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APPROXIMATION OF THE LONG-TERM DYNAMICS OF THE DYNAMICAL SYSTEM GENERATED BY THE TWO-DIMENSIONAL THERMOHYDRAULICS EQUATIONS

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Abstract. Pursuing our work in [18], [17], [20], [5], we consider in this article the two-dimensional thermohydraulics equations. We discretize these equations in time using the implicit Euler scheme and we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

Key words. Thermohydraulics equations, discrete Gronwall lemmas, implicit Euler scheme, global attractors

1. Introduction

In this article we discretize the two-dimensional thermohydraulics equations in time using the implicit Euler scheme, and we show that global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero. In order to do this, we first prove that the scheme is H^1-uniformly stable in time (see Section 4) and then we show that the long-term dynamics of the continuous system can be approximated by the discrete attractors of the dynamical systems generated by the numerical scheme (see Section 5).

In the case of the Navier–Stokes equations with Dirichlet boundary conditions, the H^1-uniform stability of the fully implicit Euler scheme has proven to be rather challenging. However, using techniques based on the classical and uniform discrete Gronwall lemmas, we have been able to show the H^1-stability for all time of the implicit Euler scheme for the Navier–Stokes equations with Dirichlet boundary conditions (see [20]). The H^2-stability has also been established. More precisely, the H^2-stability has first been proven in the simpler case of space periodic boundary conditions (see [17]), and then extended to Dirichlet boundary conditions (see [18]); the magnetohydrodynamics equations are also considered in [18].

Our first objective in this article is to extend the H^1-uniform stability proven in [20] for the Navier–Stokes equations with Dirichlet boundary conditions, to the thermohydraulics equations. In order to do so, we divide our proof into three steps. First, we prove the L^2-uniform stability of both the discrete temperature θ^n and the discrete velocity v^n (see Lemma 3.2 and Lemma 3.3 below). Then, using techniques based on the classical and uniform discrete Gronwall lemmas, we derive the H^1-uniform stability of v^n (see Proposition 4.1 below), which we will use in Subsection 4.2 in order to establish the H^1-uniform stability of θ^n (see Proposition 4.2 below). Besides the intrinsec interest of considering the thermohydraulics equations, the new technical difficulties which appear here are related to the specific treatment of the temperature with the necessary utilization of the maximum

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principle. Furthermore, we have simplified some steps of the proof as compared to [20].

Our second objective in this article is to employ the technique developed in [5] to prove that the global attractors generated by the fully implicit Euler scheme converge to the global attractor of the continuous system as the time-step approaches zero. When discretizing the two-dimensional thermohydraulics equations in time using the implicit Euler scheme, one can prove the uniqueness of the solution provided that the time step is sufficiently small. More precisely, the time restriction depends on the initial value, and thus one cannot define a single-valued attractor in the classical sense. This is why we need to use the theory of the so-called multi-valued attractors, which we briefly recall in Subsection 5.1.

2. The thermohydraulics equations

Let $\Omega = (0,1) \times (0,1)$ be the domain occupied by the fluid and let e_2 be the unit upward vertical vector. The thermohydraulics equations consist of the coupled system of the equations of fluid and temperature in the Boussinesq approximation and they read (see, e.g., [6], [15]):

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu\Delta v + \nabla p = e_2(T - T_1), \tag{2.1}$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)T - \kappa\Delta T = 0, \tag{2.2}$$

$$\operatorname{div} v = 0; \tag{2.3}$$

here $v = (v_1, v_2)$ is the velocity, p is the pressure, T is the temperature, T_1 is the temperature at the top boundary, $x_2 = 1$, and ν, κ are positive constants. We supplement these equations with the initial conditions

$$v(x,0) = v_0(x), \tag{2.4}$$

$$T(x,0) = T^0(x), \tag{2.5}$$

where $v_0 : \Omega \to \mathbb{R}^2$, $T^0 : \Omega \to \mathbb{R}$ are given, and with the boundary conditions

$$v = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.6}$$

$$T = T_0 = T_1 + 1 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad T = T_1 \quad \text{at} \quad x_2 = 1, \tag{2.7}$$

and

$$\begin{gathered} p, v, T \text{ and the first derivatives of } v \text{ and } T \text{ are periodic} \\ \text{of period 1 in the direction } x_1, \end{gathered} \tag{2.8}$$

meaning that $\phi|_{x_1=0} = \phi|_{x_1=1}$ for the corresponding functions ϕ.

Letting

$$\theta = T - T_0 + x_2, \tag{2.9}$$

and changing p to

$$p - \left(x_2 - \frac{x_2^2}{2}\right), \tag{2.10}$$

equations (2.1)–(2.3) together with the boundary conditions (2.6)–(2.8) become

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu\Delta v + \nabla p = e_2\theta, \tag{2.11}$$

$$\frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta - v_2 - \kappa\Delta\theta = 0, \tag{2.12}$$

$$\operatorname{div} v = 0, \tag{2.13}$$

$$v = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.14}$$

$$\theta = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = 1, \tag{2.15}$$

$$\text{(2.8) holds with } T \text{ replaced by } \theta. \tag{2.16}$$

These equations are supplemented with the initial conditions

$$v(x, 0) = v_0(x), \tag{2.17}$$

$$\theta(x, 0) = T^0(x) - T_0 + x_2 =: \theta_0(x). \tag{2.18}$$

For the mathematical setting of the problem we define the space $H = H_1 \times H_2$, where

$$H_1 = \left\{v \in L^2(\Omega)^2, \operatorname{div} = 0, v_2|_{x_2=0} = v_2|_{x_2=1} = 0, v_1|_{x_1=0} = v_1|_{x_1=1}\right\}, \tag{2.19}$$

$$H_2 = L^2(\Omega), \tag{2.20}$$

and we denote the scalar products and norms in H_1, H_2 and H by $(\cdot, \cdot)$ and $|\cdot|$.

We also define the space $V = V_1 \times V_2$, where

$$V_1 = \left\{v \in H^1(\Omega)^2, v|_{x_2=0} = v|_{x_2=1} = 0, v|_{x_1=0} = v|_{x_1=1}, \ \operatorname{div} v = 0\right\}, \tag{2.21}$$

$$V_2 = \left\{\theta \in H^1(\Omega), \theta|_{x_2=0} = \theta|_{x_2=1} = 0, \theta|_{x_1=0} = \theta|_{x_1=1}\right\}. \tag{2.22}$$

The space V_2 is a Hilbert space with the scalar product and the norm

$$((\phi, \psi)) = \int \nabla\phi \cdot \nabla\psi \, dx, \qquad \|\phi\| = \sqrt{((\phi, \phi))}, \tag{2.23}$$

and we have the Poincaré inequality (see, e.g., [15], page 134)

$$|\phi| \leq \|\phi\|, \quad \forall\, \phi \in V_1 \text{ or } V_2. \tag{2.24}$$

We denote both scalar products and norms in V_1 and V by $((\cdot, \cdot))$ and $\|\cdot\|$.

