^n\}_{n=1}^\infty$ are bounded in $L^2(0,T;L^2(\Omega_U))$. Thus there is a subsequence of $\{u^n\}_{n=1}^\infty$ (still denote by $\{u^n\}_{n=1}^\infty$) such that u^n converges to u weakly in $L^2(0,T;L^2(\Omega_U))$. For the subsequence $\{u^n\}_{n=1}^\infty$, we have

$$(4)\qquad \begin{aligned} &(y^n_t,w)+a(y^n_t,w)+d(y^n,w)+\int_0^t c(t,\tau;y^n(\tau),w)d\tau\\ =&(f+Bu^n,w)\ \ \forall\ w\in V,\quad t\in\ (0,T].\end{aligned}$$

Taking $w=y^n$ in (4) gives

$$(y^n_t,y^n)+a(y^n_t,y^n)+d(y^n,y^n)+\int_0^t c(t,\tau;y^n(\tau),y^n(t))d\tau=(f+Bu^n,y^n)$$

such that for $0\le t\le T$,

$$(5)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|y^n\|^2_{0,\Omega}+a(y^n,y^n)\big)+d(y^n,y^n)+\int_0^t c(t,\tau;y^n(\tau),y^n(t))d\tau\\ =&(f+Bu^n,y^n).\end{aligned}$$

By integrating from 0 to t in (5), we obtain

$$(6)\qquad \begin{aligned} &\|y^n(t)\|^2_{1,\Omega}+\int_0^t\|y^n\|^2_{1,\Omega}d\tau\\ \le& C\Big\{\|y^0\|^2_{1,\Omega}+C\int_0^t(\|f\|^2_{-1,\Omega}+\|u^n\|^2_{0,\Omega_U})d\tau+\int_0^t\int_0^\tau\|y(s)\|^2_{1,\Omega}dsd\tau\Big\}.\end{aligned}$$

Applying Gronwall's inequality to (6) yields

$$\|y^n\|^2_{L^\infty(0,T;H^1(\Omega))} + \|y^n\|^2_{L^2(0,T;H^1(\Omega))} \le C\Big\{\|y^0\|^2_{1,\Omega} + \int_0^T \left(\|f\|^2_{-1,\Omega} + \|u^n\|^2_{0,\Omega_U}\right)\Big\}.$$

Thus we have that $\{y^n\}_{n=1}^\infty$ is a bounded set in $L^\infty(0,T;H^1(\Omega))\bigcap L^2(0,T;H^1(\Omega))$. Hence

$$\begin{cases} u^n \longrightarrow u & \text{weakly in } \ L^2(0,T;L^2(\Omega_U)),\\ y^n \longrightarrow y & \text{weakly in } \ L^2(0,T;H^1(\Omega)),\\ y^n(T) \longrightarrow y(T) & \text{weakly in } \ H^1(\Omega).\end{cases}$$

Let $W = \{w;\ w \in L^2(0,T;H^1(\Omega)), w_t' \in L^2(0,T;H^1(\Omega))\}$. Integrating (4) from 0 to T, we obtain

$$\begin{aligned}&(y^n(T),w(T)) + a(y^n(T),w(T)) - \int_0^T \Big[(y^n,w_t') + a(y^n,w_t') + d(y^n,w)\Big]\\ &+\int_0^T\int_0^t c(t,\tau;y^n(\tau),w(t))d\tau dt \\ =&(y^0,w(0)) + a(y^0,w(0)) + \int_0^T (f+Bu^n,w) \ \ \forall\ w\in W.\end{aligned} \tag{7}$$

Taking the limit in (7) as $n\to\infty$, we have

$$\begin{aligned}&(y(T),w(T)) + a(y(T),w(T)) - \int_0^T \Big[(y,w_t') + a(y,w_t') + d(y,w)\Big]\\ &+\int_0^T\int_0^t c(t,\tau;y(\tau),w(t))d\tau dt \\ =&(y^0,w(0)) + a(y^0,w(0)) + \int_0^T (f+Bu,w) \ \ \forall\ w\in W\end{aligned}$$

such that

$$(y_t,w)+a(y_t,w)+d(y,w)+\int_0^t c(t,\tau;y(\tau),w)d\tau = (f+Bu,w) \ \ \forall\ w\in V,\ \ t\in(0,T].$$

Furthermore, we have

$$\int_0^T \Big[(y_t,y_t) + a(y_t,y_t) + d(y,y_t) + \int_0^t c(t,\tau;y(\tau),y_t(t))d\tau\Big]dt = \int_0^T (f+Bu,y_t)$$

such that

$$\int_0^T \|y_t\|^2_{1,\Omega} \le C\int_0^T \Big[\|f\|^2_{-1,\Omega} + \|u\|^2_{0,\Omega_U} + \|y\|^2_{1,\Omega} + \int_0^t \|y\|^2_{1,\Omega}d\tau\Big].$$

This means $y_t \in L^2(0,H^1(\Omega))$. So (u,y) is one solution of (OCP).

Since $\int_0^T \|y-z_d\|^2_{0,\Omega}$ is a convex function on space $L^2(0,T;L^2(\Omega))$ and $\frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Omega_U}$ is a strictly convex function on U, hence $J(u,y(u))$ is a strictly convex function on U. Thus the solution of the minimization problem (OCP) is unique. □

The following theorem states the regularity of the solution of (OCP).