Let $D(A) = D(A_1) \times D(A_2)$, where

$$D(A_i) = \left\{v \in V_i \cap H^2(\Omega)^2, \frac{\partial v}{\partial x_1}\Big|_{x_1=0} = \frac{\partial v}{\partial x_1}\Big|_{x_1=1}\right\}, \ i = 1, 2, \tag{2.25}$$

and let A be the linear operator from $D(A)$ into H and from V into V' defined by

$$(Au_1, u_2) = a(u_1, u_2), \ \forall\, u_i = \{v_i, \theta_i\} \in D(A), \ i = 1, 2, \tag{2.26}$$

with

$$a(u_1, u_2) = \nu((v_1, v_2)) + \kappa((\theta_1, \theta_2)). \tag{2.27}$$

We consider the trilinear continuous form b on V, defined by

$$b(u_1, u_2, u_3) = b_1(v_1, v_2, v_3) + b_2(v_1, \theta_2, \theta_3), \forall u_i = \{v_i, \theta_i\} \in V, \tag{2.28}$$

where

$$b_1(y, w, z) = \sum_{i,j=1,2} \int_\Omega y_i \frac{\partial w_j}{\partial x_i} z_j \, \mathrm{d}x, \forall\, y, w, z \in H^1(\Omega)^2, \tag{2.29}$$

$$b_2(y,\phi,\psi) = \sum_{i=1}^{2} \int_\Omega y_i \frac{\partial \phi}{\partial x_i} \psi \, \mathrm{d}x, \forall\, y \in H^1(\Omega)^2, \phi, \psi \in H^1(\Omega). \tag{2.30}$$

The form b_1 is trilinear continuous on $V_1 \times V_1 \times V_1$ and enjoys the following properties:

$$|b_1(y,w,z)| \le c_b |y|^{1/2} \|y\|^{1/2} \|w\| |z|^{1/2} \|z\|^{1/2}, \quad \forall\, y,w,z \in V_1, \tag{2.31}$$

$$\begin{gathered} |b_1(y,w,z)| \le c_b |y|^{1/2} |A_1 y|^{1/2} \|w\| |z|, \\ \forall\, y \in D(A_1),\ w \in V_1,\ z \in H_1, \end{gathered} \tag{2.32}$$

$$\begin{gathered} |b_1(y,w,z)| \le c_b |y|^{1/2} \|y\|^{1/2} \|w\|^{1/2} |A_1 w|^{1/2} |z|, \\ \forall\, y \in V_1, w \in D(A_1), z \in H_1, \end{gathered} \tag{2.33}$$

$$b_1(y,w,w) = 0, \quad \forall\, y, w \in V_1, \tag{2.34}$$

the last equation implying

$$b_1(y,w,z) = -b_1(y,z,w), \quad \forall\, y,w,z \in V_1. \tag{2.35}$$

The form b_2 is trilinear continuous on $V_1 \times V_2 \times V_2$ and enjoys the following properties, similar to (2.31)–(2.35):

$$|b_2(y,\phi,\psi)| \le c_b |y|^{1/2} \|y\|^{1/2} \|\phi\| |\psi|^{1/2} \|\psi\|^{1/2}, \quad \forall\, y,\phi,\psi \in V_2, \tag{2.36}$$

$$\begin{gathered} |b_2(y,\phi,\psi)| \le c_b |y|^{1/2} |A_2 y|^{1/2} \|\phi\| |\psi|, \\ \forall\, y \in D(A_2),\ \phi \in V_2,\ \psi \in H_2, \end{gathered} \tag{2.37}$$

$$\begin{gathered} |b_2(y,\phi,\psi)| \le c_b |y|^{1/2} \|y\|^{1/2} \|\phi\|^{1/2} |A_2 \phi|^{1/2} |\psi|, \\ \forall\, y \in V_1,\ \phi \in D(A_2),\ \psi \in H_2, \end{gathered} \tag{2.38}$$

$$b_2(y,\phi,\phi) = 0, \quad \forall\, y \in V_1,\ \phi \in V_2, \tag{2.39}$$

the last equation implying

$$b_2(y,\phi,\psi) = -b_2(y,\psi,\phi), \quad \forall\, y \in V_1,\ \phi,\psi \in V_2. \tag{2.40}$$

We associate with b the bilinear continuous operator B from $V \times V$ into V' and from $D(A) \times D(A)$ into H, such that

$$\langle B(u_1,u_2), u_3 \rangle_{V',V} = b(u_1,u_2,u_3), \quad \forall\, u_1,u_2,u_3 \in V. \tag{2.41}$$

We also define the continuous operator in H

$$Ru = -\{e_2\theta, v_2\},\ u = \{v, \theta\}. \tag{2.42}$$

For more details about the function spaces $D(A)$, V and H, as well as the operators A, B, R and b, the reader is referred to, e.g., [15].

In the above notation, the system (2.11)–(2.13) can be written as the functional evolution equation

$$u_t + Au + B(u) + Ru = 0, \qquad u(0) = u_0 = \{v_0, \theta_0\}. \tag{2.43}$$

In the two-dimensional case under consideration, the solution to the thermohydraulics equations is known to be smooth for all time (cf. [15]). Using the maximum principle for parabolic equations, one can show that $\theta \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ and the velocity u is bounded uniformly for all time by

$$|v(t)|^2_{L^2(\Omega)^2} \le \mathrm{e}^{-\nu t} |v_0|^2_{L^2(\Omega)^2} + \frac{\theta_\infty^2}{\nu^2}\big(1 - \mathrm{e}^{-\nu t}\big), \tag{2.44}$$

where $\theta_\infty = |\theta|_{L^\infty(\mathbb{R}_+;L^2(\Omega))}$. Furthermore, using techniques based on the uniform Gronwall lemma (cf. [15]), one can bound the solution u of (2.43) uniformly in V for all $t \geq 0$.