Theorem 2.2. *Assume that the second order differential operator A is H^2-regularity operator on Ω and that the condition (3) holds and $f\in L^2(0,T;L^2(\Omega))$ and $y^0 \in H_0^1(\Omega)\bigcap H^2(\Omega)$. Then the solution of (OCP) obeys $y\in L^2(0,T;H^2(\Omega))$ and $y_t\in L^2(0,T;H^2(\Omega))$.*

Proof. From (2), we have

$$\begin{aligned}&(A\nabla y_t,\nabla y)+(\nabla\cdot(A\nabla y_t),\nabla\cdot(A\nabla y))+(\nabla\cdot(D\nabla y),\nabla\cdot(A\nabla y))\\&+\int_0^t(\nabla\cdot(C(t,\tau)\nabla y(\tau)),\nabla\cdot(A\nabla y))d\tau\\=&-(f+Bu,\nabla\cdot(A\nabla y))\end{aligned}$$

such that

$$\begin{aligned}&\frac{d}{dt}\Big[(A\nabla y,\nabla y)+(\nabla\cdot(A\nabla y),\nabla\cdot(A\nabla y))\Big]\\\le& C\Big\{\|y\|^2_{2,\Omega}+\|f\|_{0,\Omega}+\|u\|_{0,\Omega_U}+\int_0^t\|y\|^2_{2,\Omega}d\tau\Big\}.\end{aligned}\tag{8}$$

Since $\|y\|^2_{2,\Omega}\le C\|\nabla\cdot(A\nabla y)\|^2_{0,\Omega}$, hence

$$\|y\|^2_{2,\Omega}\le C\Big\{\|y^0\|^2_{2,\Omega}+\int_0^t\Big[\|f\|^2_{0,\Omega}+\|u\|^2_{0,\Omega_U}+\|y\|^2_{2,\Omega}+\int_0^\tau\|y\|^2_{2,\Omega}\Big]\Big\}$$

such that

$$\begin{aligned}&\|y\|^2_{2,\Omega}+\int_0^t\|y\|^2_{2,\Omega}\\\le& C\Big\{\|y^0\|^2_{2,\Omega}+\int_0^T\Big(\|f\|^2_{0,\Omega}+\|u\|^2_{0,\Omega_U}\Big)+\int_0^t\Big(\|y\|^2_{2,\Omega}+\int_0^\tau\|y\|^2_{2,\Omega}\Big)\Big\}.\end{aligned}\tag{9}$$

Applying the Gronwall inequality to (9) yields

$$\begin{aligned}&\|y\|^2_{L^\infty(0,T;H^2(\Omega))}+\|y\|^2_{L^2(0,T;H^2(\Omega))}\\\le& C\Big\{\|y^0\|^2_{2,\Omega}+\|f\|^2_{L^2(0,T;L^2(\Omega)}+\|u\|^2_{L^2(0,T;L^2(\Omega_U))}\Big\}.\end{aligned}\tag{10}$$

From (2) and (10), we obtain

$$\begin{aligned}&\|y_t\|^2_{L^2(0,T;H^2(\Omega))}\\\le& C\Big\{\|f\|^2_{L^2(0,T;L^2(\Omega))}+\|u\|^2_{L^2(0,T;L^2(\Omega_U))}+\|y_t\|_{L^2(0,T;L^2(\Omega))}+\|\,y\,\|_{L^2(0,T;H^2(\Omega))}\Big\}.\end{aligned}$$

Thus $y\in L^\infty(0,T;H^2(\Omega))$ and $y_t\in L^2(0,T;H^2(\Omega))$. This complete the proof of Theorem 2.2. □

Remark 2.1. Here we suppose that A and D are independent of time variable t. The above results can also be applied to the case $A=A(\mathbf{x},t)$ and $D=D(\mathbf{x},t)$ provided suitable conditions for the operators A and D to be imposed.

3. The Optimality conditions and its finite element approximation

In this section, we study the optimality conditions and the finite element approximation for optimal control problems governed by quasi-parabolic integro-differential equation .

3.1. The Optimality conditions. The following theorem states the optimality conditions of the problem (OCP).

Theorem 3.1. *A pair* $(y,u)\in L^2(0,T;H^1_0(\Omega))\times L^2(0,T;L^2(\Omega_U))$ *is the solution of the optimal control problem (OCP), if and only if there exists a co-state*

$p \in L^2(0,T;H_0^1(\Omega))$ such that the triple (y,p,u) satisfies the following optimality conditions:

$$\begin{cases} (y_t,w)+a(y_t,w)+d(y,w)+\displaystyle\int_0^t c(t,\tau;y(\tau),w)d\tau \\ =(f+Bu,w) \quad \forall\ w\in V, \quad t\in(0,T], \\ y|_{t=0}=y^0; \end{cases} \tag{11}$$

$$\begin{cases} -(q,p_t)-a(q,p_t)+d(q,p)+\displaystyle\int_t^T c(\tau,t;q,p(\tau))d\tau \\ =(y-z_d,q) \quad \forall\ q\in V, \quad t\in[0,T), \\ p|_{t=T}=0; \end{cases} \tag{12}$$

$$\int_0^T (\alpha u+B^*p, v-u)_U \geq 0 \quad \forall\ v\in U_{ad}, \tag{13}$$

where B^* is the adjoint operator of the operator B.