In this article we discretize (2.43) in time using the fully implicit Euler scheme, and define recursively the elements $u^n = \{v^n, \theta^n\}$ of V as follows:

$$\begin{aligned} u^0 = \{v^0, \theta^0\}, \text{ where } v^0(x) = v_0(x), \text{ and} \\ \theta^0(x) = \theta_0(x) := T^0(x) - T_0 + x_2 \text{ are given;} \end{aligned} \tag{2.45}$$

then when $u^0 = \{v^0, \theta^0\}, \cdots, u^{n-1} = \{v^{n-1}, \theta^{n-1}\}$ are known, we define $u^n = \{v^n, \theta^n\} \in V$ such that

$$\frac{1}{k}(v^n - v^{n-1}, v) + \nu((v^n, v)) + b_1(v^n, v^n, v) = (e_2\theta^n, v), \; \forall v \in V_1, \tag{2.46}$$

$$\frac{1}{k}(\theta^n - \theta^{n-1}, \theta) + \kappa((\theta^n, \theta)) + b_2(v^n, \theta^n, \theta) - (v_2^n, \theta) = 0, \; \forall \theta \in V_2. \tag{2.47}$$

The above system is very similar to the stationary Navier–Stokes equations and the existence of solutions is proven e.g. by the Galerkin method, as in [16]. Uniqueness can also be derived as in [16] under some conditions. Let us explain this point, which somehow motivates the developments in Section 5. For that, we rewrite the system (2.45)–(2.47) in the form

$$(v^n, v) + \nu k((v^n, v)) + kb_1(v^n, v^n, v) - k(e_2\theta^n, v) = (v^{n-1}, v), \; \forall v \in V_1, \tag{2.48}$$

$$(\theta^n, \theta) + \kappa k((\theta^n, \theta)) + kb_2(v^n, \theta^n, \theta) - k(v_2^n, \theta) = (\theta^{n-1}, \theta), \; \forall \theta \in V_2, \tag{2.49}$$

and assume that $\{v^n, \theta^n\}$ and $\{\bar{v}^n, \bar{\theta}^n\}$ are two solutions corresponding to the same initial data $\{v_0, \theta_0\} \in V$. Setting $\tilde{v}^n = v^n - \bar{v}^n$ and $\tilde{\theta}^n = \theta^n - \bar{\theta}^n$, we obtain that $\{\tilde{v}^n, \tilde{\theta}^n\}$ is a solution to the following system:

$$(\tilde{v}^n, v) + \nu k((\tilde{v}^n, v)) + kb_1(\tilde{v}^n, v^n, v) + kb_1(\bar{v}^n, \tilde{v}^n, v) - k(e_2\tilde{\theta}^n, v) = 0, \; \forall v \in V_1, \tag{2.50}$$

$$(\tilde{\theta}^n, \theta) + \kappa k((\tilde{\theta}^n, \theta)) + kb_2(\tilde{v}^n, \theta^n, \theta) + kb_2(\bar{v}^n, \tilde{\theta}^n, \theta) - k(\tilde{v}_2^n, \theta) = 0. \forall \theta \in V_2, \tag{2.51}$$

Taking $v = \tilde{v}^n$ in (2.50) and using (2.34), we obtain

$$|\tilde{v}^n|^2 + \nu k\|\tilde{v}^n\|^2 + kb_1(\tilde{v}^n, v^n, \tilde{v}^n) - k(e_2\tilde{\theta}^n, \tilde{v}^n) = 0. \tag{2.52}$$

Using property (2.31) of the trilinear form b_1 and the bound (4.52) below on $\|v^n\|$, we obtain (for $k \leq \kappa_4(\|\{v_0, \theta_0\}\|)$, with $\kappa_4(\|\{v_0, \theta_0\}\|)$ given in Theorem 4.1 below):

$$\begin{aligned} kb_1(\tilde{v}^n, v^n, \tilde{v}^n) &\leq c_b k|\tilde{v}^n|\|\tilde{v}^n\|\|v^n\| \leq c_b K_6 k|\tilde{v}^n|\|\tilde{v}^n\| \\ &\leq \frac{\nu}{4}k\|\tilde{v}^n\|^2 + \frac{c_b}{\nu}K_6^2 k|\tilde{v}^n|^2. \end{aligned} \tag{2.53}$$

We also have

$$\begin{aligned} k(e_2\tilde{\theta}^n, \tilde{v}^n) &\leq k|e_2\tilde{\theta}^n||\tilde{v}^n| \leq k|\tilde{\theta}^n|\|\tilde{v}^n\| \\ &\leq \frac{\nu}{4}k\|\tilde{v}^n\|^2 + \frac{1}{\nu}k|\tilde{\theta}^n|^2. \end{aligned} \tag{2.54}$$

Relations (2.52)–(2.54) imply

$$\left(1 - \frac{c_b}{\nu}K_6^2 k\right)|\tilde{v}^n|^2 + \frac{\nu}{2}k\|\tilde{v}^n\|^2 \leq \frac{1}{\nu}k|\tilde{\theta}^n|^2. \tag{2.55}$$

Now taking $\theta = \tilde{\theta}^n$ in (2.51) and using (2.39), we obtain

$$|\tilde{\theta}^n|^2 + \kappa k\|\tilde{\theta}^n\|^2 + kb_2(\tilde{v}^n, \theta^n, \tilde{\theta}^n) - k(\tilde{v}_2^n, \tilde{\theta}^n) = 0. \tag{2.56}$$

Using property (2.36) of the trilinear form b_2 and the bound (4.52) below on $\|\theta^n\|$, we obtain

$$
\begin{aligned}
kb_2(\tilde{v}^n, \theta^n, \tilde{\theta}^n) &\le c_b k|\tilde{v}^n|^{1/2}\|\tilde{v}^n\|^{1/2}\|\theta^n\||\tilde{\theta}^n|^{1/2}\|\tilde{\theta}^n\|^{1/2} \\
&\le \frac{\nu}{4}k\|\tilde{v}^n\|^2 + \frac{\kappa}{4}k\|\tilde{\theta}^n\|^2 + cK_6^2 k|\tilde{v}^n|^2 + cK_6^2 k|\tilde{\theta}^n|^2.
\end{aligned}
\tag{2.57}
$$

We also have

$$
k(\tilde{v}_2^n, \tilde{\theta}^n) \le k|\tilde{v}_2^n||\tilde{\theta}^n| \le k|\tilde{v}^n|\|\tilde{\theta}^n\| \le \frac{\kappa}{4}k\|\tilde{\theta}^n\|^2 + \frac{1}{\kappa}k|\tilde{v}^n|^2.
\tag{2.58}
$$

Relations (2.56)–(2.58) yield

$$
(1 - cK_6^2 k)|\tilde{\theta}^n|^2 + \frac{\kappa}{2}k\|\tilde{\theta}^n\|^2 \le \frac{\nu}{4}k\|\tilde{v}^n\|^2 + cK_6^2 k|\tilde{v}^n|^2 + \frac{1}{\kappa}k|\tilde{v}^n|^2.
\tag{2.59}
$$

Adding relations (2.55) and (2.59), we obtain

$$
\begin{aligned}
&\left(1 - \frac{c_b}{\nu}K_6^2 k - cK_6^2 k - \frac{1}{\kappa}k\right)|\tilde{v}^n|^2 + \left(1 - cK_6^2 k - \frac{c}{\nu}k\right)|\tilde{\theta}^n|^2 \\
&+ \frac{\nu}{4}k\|\tilde{v}^n\|^2 + \frac{\kappa}{2}k\|\tilde{\theta}^n\|^2 \le 0.
\end{aligned}
\tag{2.60}
$$

Assuming k is sufficiently small, that is

$$
k \le \min\left\{\kappa_4(\|\{v_0, \theta_0\}\|), \frac{1}{2\left(\frac{c_b}{\nu}K_6^2 + cK_6^2 + \frac{1}{\kappa}\right)}, \frac{1}{2\left(cK_6^2 + \frac{c}{\nu}\right)}\right\},
\tag{2.61}
$$

relation (2.60) implies $\tilde{v}^n = \tilde{\theta}^n = 0$. Hence, the system (2.45)–(2.47) possesses a unique solution, provided that the time-step satisfies the constraint (2.61). This is enough to uniquely define the sequence $\{v^n, \theta^n\}$ for k small enough, but the dependence of the time step k on the initial data prevents us from defining a single-valued attractor in the classical sense, and this is why we need the theory of the multi-valued attractors, that we discuss in Subsection 5.1.

Our next aims are to prove that the solution $u^n = \{v^n, \theta^n\}$ to the discrete system (2.45)–(2.47) is uniformly bounded in the V-norm and then to show that the global attractors generated by the numerical scheme (2.45)–(2.47) converge to the global attractor of the continuous system as the time-step approaches zero.

In this article we only consider time discretization, we do not consider space discretization. Important background information on space discretization and on various computational methods can be found in some of the books and articles available in the literature. On finite elements, see, e.g., [7], [9]; on finite differences and finite elements, [10], [16]; on spectral methods, [3], [8].

3. H-Uniform Boundedness of v^n and θ^n

In proving the H-uniform boundedness of v^n and θ^n, we need first to prove a variant of the maximum principle for θ^n. In order to do so, we introduce the following truncation operators (cf. [15]), that associate with the function φ, the functions φ_+ and φ_-, given by

$$
\varphi_+(x) = \max(\varphi(x), 0), \quad \varphi_-(x) = \max(-\varphi(x), 0).
\tag{3.1}
$$

Note that, with this notation, we have $\varphi = \varphi_+ - \varphi_-$, the absolute value $|\varphi|$ of φ is $\varphi_+ + \varphi_-$ and $\varphi_+\varphi_- = 0$. Using these operators, we can prove the following preliminary lemma

Lemma 3.1. *If $\varphi, \psi \in L^2(\Omega)$, then*

$$2(\varphi - \psi, \varphi_+) \geq |\varphi_+|^2 - |\psi_+|^2 + |\varphi_+ - \psi_+|^2, \tag{3.2}$$

$$-2(\varphi - \psi, \varphi_-) \geq |\varphi_-|^2 - |\psi_-|^2 + |\varphi_- - \psi_-|^2. \tag{3.3}$$

Proof. We have

$$\begin{aligned} 2(\varphi - \psi, \varphi_+) &= 2(\varphi_+ - \varphi_- - \psi_+ + \psi_-, \varphi_+) \\ &= 2(\varphi_+ - \psi_+, \varphi_+) - 2(\varphi_- - \psi_-, \varphi_+) \\ &= |\varphi_+|^2 - |\psi_+|^2 + |\varphi_+ - \psi_+|^2 + 2\int_\Omega \psi_- \varphi_+ \, dx \\ &\geq |\varphi_+|^2 - |\psi_+|^2 + |\varphi_+ - \psi_+|^2, \end{aligned} \tag{3.4}$$

since $\psi_- \varphi_+ \geq 0$. The proof is similar for (3.3) and the lemma is proved. □

We are now able to prove the following variant of the maximum principle for θ^n:

Lemma 3.2. *If v^n and θ^n satisfy* (2.46) *and* (2.47)*, then*

$$\theta^n = \tilde{\theta}^n + \bar{\theta}^n, \tag{3.5}$$

with

$$x_2 - 1 \leq \tilde{\theta}^n \leq x_2, \tag{3.6}$$

$$|\bar{\theta}^n| \leq \left(|\theta_+^0| + |\theta_-^0|\right)(1 + 2\kappa k)^{-\frac{n}{2}}. \tag{3.7}$$

Moreover, there exists $M_1 = M_1(|\theta_0|)$, given in (3.26) *below, such that*

$$|\theta^n| \leq M_1, \forall n \geq 1. \tag{3.8}$$

Proof. Rewriting (2.47) in terms of $T^n = \theta^n + T_0 - x_2$, we find:

$$\frac{1}{k}(T^n - T^{n-1}, T) + \kappa((T^n, T)) + b_2(v^n, T^n, T) = 0, \, \forall T \in V_2, = 0, \, n \geq 1. \tag{3.9}$$

Replacing T by $2k(T^n - T_0)_+$ in the above equation and using (3.2), we obtain:

$$\begin{aligned} &|(T^n - T_0)_+|^2 - |(T^{n-1} - T_0)_+|^2 \\ &+ |(T^n - T_0)_+ - (T^{n-1} - T_0)_+|^2 + 2k\kappa\|(T^n - T_0)_+\|^2 \leq 0. \end{aligned} \tag{3.10}$$

Using the Poincaré inequality (2.24), we find

$$|(T^n - T_0)_+|^2 \leq \frac{1}{\alpha}|(T^{n-1} - T_0)_+|^2, \tag{3.11}$$

where

$$\alpha = 1 + 2\kappa k. \tag{3.12}$$

Using recursively (3.11), we find

$$|(T^n - T_0)_+|^2 \leq (1 + 2\kappa k)^{-n}|(T^0 - T_0)_+|^2. \tag{3.13}$$

Similarly, using (3.3), we obtain

$$|(T^n - T_1)_-|^2 \leq (1 + 2\kappa k)^{-n}|(T^0 - T_1)_-|^2. \tag{3.14}$$

Setting

$$T^n = \tilde{T}^n + \bar{T}^n, \text{ with } \bar{T}^n = (T^n - T_0)_+ - (T^n - T_1)_-, \tag{3.15}$$

we find that $\tilde{T}^n = T^n - (T^n - T_0)_+ + (T^n - T_1)_-$, so that $\tilde{T}^n = T_1$, for $T^n \leq T_1$, $\tilde{T}^n = T^n$, for $T_1 \leq T^n \leq T_0$, and $\tilde{T}^n = T_0$, for $T^n > T_0$; in all cases

$$T_1 \leq \tilde{T}^n \leq T_0. \tag{3.16}$$

Rewriting (3.13)–(3.15) in terms of θ, we obtain

$$|(\theta^n - x_2)_+|^2 \leq (1+2\kappa k)^{-n}|(\theta^0 - x_2)_+|^2, \tag{3.17}$$

$$|(\theta^n - x_2 + 1)_-|^2 \leq (1+2\kappa k)^{-n}|(\theta^0 - x_2 + 1)_-|^2, \tag{3.18}$$

$$\theta^n + T_0 - x_2 = \tilde{T}^n + (\theta^n - x_2)_+ - (\theta^n - x_2 + 1)_-. \tag{3.19}$$