Proof. Let $J(u,y)=g(y(u))+j(u)$. By the standard method as in [8], the optimal condition reads

$$j'(u)(v-u)+(g(y(u)))'(v-u)\geq 0, \quad \forall\ v\in U_{ad}. \tag{14}$$

It is clear that

$$\begin{aligned} j'(u)(v-u) &= \lim_{s\to0^+}\frac{1}{s}\Big(\frac{\alpha}{2}\int_0^T \big[\|u+s(v-u)\|^2_{0,\Omega_U}-\|u\|^2_{0,\Omega_U}\big]\Big) \\ &= \int_0^T (\alpha u, v-u)_U \end{aligned} \tag{15}$$

and

$$\begin{aligned} (g(y(u)))'(v-u) &= \lim_{s\to0^+}\frac{1}{s}\big(g(y(u+s(v-u)))-g(y(u))\big) \\ &= \lim_{s\to0^+}\frac{1}{2s}\int_0^T \big[\|y(u+s(v-u))-y(u)\|^2_{0,\Omega} \\ &\quad +2(y(u+s(v-u))-y(u),y-z_d)\big] \\ &= \int_0^T (y'(u)(v-u),y-z_d). \end{aligned} \tag{16}$$

Next, differentiating the state equation (OCP) at u in the direction v, we have

$$\begin{aligned} &\frac{1}{s}\Big(\int_0^T (y_t(u+sv)-y_t(u),w)+\int_0^T a(y_t(u+sv)-y_t(u),w) \\ &+\int_0^T d(y(u+sv)-y(u),w) \\ &+\int_0^T\int_0^t c(t,\tau;y(u+sv)(\tau)-y(u)(\tau),w)d\tau\Big) \\ =&\int_0^T (Bv,w) \quad \forall\ w\in W. \end{aligned} \tag{17}$$

Taking the limit in (17) as $s \to 0$, we obtain

$$\begin{aligned}
&\int_0^T (y_t'(u)(v), w) + \int_0^T a(y_t'(u)(v), w) + \int_0^T d(y'(u)(v), w) \\
&+ \int_0^T \int_0^t c(t, \tau; (y'(u)(v))(\tau), w) \\
=& \int_0^T (Bv, w) \quad \forall\, v \in U_{ad}, \quad w \in W,
\end{aligned} \tag{18}$$

where we used the equality that for any v and w in $L^2(0,T;H^1(\Omega))$

$$\int_0^T \int_0^t c(t, \tau; z(\tau), w(t)) d\tau dt = \int_0^T \int_\tau^T c(t, \tau; z(\tau), w(t)) dt d\tau. \tag{19}$$

Then (18) is equivalent to

$$\begin{aligned}
&\int_0^T (y_t'(u)(v), w) + \int_0^T a(y_t'(u)(v), w) + \int_0^T d((y'(u)(v)), w) \\
&+ \int_0^T \int_t^T c(\tau, t; (y'(u)(v))(t), w(\tau)) d\tau dt \\
=& \int_0^T (Bv, w) \quad \forall\, v \in U_{ad}, \quad w \in W.
\end{aligned} \tag{20}$$

Define the co-state $p \in W$ satisfying

$$\begin{cases}
\displaystyle\int_0^T [-(q, p_t) - a(q, p_t) + d(q, p) + \int_t^T c(\tau, t; q(t), p(\tau)) d\tau] \\
\displaystyle= \int_0^T (y - z_d, q), \;\; \forall\, q \in W \\
p|_{t=T} = 0.
\end{cases} \tag{21}$$

Letting $w = p$ in (20), we have

$$\begin{aligned}
&\int_0^T (B(v-u), p) dt = \int_0^T (v - u, B^* p)_U dt \\
=& \int_0^T [-(y'(u)(v-u), p_t) - a(y'(u)(v-u), p_t) + d(y'(u)(v-u), p) \\
&+ \int_t^T c(\tau, t; y'(u)(v-u)(t), p(\tau)) d\tau] \\
=& \int_0^T (y - z_d, y'(u)(v-u)) \;\; \forall\, v \in U_{ad}.
\end{aligned} \tag{22}$$

It follows from (16) and (22) that

$$(g(y(u)))'(v-u) = \int_0^T (y'(u)(v-u), y - z_d) = \int_0^T (v - u, B^* p)_U, \qquad \forall\, v \in U_{ad}. \tag{23}$$

By (14)-(15) and (23), the optimality condition reads

$$J'(u)(v-u) = \int_0^T (\alpha u + B^* p, v - u)_U \geq 0, \qquad \forall\, v \in U_{ad}, \tag{24}$$

where p is defined in (21). This completes the proof of Theorem 3.1. $\square$

3.2. Finite element approximation. Let us discuss the finite element approximation of the control problem (OCP). Here we only consider triangular and conforming elements. For simplicity, we assume that Ω is a convex polygon. Let T^h be a partitioning of Ω into disjoint regular d-simplices τ, so that $\bar{\Omega} = \bigcup_{\tau \in T^h} \bar{\tau}$. Each element has at most one face on $\partial\Omega$, and $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if $\bar{\tau}$ and $\bar{\tau}' \in T^h$. As usual, h denotes the diameter of the triangulation T^h. Associated with T^h is a finite-dimensional subspace S^h of $C(\bar{\Omega})$, such that $\chi|_\tau$ are polynomials of order m $(m \geq 1)$ for all $\chi \in S^h$ and $\tau \in T^h$. Let $V^h = \{v_h \in S_h : \ v_h\big|_{\partial\Omega} = 0\}$ and $W^h = L^2(0, T; V^h)$. It is easy to be seen that $V^h \subset V$ and $W^h \subset W$.

For simplicity, we again assume that Ω_U is a convex polygon. Let T_U^h be a partitioning of Ω_U into disjoint regular d-simplices τ_U, so that $\bar{\Omega}_U = \bigcup_{\tau_U \in T_U^h} \bar{\tau}_U$. $\bar{\tau}_U$ and $\bar{\tau}_U'$ have either only one common vertex or a whole edge or face if $\bar{\tau}_U$ and $\bar{\tau}_U' \in T_U^h$. Associated with T_U^h is another finite-dimensional subspace U^h of $L^2(\Omega_U^h)$, such that $\chi|_{\tau_U}$ are polynomials of order m $(m \geq 0)$ for all $\chi \in U^h$ and $\tau_U \in T_U^h$. Here there is no requirement for the continuity. Let $X^h = L^2(0, T; U^h)$. It is easy to be seen that $X^h \subset X$.

Let $h_\tau(h_{\tau_U})$ denote the maximum diameter of the element τ (τ_U) in T^h (T_U^h). Due to the limited regularity of the optimal control u in general, there will be no advantage in considering higher-order finite element spaces than the piecewise constant space for the control. We therefore only consider the piecewise constant finite element space for the approximation of the control, though higher-order finite element spaces will be used to approximate the state and the co-state. Let $P_0(\Omega)$ denote all the 0-order polynomial over Ω. Therefore we always take$X^h = \{u \in X : u(x,t) \mid_{x \in \tau_U} \in P_0(\tau_U), \ \forall \ t \in [0, T]\}$. U_{ad}^h is a closed convex set in X^h. For ease of exposition, in this paper we assume that$U_{ad}^h \subset U_{ad} \cap X^h$.

Then the finite element approximation of (OCP) is thus defined by $(OCP)^h$:

$$\min_{u_h \in U_{ad}^h} \Big\{ \frac{1}{2} \int_0^T \|y_h - z_d\|_{0,\Omega}^2 + \frac{\alpha}{2} \int_0^T \|u_h\|_{0,\Omega_U}^2 \Big\}, \tag{25}$$

where $y_h \in W^h$ and $y_h^0 \in V^h$, an approximation of y^0, such that

$$\begin{cases} \quad (y_{h,t}, w_h) + a(y_{h,t}, w_h) + d(y_h, w_h) + \displaystyle\int_0^t c(t, \tau; y_h(\tau), w_h) d\tau \\ = (f + Bu_h, w_h), \ \forall \ w_h \in V^h, \\ \quad y_h|_{t=0} = y_h^0. \end{cases} \tag{26}$$

Similarly to the continuous case, a pair $(y_h, u_h) \in W^h \times U_{ad}^h$ is a solution of (25) - (26), if and only if there exists a co-state $p_h \in W^h$ such that the triple (y_h, p_h, u_h) satisfies the following optimality conditions:

$$\begin{cases} \quad (y_{h,t}, w_h) + a(y_{h,t}, w_h) + d(y_h, w_h) + \displaystyle\int_0^t c(t, \tau; y_h(\tau), w_h) d\tau \\ = (f + Bu_h, w_h) \quad \forall \ w_h \in V^h, \\ \quad y_h|_{t=0} = y_h^0, \end{cases} \tag{27}$$

$$(28)\qquad \begin{cases} \quad -(q_h, p_{ht}) - a(q_h, p_{h,t}) + d(q_h, p_h) + \displaystyle\int_t^T c(\tau, t; q_h, p_h(\tau))d\tau \\ =(y_h - z_d, q_h) \quad \forall\ q_h \in V^h, \\ \quad p_h|_{t=T} = 0 \end{cases}$$

$$(29)\qquad \int_0^T (\alpha u_h + B^* p_h, v_h - u_h)_U \geq 0, \quad \forall\ v_h \in U_{ad}^h.$$

The optimality conditions (27)-(29) are the semi-discrete approximation to the problem (11)-(13). Let π_{h_U} be the local averaging operator given by

$$(30)\qquad (\pi_{h_U} w)|_{\tau_U} := \frac{\int_{\tau_U} w}{\int_{\tau_U} 1} \quad \forall\ \tau_U \in T_U^h.$$

It is the obvious fact that $\int_{\Omega_U} w = \int_{\Omega_U} \pi_{h_U} w$ for any $w \in L^2(\Omega_U)$. By the operator π_{h_U}, (29) is equivalent to

$$(31)\qquad \int_0^T (\alpha u_h + \pi_{h_U}(B^* p_h), v_h - u_h)_U \geq 0, \quad \forall\ v_h \in U_{ad}^h.$$

In next sections, we will analyze the a priori error estimates of approximation solution.