Setting

$$\bar{\theta}^n = (\theta^n - x_2)_+ - (\theta^n - x_2 + 1)_-, \tag{3.20}$$

$$\tilde{\theta}^n = \tilde{T}^n - T_0 + x_2, \tag{3.21}$$

equation (3.19) becomes

$$\theta^n = \tilde{\theta}^n + \bar{\theta}^n. \tag{3.22}$$

By (3.16), we have

$$x_2 - 1 \leq \tilde{\theta}^n \leq x_2, \tag{3.23}$$

and by (3.20), (3.17) and (3.18) we derive

$$\begin{aligned} |\bar{\theta}^n| &\leq |(\theta^n - x_2)_+| + |(\theta^n - x_2 + 1)_-| \\ &\leq (1+2\kappa k)^{-\frac{n}{2}}(|\theta^0_+| + |\theta^0_-|). \end{aligned} \tag{3.24}$$

To complete the proof of the lemma, we note that (3.22), (3.23) and (3.24) yield

$$|\theta^n| \leq |\Omega|^{1/2} + \left(|\theta^0_+| + |\theta^0_-|\right)(1+2\kappa k)^{-\frac{n}{2}}, \forall n \geq 1, \tag{3.25}$$

and setting

$$M_1(|\theta_0|) = |\Omega|^{1/2} + |\theta^0_+| + |\theta^0_-|, \tag{3.26}$$

we obtain conclusion (3.8) of the lemma. □

Corollary 3.1. *If*

$$k \leq \frac{1}{2\kappa}, \tag{3.27}$$

then $B_{L^2}(0, 2|\Omega|^{1/2})$, the ball in L^2 centered at 0 and radius $2|\Omega|^{1/2}$, is an absorbing ball for θ^n in L^2.

Proof. Indeed, let $\mathcal{B}$ be any bounded set in L^2 and assume that it is included in a ball $B(0,R)$ of L^2. It is easy to deduce from (3.25) that for any $\theta_0 \in B(0,R)$,

$$|\theta^n| \leq |\Omega|^{1/2} + 2R(1+2\kappa k)^{-\frac{n}{2}}, \forall n \geq 1, \tag{3.28}$$

and using assumption (3.27) on k and the fact that $1+x \geq \exp(x/2)$ if $x \in (0,1)$, we obtain that there exists $N_0^1(R,k) := \frac{2\ln\left(\frac{2R}{|\Omega|^{1/2}}\right)}{\kappa k}$ such that $\theta^n \in B_{L^2}(0, 2|\Omega|^{1/2}), \forall n \geq N_0^1$. This completes the proof of the corollary. □

We are now able to prove the H-uniform boundedness of v^n. More precisely, we have the following:

Lemma 3.3. *Let $\{v^n, \theta^n\}$ be the solution of the numerical scheme (2.46)–(2.47). Then for every $k > 0$, we have*

$$|v^n|^2 \leq (1+\nu k)^{-n}|v_0|^2 + \frac{M_1^2}{\nu^2}\left[1 - (1+\nu k)^{-n}\right], \forall\, n \geq 0. \tag{3.29}$$

Moreover, there exists $K_1 = K_1(|v_0|, |\theta_0|)$, such that

$$|v^n| \leq K_1, \quad \forall\, n \geq 0, \tag{3.30}$$

and

$$\nu k \sum_{j=i}^{m} \|v^j\|^2 \le |v^{i-1}|^2 + \frac{1}{\nu} k \sum_{j=i}^{m} |\theta^j|^2, \quad \forall\, i = 1, \cdots, m, \tag{3.31}$$

$$\kappa k \sum_{j=i}^{m} \|\theta^j\|^2 \le |\theta^{i-1}|^2 + \frac{1}{\kappa} k \sum_{j=i}^{m} |v^j|^2, \quad \forall\, i = 1, \cdots, m. \tag{3.32}$$

Proof. Taking v to be $2kv^n$ in (2.46) and using the relation

$$2(\varphi - \psi, \varphi) = |\varphi|^2 - |\psi|^2 + |\varphi - \psi|^2, \tag{3.33}$$

as well as the skew property (2.34), we obtain

$$|v^n|^2 - |v^{n-1}|^2 + |v^n - v^{n-1}|^2 + 2\nu k\, \|v^n\|^2 = 2k(e_2\theta^n, v^n). \tag{3.34}$$

Using the Cauchy–Schwarz inequality and the Poincaré inequality (2.24), we majorize the right-hand side of (3.34) by

$$\begin{aligned} 2k(e_2\theta^n, v^n) &\le 2k|e_2\theta^n||v^n| \le 2k|\theta^n||v^n| \\ &\le 2k|\theta^n|\|v^n\| \le \nu k\|v^n\|^2 + \frac{1}{\nu} k\, |\theta^n|^2. \end{aligned} \tag{3.35}$$

Relations (3.34) and (3.35) imply

$$|v^n|^2 - |v^{n-1}|^2 + |v^n - v^{n-1}|^2 + \nu k\, \|v^n\|^2 \le \frac{1}{\nu} k\, |\theta^n|^2. \tag{3.36}$$

Using again the Poincaré inequality (2.24), we find

$$|v^n|^2 \le \frac{1}{\alpha} |v^{n-1}|^2 + \frac{1}{\alpha\nu} k\, |\theta^n|^2, \tag{3.37}$$

where

$$\alpha = 1 + \nu k. \tag{3.38}$$

Using recursively (3.37), we find

$$\begin{aligned} |v^n|^2 &\le \frac{1}{\alpha^n} |v^0|^2 + \frac{1}{\nu} k \sum_{i=1}^{n} \frac{1}{\alpha^i} |\theta^{n+1-i}|^2 \\ &\le (1 + \nu k)^{-n}\, |v^0|^2 + \frac{M_1^2}{\nu^2} \left[1 - (1 + \nu k)^{-n} \right], \end{aligned} \tag{3.39}$$

which proves (3.29).