4. A priori error analysis

In this section, we consider the zero obstacle problem:

$$(32)\qquad U_{ad} = \big\{ v \in X;\ v \geq 0,\ a.e. \text{ in } \Omega_U, \quad t \in [0, T]\ \big\},$$

or the integration obstacle problem:

$$(33)\qquad U_{ad} = \big\{ v \in X;\ \int_{\Omega_U} v \geq 0, \quad t \in [0, T]\ \big\}.$$

In the case of (32), (13) and (29) yields

$$(34)\qquad \alpha u = \max\{0, -B^* p\}, \quad \alpha u_h = \max\{0, -\pi_{h_U}(B^* p_h)\}.$$

In the case of (33), (13) and (29) yields

$$(35)\qquad \begin{cases} \alpha u = -B^* p + \max\Big\{0, \dfrac{1}{|\Omega_U|} \displaystyle\int_{\Omega_U} B^* p\ \Big\}, \\ \alpha u_h = -\pi_{h_U}(B^* p_h) + \max\Big\{0, \dfrac{1}{|\Omega_U|} \displaystyle\int_{\Omega_U} B^* p_h\ \Big\}. \end{cases}$$

In these two cases, it follows from Theorem 2.2 that

$$(36)\qquad y, p \in H^1(0, T; H^2(\Omega)), \quad u \in L^2(0, T; H^1(\Omega_U)).$$

Lemma 4.1. *Let U_{ad} be given by (32) or (33). Then $\pi_{h_U} w \in U_{ad}^h$ for any $w \in U_{ad}$.*

In next two subsections, we will give the a priori error estimates in $H^1(H^1)$-norm and in $L^2(L^2)$-norm respectively.

4.1. Convergent rate in $H^1(H^1)$-norm. The following theorem gives the a priori error estimate in $H^1(H^1)$-norm.

Theorem 4.2. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of the systems (11) - (13) and (27) - (29). Then there holds the a priori error estimate:*

$$\|y - y_h\|_{H^1(0,T;H^1(\Omega))} + \|p - p_h\|_{H^1(0,T;H^1(\Omega))} \leq C(h_U + h). \tag{37}$$

In order to prove the a priori error estimate given in Theorem 4.2, introduce the auxiliary functions $(y_h(u), p_h(u))$ such that

$$\begin{cases} \quad (y_{h,t}(u), w_h) + a(y_{h,t}(u), w_h) + d(y_h(u), w_h) \\ \quad + \displaystyle\int_0^t c(t, \tau; y_h(u)(\tau), w_h) d\tau \\ =(f + Bu, w_h), \quad \forall\ w_h \in V^h, \\ \quad y_h(u)|_{t=0} = y_h^0 \end{cases} \tag{38}$$

and

$$\begin{cases} \quad -(q_h, p_{h,t}(u)) - a(q_h, p_{h,t}(u)) + d(q_h, p_h(u)) \\ \quad + \displaystyle\int_t^T c(\tau, t; q_h, p_h(u)(\tau)) d\tau \\ =(y - z_d, q_h), \quad \forall\ q_h \in V^h, \\ \quad p_h(u)|_{t=T} = 0. \end{cases} \tag{39}$$

The proof of Theorem 4.2 will be complete by use of the following Lemmas.

Lemma 4.3. *Let (y_h, p_h, u_h) be the solutions of the systems (27) - (29). Then there holds the estimate:*

$$\begin{aligned} &\|y_h - y_h(u)\|_{L^\infty(0,T;H^1(\Omega))} + \|p_h - p_h(u)\|_{L^\infty(0,T;H^1(\Omega))} \\ &+ \|y_{h,t} - y_{h,t}(u)\|_{L^2(0,T;H^1(\Omega))} + \|p_{h,t} - p_{h,t}(u)\|_{L^2(0,T;H^1(\Omega))} \\ \leq& C\Big\{\|u_h - \pi_{h_U} u\|_{L^2(0,T;L^2(\Omega_U))} + h_U\|u - \pi_{h_U} u\|_{L^2(0,T;L^2(\Omega_U))} \\ &+ \|y - y_h(u)\|_{L^2(0,T;L^2(\Omega))}\Big\}. \end{aligned} \tag{40}$$

Proof. From (38) and (27), we obtain

$$\begin{cases} \quad (y_{h,t} - y_{h,t}(u), w_h) + a(y_{h,t} - y_{h,t}(u), w_h) + d(y_h - y_h(u), w_h) \\ \quad + \displaystyle\int_0^t c(t, \tau; (y_h - y_h(u))(\tau), w_h) d\tau \\ =(B(u_h - u), w_h) \ \ \forall\ w_h \in V^h, \\ \quad (y_h - y_h(u))|_{t=0} = 0. \end{cases} \tag{41}$$

Similarly, from (39) and (28) , we have

$$\begin{cases} \quad -(q_h, p_{h,t} - p_{h,t}(u)) - a(q_h, p_{h,t} - p_{h,t}(u)) + d(q_h, (p_h - p_h(u))) \\ \quad + \displaystyle\int_t^T c(\tau, t; q_h, (p_h - p_h(u))(\tau)) d\tau \\ =(y_h - y, q_h) \ \ \forall\ q_h \in V^h, \\ \quad (p_h - p_h(u))|_{t=T} = 0. \end{cases} \tag{42}$$

Taking $w_h = y_h - y_h(u)$ in (41) , we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|y_h - y_h(u)\|_{0,\Omega}^2 + a(y_h - y_h(u), y_h - y_h(u))\big)\\
&+ d(y_h - y_h(u), y_h - y_h(u)) + \int_0^t c(t,\tau;(y_h - y_h(u))(\tau), y_h - y_h(u))d\tau \\
=&(B(u_h - u), y_h - y_h(u)).
\end{aligned} \tag{43}
$$

Noting that for any $w_h \in V^h$ and $0 < \varepsilon < 1$,

$$
\begin{aligned}
&(B(u_h - u), w_h) = (B(u_h - \pi_{h_U} u), w_h) + (\pi_{h_U} u - u, (\mathcal{I} - \pi_{h_U})(B^* w_h))_U\\
\le& C(\varepsilon)\Big\{\|\pi_{h_U} u - u_h\|_{0,\Omega_U}^2 + h_U^2\|\pi_{h_U} u - u\|_{0,\Omega_U}^2\Big\} + \varepsilon\|w_h\|_{1,\Omega}^2
\end{aligned} \tag{44}
$$

and integrating (43) from 0 to t, by (3) and (44), we obtain

$$
\begin{aligned}
&\|y_h - y_h(u)\|_{1,\Omega}^2 + \int_0^t \|y_h - y_h(u)\|_{1,\Omega}^2 d\tau\\
\le& C\int_0^t \big(\|\pi_{h_U} u - u_h\|_{0,\Omega_U}^2 + h_U^2\|\pi_{h_U} u - u\|_{0,\Omega_U}^2\\
&+ \int_0^\tau \|(y_h - y_h(u))(s)\|_{1,\Omega}^2 ds\big)d\tau.
\end{aligned} \tag{45}
$$

By Gronwall Lemma, from (45), we obtain

$$
\begin{aligned}
&\|y_h - y_h(u)\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|y_h - y_h(u)\|_{L^2(0,T;H^1(\Omega))}^2\\
\le& C\Big\{\|\pi_{h_U} u - u_h\|_{L^2(0,T;H^1(\Omega_U))}^2 + h_U^2\|\pi_{h_U} u - u\|_{L^2(0,T;H^1(\Omega_U))}^2\Big\}.
\end{aligned} \tag{46}
$$

Taking $w_h = (y_h - y_h(u))_t$ in (41), we have

$$
\begin{aligned}
&\|y_{h,t} - y_{h,t}(u)\|_{1,\Omega}^2 + \frac{1}{2}\frac{d}{dt}d(y_h - y_h(u), y_h - y_h(u))\\
\le& C\Big\{\|\pi_{h_U} u - u_h\|_{0,\Omega_U}^2 + h_U^2\|\pi_{h_U} u - u\|_{0,\Omega_U}^2 + \int_0^t \|(y_h - y_h(u))(\tau)\|_{1,\Omega}^2 d\tau\Big\}
\end{aligned}
$$

such that

$$
\begin{aligned}
&\|y_{h,t} - y_{h,t}(u)\|_{L^2(0,T;H^1(\Omega))}^2\\
\le& C\Big\{\|\pi_{h_U} u - u_h\|_{L^2(0,T;H^1(\Omega_U))}^2 + h_U^2\|\pi_{h_U} u - u\|_{L^2(0,T;H^1(\Omega_U))}^2\Big\}.
\end{aligned} \tag{47}
$$

Similarly taking $q_h = p_h - p_h(u)$ and $q_h = (p_h - p_h(u))_t$ in (42) respectively, we have

$$
\begin{aligned}
&\|p_h - p_h(u)\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|(p_h - p_h(u))_t\|_{L^2(0,T;H^1(\Omega))}^2\\
\le& C\|y - y_h\|_{L^2(0,T;L^2(\Omega))}^2.
\end{aligned} \tag{48}
$$

The proof of Lemma 4.3 is complete. □

Since $y_h(u)$ and $p_h(u)$ are the standard finite element solutions of y and p, from [2], we cite the following result.

Lemma 4.4. ([2]) *Let $(y_h(u), p_h(u))$ be the solutions of the systems* (38)-(39)*. Then there holds the a priori error estimate:*

$$
\|y - y_h(u)\|_{L^2(0,T;H^1(\Omega))} + \|p - p_h(u)\|_{L^2(0,T;H^1(\Omega))} \le Ch \tag{49}
$$

and

$$
\|y - y_h(u)\|_{L^2(0,T;L^2(\Omega))} + \|p - p_h(u)\|_{L^2(0,T;L^2(\Omega))} \le Ch^2 \tag{50}
$$

From Lemma 4.1- 4.4, we derive the following result.