Taking $K_1^2 = |v^0|^2 + \frac{M_1^2}{\nu^2}$ relation (3.30) follows right away.

Adding inequalities (3.36) with n from i to m we obtain (3.31).

Now, replacing θ by $2k\theta^n$ in (2.47) and using the skew property (2.39), we obtain

$$|\theta^n|^2 - |\theta^{n-1}|^2 + |\theta^n - \theta^{n-1}|^2 + 2\kappa k\, \|\theta^n\|^2 = 2k(v_2^n, \theta^n). \tag{3.40}$$

Using again the Cauchy–Schwarz inequality and the Poincaré inequality (2.24), we majorize the right-hand side of (3.40) by

$$\begin{aligned} 2k(v_2^n, \theta^n) &\le 2k|v_2^n||\theta^n| \le 2k|v^n|\|\theta^n\| \\ &\le \kappa k\|\theta^n\|^2 + \frac{1}{\kappa} k\, |v^n|^2. \end{aligned} \tag{3.41}$$

Relations (3.40) and (3.41) imply

$$|\theta^n|^2 - |\theta^{n-1}|^2 + |\theta^n - \theta^{n-1}|^2 + \kappa k\, \|\theta^n\|^2 \le \frac{1}{\kappa} k\, |v^n|^2. \tag{3.42}$$

Summing inequalities (3.42) with n from i to m we obtain (3.32). □

Corollary 3.2. *Let*

$$k \leq \min\left\{\frac{1}{2\kappa}, \frac{1}{\nu}\right\} =: \kappa_1, \tag{3.43}$$

and set $\rho_0 = 2|\Omega|^{1/2} + \frac{\sqrt{5}|\Omega|^{1/2}}{\nu}$*. Then* $B_H(0,\rho_0)$*, the ball in* H *centered at* 0 *and radius* ρ_0*, is an absorbing ball for* $\{v^n, \theta^n\}$ *in* H*.*

Proof. Let $\mathcal{B}$ be any bounded set in H and assume that it is included in a ball $B(0,R)$ of H. For any initial data $\{v^0, \theta^0\} \in \mathcal{B}$, Corollary 3.1 implies that

$$|\theta^n| < 2|\Omega|^{1/2}, \forall n \geq N_0^1(R,k), \tag{3.44}$$

and then (3.37) becomes

$$|v^n|^2 \leq \frac{1}{\alpha}|v^{n-1}|^2 + \frac{4}{\alpha\nu}|\Omega|k, \, \forall n \geq N_0^1(R,k), \tag{3.45}$$

where

$$\alpha = 1 + \nu k. \tag{3.46}$$

Iterating the above inequality, we find (for any $n \geq N_0^1(R,k)$)

$$\begin{aligned} |v^n|^2 &\leq \frac{1}{\alpha^{(n-N_0^1)}}|v^{N_0^1}|^2 + \frac{4}{\nu}|\Omega|k \sum_{i=1}^{n-N_0^1} \frac{1}{\alpha^i} \\ &= (1+\nu k)^{-(n-N_0^1)}|v^{N_0^1}|^2 + \frac{4}{\nu^2}|\Omega|\left[1 - (1+\nu k)^{-(n-N_0^1)}\right], \\ &\leq (1+\nu k)^{-(n-N_0^1)}\left[R^2 + \frac{4}{\nu^2}(|\Omega| + 2R^2)\right] + \frac{4}{\nu^2}|\Omega| \\ &\qquad \text{(by (3.29) and (3.26))}, \end{aligned} \tag{3.47}$$

and using assumption (3.43) on k and the fact that $1 + x \geq \exp(x/2)$ if $x \in (0,1)$, we obtain that there exists $N_0^2(R,k)$,

$$N_0^2(R,k) := \frac{2}{\nu k} \ln \frac{\nu^2\left[R^2 + \frac{4}{\nu^2}(|\Omega| + 2R^2)\right]}{|\Omega|}, \tag{3.48}$$

such that $|v^n| \leq \sqrt{5}|\Omega|^{1/2}/\nu$, $\forall n \geq N_0^1 + N_0^2 =: N_0(R,k)$.

We, therefore, have that $\{v^n, \theta^n\} \in B_H(0,\rho_0)$, for all $n \geq N_0(R,k)$, which completes the proof of the corollary. □

4. V-Uniform Boundedness of v^n and θ^n

We now seek to obtain uniform bounds for v^n and θ^n in V, similar to those we have already obtained in H (see (3.30) and (3.8) above). In order to do this, we will first use the discrete Gronwall lemma to derive an upper bound on $\|v^n\|$, $n \leq N$, for some $N > 0$, and then we will use the discrete uniform Gronwall lemma to obtain an upper bound on $\|v^n\|$, $n \geq N$. Once we have obtained the V-uniform bounds on v^n, we can use those, together with a new version of the discrete uniform Gronwall lemma, to derive the V-uniform boundedness of θ^n.

4.1. H^1-Uniform Boundedness of v^n.

Lemma 4.1. *For every* $k > 0$*, we have*

$$\|v^n\|^2 \leq K_2\|v^{n-1}\|^2 + \frac{4}{\nu^2}M_1^2, \, \forall n \geq 1, \tag{4.1}$$

where $K_2 = K_2(|v_0|, |\theta_0|) = 2(1 + 2c_b^2 K_1^2/\nu^2)$*.*

Proof. Replacing v by $2k(v^n - v^{n-1})$ in (2.46), we obtain

$$\begin{aligned} 2|v^n - v^{n-1}|^2 + \nu k\|v^n\|^2 - \nu k\|v^{n-1}\|^2 + \nu k\|v^n - v^{n-1}\|^2 \\ + 2k\, b_1(v^n, v^n, v^n - v^{n-1}) = 2k\,(e_2\theta^n, v^n - v^{n-1}). \end{aligned} \tag{4.2}$$

Using properties (2.34), (2.35) and (2.31) of the trilinear form b_1 and recalling (3.30), we bound the nonlinear term as

$$\begin{aligned} 2kb_1(v^n, v^n, v^n - v^{n-1}) &= 2kb_1(v^n, v^{n-1}, v^n) \quad \text{(by (2.34), (2.35))} \\ &\le 2c_b k|v^n|\|v^n\|\|v^{n-1}\| \quad \text{(by (2.31))} \\ &\le \frac{\nu}{2}k\|v^n\|^2 + \frac{2c_b^2}{\nu}K_1^2 k\|v^{n-1}\|^2. \end{aligned} \tag{4.3}$$