Lemma 4.5. *Let $(y_h(u), p_h(u))$ and (y_h, p_h, u_h) be the solutions of the systems (38) - (39) and (27) - (29). Then there holds the estimate:*

$$\|y_h - y_h(u)\|_{H^1(0,T;H^1(\Omega))} + \|p_h - p_h(u)\|_{H^1(0,T;H^1(\Omega))} \leq C(h_U + h^2). \tag{51}$$

Proof. Taking $w_h = p_h - p_h(u)$ in (41) and $q_h = y_h - y_h(u)$ in (42) and then integrating from 0 to T, we obtain

$$\begin{aligned}
&\int_0^T \Big((B(u_h - u), p_h - p_h(u)) - (y_h - y, y_h - y_h(u))\Big) \\
=&(y_h - y_h(u)), p_h - p_h(u))\big|_{t=0}^{t=T} \\
&+ \int_0^T \int_0^t c(t, \tau; (y_h - y_h(u))(\tau), (p_h - p_h(u))(t)) d\tau dt \\
&- \int_0^T \int_t^T c(\tau, t; (y_h - y_h(u))(t), (p_h - p_h(u))(\tau)) d\tau dt = 0
\end{aligned}$$

such that

$$\begin{aligned}
&\int_0^T \big((y_h - y, y_h - y_h(u)) + \alpha(u_h - \pi_{h_U} u, u_h - \pi_{h_U} u)_U\big) \\
=&\int_0^T \big((u_h - \pi_{h_U} u, \alpha u_h + B^* p_h)_U + (u - u_h, \alpha u + B^* p)_U \\
&+ (u_h - \pi_{h_U} u, B^*(p - p_h(u)))_U + (\pi_{h_U} u - u, \alpha u + B^* p)_U \\
&+ (\pi_{h_U} u - u, B^*(p_h - p_h(u)))_U + (y - y_h(u), y_h - y_h(u))\big) \\
\leq&\int_0^T \big((u_h - \pi_{h_U} u, B^*(p - p_h(u)))_U + (\pi_{h_U} u - u, \alpha u + B^* p)_U \\
&+ (\pi_{h_U} u - u, B^*(p_h - p_h(u)))_U + (y - y_h(u), y_h - y_h(u))\big)
\end{aligned} \tag{52}$$

Thus we have

$$\begin{aligned}
&\|y_h - y_h(u)\|^2_{L^2(0,T;L^2(\Omega))} + \alpha\|u_h - \pi_{h_U} u\|^2_{L^2(0,T;L^2(\Omega_U))} \\
\leq&\varepsilon\|p_h - p_h(u)\|^2_{L^2(0,T;H^1(\Omega))} + C(\varepsilon)\Big\{h_U^2\|\pi_{h_U} u - u\|^2_{L^2(0,T;L^2(\Omega_U))} \\
&+ \|y - y_h(u)\|^2_{L^2(0,T;L^2(\Omega))} + \|p - p_h(u)\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ (\pi_{h_U} u - u, (\mathcal{I} - \pi_{h_U})(\alpha u + B^* p))_U\Big\}
\end{aligned} \tag{53}$$

such that

$$\|u_h - \pi_{h_U} u\|_{L^2(0,T;L^2(\Omega_U))} \leq C(h_U + h^2). \tag{54}$$

By (54) and (40), (51) is derived. □

Thus from Lemmas 4.1- Lemma 4.5 and triangle inequality, we have proved Theorem 4.2.

4.2. Convergent rate in $L^2(L^2)$-norm. Further, we concern with the a priori error estimate in L^2-norm with the respect to the state. In many case in engineering applications, the boundary of the contacting set of the optimal control is some curves with finite lengths in 2-D case or surfaces with finite area in 3-D case. In these cases, one can obtain higher order accuracy.

Theorem 4.6. *Assume that U_{ad} is given by (32). Let (y,p,u) and (y_h,p_h,u_h) be the solutions of the systems (11) - (13) and (27) - (29). Then there holds the a priori error estimate:*

$$\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+\|p-p_h\|_{L^2(0,T;L^2(\Omega))}\le C(h_U+h^2). \tag{55}$$

Further, let

$$\Omega_h^+(t)=\{\tau\in T_U^h;\ u>0 \text{ in } \tau\},\quad \Omega_h^0(t)=\{\tau\in T_U^h;\ u=0 \text{ in } \tau\},$$
$$\Omega_h^b(t)=\Omega_U\backslash(\Omega_h^+(t)\cup\Omega_h^0(t)).$$

And assume that

$$meas(\Omega_h^b(t))\le Ch,\quad \forall\ t\in[0,T]. \tag{56}$$

Then there holds the a priori error estimate:

$$\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+\|p-p_h\|_{L^2(0,T;L^2(\Omega))}\le C(h_U^{\frac{3}{2}}+h^2). \tag{57}$$

Proof. In (53), noting

$$\begin{aligned}&\left|(\pi_{h_U}u-u,(\mathcal{I}-\pi_{h_U})(\alpha u+B^*p))_U\right|\\ \le&C\big(\|B^*p-\pi_{h_U}(B^*p)\|^2_{L^2(\Omega_h^b)}+\|u-\pi_{h_U}u\|^2_{L^2(\Omega_h^b)}\big)\\ \le&Ch_U^2meas(\Omega_h^b)\|p\|^2_{W^{1,\infty}(\Omega_U)},\end{aligned}$$

we have

$$\|u_h-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}\le C\big(h_U\sqrt{meas(\Omega_h^b)}+h^2\big) \tag{58}$$

such that

$$\begin{aligned}&\|y_h-y_h(u)\|_{L^2(0,T;H^1(\Omega))}+\|p_h-p_h(u)\|_{L^2(0,T;H^1(\Omega))}\\ \le&C(h_U\sqrt{meas(\Omega_h^b)}+h^2).\end{aligned} \tag{59}$$

By using (59) and the triangle inequality

$$\begin{aligned}&\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+\|p-p_h\|_{L^2(0,T;L^2(\Omega))}\\ \le&\|y-y_h(u)\|_{L^2(0,T;L^2(\Omega))}+\|p-p_h(u)\|_{L^2(0,T;L^2(\Omega))}\\ &+\|y_h-y_h(u)\|_{L^2(0,T;L^2(\Omega))}+\|p_h-p_h(u)\|_{L^2(0,T;L^2(\Omega))},\end{aligned}$$

we derive (55) and (57) □

Theorem 4.7. *Assume that U_{ad} is given by (33). Let (y,p,u) and (y_h,p_h,u_h) be the solutions of the systems (11) - (13) and (27) - (29). Then there holds the a priori error estimate:*

$$\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+\|p-p_h\|_{L^2(0,T;L^2(\Omega))}\le C(h_U^2+h^2). \tag{60}$$

Proof. In this case, we see that

$$(B^*p+\alpha u,\pi_{h_U}u-u)_U=0,$$

since $\alpha u+B^*p$ is a constant. So we have

$$\|u_h-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}\le C(h_U^2+h^2).$$

Furthermore, we get

$$\|y_h-y_h(u)\|_{L^2(0,T;H^1(\Omega))}+\|p_h-p_h(u)\|_{L^2(0,T;H^1(\Omega))}\le C(h_U^2+h^2). \tag{61}$$

Noting (61) and the triangle inequality, we derive (60). □

5. Numerical Experiment

In this section, we carry out some numerical experiments to check if the numerical algorithm is effective and if the a priori error estimates derived in Section 4 is reliable and accurate. The numerical tests were done by using AFEpack software package (see [9]). In the numerical examples, $\Omega = \Omega_U = [0,1]^2$. We use linear finite element spaces to treat the state and co-state and the piecewise constant finite element spaces to treat the control. For time variable, a Euler backward-difference procedure is used to solve semi-discrete system. Here time step size is controlled to demonstrate the relation between the error function and spacial sizes.

We solve the following control problem

$$\min_{u\geq 0} \frac{1}{2}\int_0^1 \Big(\int_\Omega (y-z_d)^2 + \int_\Omega u^2\Big) \tag{62}$$

subject to

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y_t - \Delta y - \displaystyle\int_0^t (t-\tau)\Delta y = u + f \quad \text{in } \Omega, \quad 0 < t \leq 1; \\ y|_{\partial\Omega} = 0. \end{cases} \tag{63}$$

The data and solutions are:

$$\begin{cases} p = -(T-t)\sin \pi x_1 \sin \pi x_2 \\ u = \max(-p, 0) \\ y = t x_1(1-x_1)x_2(1-x_2) \\ z_d = y + \dfrac{\partial p}{\partial t} - \triangle p_t + \triangle p + \displaystyle\int_t^T (t-\tau)\triangle p \\ f = \dfrac{\partial y}{\partial t} - \triangle y_t - \triangle y - \displaystyle\int_0^t (t-\tau)\triangle y - u \end{cases} \tag{64}$$

The numerical results are put into the following Table. In the Table 1, the error values in $L^2(0,T;L^2(\Omega))$-norm and $L^2(0,T;H^1(\Omega))$-norm are listed.

TABLE 1. Numerical result: for adaptive time steps 50

Freedom number			$L^2(H^1(\Omega))$		$L^\infty(L^2(\Omega))$		
nodes	sides	elements	$y-y_h$	$p-p_h$	$u-u_h$	$y-y_h$	$p-p_h$
7089	19074	12036	6.2e-02	8.5e-01	4.5e-02	6.4e-04	5.9e-03
26163	74256	48144	3.1e-02	4.2e-01	2.2e-02	1.6e-04	1.5e-03
100419	292944	192576	1.5e-02	2.1e-01	1.1e-02	4.1e-05	3.9e-04
393363	1163616	770304	7.8e-03	1.0e-01	5.6e-03	1.0e-05	1.0e-04

From the Table, we see that the L^2-norm convergent rate of the control variable $u - u_h$ is $O(h)$, i.e., the first order accuracy with the respect to the spacial size; that the H^1-norm convergent rate of the state and costate variables $y - y_h$ and $p-p_h$ also are $O(h)$; but that the L^2-norm convergent rate of the state and costate variables $y - y_h$ and $p - p_h$ also are $O(h^2)$, i.e., the second order accuracy, which are consist with the a priori error estimates derived in Section 4.

Conclusion. In this article, we investigate a quadratic optimal control problem governed by a linear quasi-parabolic integro-differential equation. The weak formulation is given, the existence and regularity of the solution for the optimal control problem are analyzed. Further, the a priori error estimates are derived. By super-convergence analysis, we also prove the L^2-norm convergent rate. The numerical experiments verify the theoretical results. Some further work are being done, for example, adaptive finite element methods will be very useful for this kind of problems. We will develop equivalent residual-type a posterior error estimators.

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