We bound the right-hand side of (4.26) using Cauchy–Schwarz' inequality, (2.24) and (3.8):

$$\begin{aligned} 2k(e_2\theta^n, v^n - v^{n-1}) &\le 2k|\theta^n||v^n - v^{n-1}| \\ &\le k|\theta^n|\|v^n - v^{n-1}\| \\ &\le \frac{\nu}{2}k\|v^n - v^{n-1}\|^2 + \frac{2}{\nu}kM_1^2. \end{aligned} \tag{4.4}$$

Gathering relations (4.26) through (4.4), we find

$$\begin{aligned} 2|v^n - v^{n-1}|^2 + \frac{\nu}{2}k\|v^n\|^2 - \Big(\nu + \frac{2c_b^2}{\nu}K_1^2\Big)k\|v^{n-1}\|^2 \\ + \frac{\nu}{2}k\,\|v^n - v^{n-1}\|^2 \le \frac{2}{\nu}kM_1^2, \end{aligned} \tag{4.5}$$

We thus obtain

$$\|v^n\|^2 \le K_2\|v^{n-1}\|^2 + \frac{4}{\nu^2}M_1^2, \tag{4.6}$$

which is exactly conclusion (4.1) of the lemma. □

Lemma 4.2. *For every $k > 0$, we have*

$$c_1K_1^2k\|v^n\|^4 - \|v^n\|^2 + \|v^{n-1}\|^2 + \frac{2}{\nu}kM_1^2 \ge 0,\ \forall n \ge 1, \tag{4.7}$$

where $c_1 = 27c_b^4/(2\nu^3)$.

Proof. Replacing v by $2kA_1v^n$ in (2.46), we obtain

$$\begin{aligned} \|v^n\|^2 - \|v^{n-1}\|^2 + \|v^n - v^{n-1}\|^2 + 2kb_1(v^n, v^n, A_1v^n) \\ + 2\nu k|A_1v^n|^2 = 2k(e_2\theta^n, A_1v^n). \end{aligned} \tag{4.8}$$

Using property (2.32) of the trilinear form b_1 and recalling (3.30), we have the following bound of the nonlinear term,

$$\begin{aligned} 2kb_1(v^n, v^n, A_1v^n) &\le 2\, c_b\, k\, |v^n|^{1/2}\|v^n\||A_1v^n|^{3/2} \\ &\le \frac{\nu}{2}k|A_1v^n|^2 + \frac{27c_b^4}{2\nu^3}K_1^2k\|v^n\|^4. \end{aligned} \tag{4.9}$$

Using the Cauchy–Schwarz inequality and recalling (3.8), we bound the right-hand side of (4.8) by

$$\begin{aligned} 2k(e_2\theta^n, A_1v^n) &\le 2k|\theta^n||A_1v^n| \\ &\le \frac{\nu}{2}k|A_1v^n|^2 + \frac{2}{\nu}kM_1^2. \end{aligned} \tag{4.10}$$

Relations (4.8)–(4.10) imply

$$
\begin{aligned}
\|v^n\|^2 - \|v^{n-1}\|^2 + \|v^n - v^{n-1}\|^2 + \nu k |A_1 v^n|^2 \\
\le \frac{27 c_b^4}{2\nu^3} K_1^2 k \|v^n\|^4 + \frac{2}{\nu} k M_1^2,
\end{aligned} \tag{4.11}
$$

from which we obtain conclusion (4.7) of Lemma 4.1. □

In what follows, we will make use of the following two lemmas, whose proofs can be found in [14]:

Lemma 4.3. *Given $k > 0$ and positive sequences ξ_n, η_n and ζ_n such that*

$$
\xi_n \le \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n, \qquad \text{for } n \ge 1, \tag{4.12}
$$

we have, for any $n \ge 2$,

$$
\xi_n \le \left(\xi_0 + \sum_{i=1}^{n} k\zeta_i \right) \exp\left(\sum_{i=0}^{n-1} k\eta_i \right). \tag{4.13}
$$

Lemma 4.4. *Given $k > 0$, a positive integer n_0, positive sequences ξ_n, η_n and ζ_n such that*

$$
\xi_n \le \xi_{n-1}(1 + k\eta_{n-1}) + k\zeta_n, \qquad \text{for } n \ge n_0, \tag{4.14}
$$

and given the bounds

$$
\begin{aligned}
&\sum_{n=k_0}^{N+k_0} k\eta_n \le a_1, \qquad \sum_{n=k_0}^{N+k_0} k\zeta_n \le a_2, \\
&\sum_{n=k_0}^{N+k_0} k\xi_n \le a_3,
\end{aligned} \tag{4.15}
$$

for any $k_0 \ge n_0$, we have,

$$
\xi_n \le \left(\frac{a_3}{Nk} + a_2 \right) \mathrm{e}^{a_1}, \qquad \forall n \ge N + n_0. \tag{4.16}
$$

Proposition 4.1. *Let $T > 0$ be arbitrarily fixed and let $\{v^n, \theta^n\}$ be the solution of the numerical scheme (2.46)–(2.47). Then there exists $K_5 = K_5(\|v_0\|, |\theta_0|, T)$, such that for every $k \le \kappa_1$, we have*

$$
\|v^n\| \le K_5, \quad \forall n \ge 0, \tag{4.17}
$$

$$
\begin{aligned}
\sum_{n=i}^{m} \|v^n - v^{n-1}\|^2 \le & K_5^2 + \frac{27 c_b^4}{2\nu^3} K_1^2 K_5^4 (m - i + 1)k \\
& + \frac{2}{\nu} M_1^2 (m - i + 1)k, \quad \forall\, i = 1, \cdots, m.
\end{aligned} \tag{4.18}
$$

Moreover, for any initial data from H, there exists $K_4(T)$ such that

$$
\|v^n\| \le K_4, \quad \forall n \ge N + N_0 + 1, \tag{4.19}
$$

where $N := \lfloor T/k \rfloor$ and $T_0 = N_0 k$ is the time the approximate solution $\{v^n, \theta^n\}$ enters the absorbing ball $B(0, \rho_0)$ in H.

Proof. Using (4.1), we infer from (4.7)

$$
\begin{aligned}
\|v^n\|^2 &\le c_1K_1^2k\left(K_2\|v^{n-1}\|^2+\frac{4}{\nu^2}M_1^2\right)^2+\|v^{n-1}\|^2+\frac{2}{\nu}kM_1^2\\
&\le \|v^{n-1}\|^2\left(1+c_1K_1^2K_2^2k\|v^{n-1}\|^2+\frac{8}{\nu^2}c_1K_1^2K_2M_1^2k\right)\\
&\quad+\frac{1}{\nu}kM_1^2\left(\frac{16}{\nu^3}c_1K_1^2M_1^2+2\right).
\end{aligned}
\tag{4.20}
$$

We rewrite (4.20) in the form

$$\xi_n\le\xi_{n-1}(1+k\eta_{n-1})+k\zeta_n, \tag{4.21}$$

with

$$\xi_n=\|v^n\|^2,\quad \eta_n=c_1K_1^2K_2^2\|v^n\|^2+\frac{8}{\nu^2}c_1K_1^2K_2M_1^2,\quad \zeta_n=\frac{1}{\nu}M_1^2\left(\frac{16}{\nu^3}c_1K_1^2M_1^2+2\right), \tag{4.22}$$

and recalling (3.8) and (3.30), we compute the following:

$$\sum_{i=1}^{n}k\zeta_i=\frac{1}{\nu}M_1^2\left(\frac{16}{\nu^3}c_1K_1^2M_1^2+2\right)nk, \tag{4.23}$$

$$
\begin{aligned}
\sum_{i=0}^{n-1}k\eta_i&=c_1K_1^2K_2k\sum_{i=0}^{n-1}\left(K_2\|v^n\|^2+\frac{8}{\nu^2}M_1^2\right)\\
&\le\frac{c_1}{\nu}K_1^2K_2^2\left[K_1^2+\frac{M_1^2}{\nu}(n-1)k\right]+c_1K_1^2K_2k\|v^0\|^2+\frac{8}{\nu^2}c_1K_1^2K_2M_1^2nk\\
&\qquad\text{(by (3.31))}.
\end{aligned}
\tag{4.24}
$$

Then conclusion (4.13) of Lemma 4.3 yields

$$
\begin{aligned}
&\|v^n\|^2\\
&\le\left(\|v^0\|^2+\frac{1}{\nu}M_1^2\left(\frac{16}{\nu^3}c_1K_1^2M_1^2+2\right)nk\right)\exp\left\{\frac{c_1}{\nu}K_1^2K_2\left[K_1^2K_2+\frac{M_1^2}{\nu}(K_2+8)nk\right]\right\}\\
&\exp\left\{c_1K_1^2K_2k\|v^0\|^2\right\}=:K_3^2(\|v_0\|,|\theta_0|,nk),
\end{aligned}
\tag{4.25}
$$

and thus

$$\|v^n\|^2\le K_3^2(\|v_0\|,|\theta_0|,T+T_0),\forall n=0,\cdots,N+N_0. \tag{4.26}$$

In order to derive a bound on $\|v^n\|^2$ valid for $n\ge N+N_0+1$, we will apply (the discrete uniform Gronwall) Lemma 4.4. In order to do so, we recall that $|v^n|<\rho_0$, $|\theta^n|<\rho_0$, for $n\ge N_0$, and we compute the following (for $k_0\ge N_0+1$):

$$
\begin{aligned}
\sum_{n=k_0}^{N+k_0}k\eta_n&=c_1K_1^2K_2k\sum_{n=k_0}^{N+k_0}\left(K_2\|v^n\|^2+\frac{8}{\nu^2}M_1^2\right)\\
&\le\frac{c_1}{\nu}\rho_0^4K_2^2(\rho_0,\rho_0)\left(1+\frac{1}{\nu}(N+1)k\right)+\frac{8}{\nu^2}c_1\rho_0^4K_2(\rho_0,\rho_0)nk\\
&\qquad\text{(by (3.31))},
\end{aligned}
\tag{4.27}
$$

$$\begin{aligned}\sum_{n=k_0}^{N+k_0} k\zeta_n &= \sum_{n=k_0}^{N+k_0} \frac{1}{\nu} M_1^2 \left(\frac{16}{\nu^3} c_1 K_1^2 M_1^2 + 2 \right) nk \\ &\le \frac{1}{\nu}\rho_0^2 \left(\frac{16}{\nu^3} c_1 \rho_0^4 + 2 \right)(N+1)k,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\sum_{n=k_0}^{N+k_0} k\xi_n &= \sum_{n=k_0}^{N+k_0} k\|v^n\|^2 \\ &\le \frac{\rho_0^2}{\nu} \left(1 + \frac{1}{\nu}(N+1)k \right) \\ &\qquad \text{(by (3.31))}.\end{aligned} \tag{4.29}$$

Then conclusion (4.16) of Lemma 4.4 yields

$$\begin{aligned}\|v^n\|^2 &\le \left[\frac{\rho_0^2}{\nu N k} \left(1 + \frac{1}{\nu}(N+1)k \right) + \frac{1}{\nu}\rho_0^2 \left(\frac{16}{\nu^3} c_1 \rho_0^4 + 2 \right)(N+1)k \right] \\ &\quad \exp\left\{ \frac{c_1}{\nu} \rho_0^4 K_2^2(\rho_0, \rho_0) \left(1 + \frac{1}{\nu}(N+1)k \right) + \frac{8}{\nu^2} c_1 \rho_0^4 K_2(\rho_0,\rho_0) nk \right\} \\ &\le \left[\frac{\rho_0^2}{\nu T} \left(1 + \frac{T}{\nu} + \frac{1}{\nu^2} \right) + \frac{1}{\nu}\rho_0^2 \left(\frac{16}{\nu^3} c_1 \rho_0^4 + 2 \right) \left(T + \frac{1}{\nu} \right) \right] \\ &\quad \exp\left\{ \frac{c_1}{\nu} \rho_0^4 K_2^2(\rho_0, \rho_0) \left(1 + \frac{T}{\nu} + \frac{1}{\nu^2} \right) + \frac{8}{\nu^2} c_1 \rho_0^4 K_2(\rho_0,\rho_0) T \right\} \\ &=: K_4^2(T), \quad \forall n \ge N + N_0 + 1.\end{aligned} \tag{4.30}$$

Combining the above bound with (4.26), we obtain both conclusion (4.17) and conclusion (4.19) of the proposition.

Taking the sum of (4.11) with n from i to m and using (4.17) gives conclusion (4.18) and thus the proof of Proposition 4.1 is complete.

□

4.2. H^1-Uniform Boundedness of θ^n. We are now going to prove the H^1-uniform boundedness of θ^n, for all $n \ge 0$. In order to do so, we will first use the discrete Gronwall lemma to derive an upper bound on $\|\theta^n\|$, $n \le N$, for some $N > 0$, and then we will use another version of the discrete uniform Gronwall lemma (see Lemma 4.6 below) to obtain an upper bound on $\|\theta^n\|$, $n \ge N$.

Lemma 4.5. *Let $\{v_0, \theta_0\} \in V$ and $\{v^n, \theta^n\}$ be the solution of the numerical scheme (2.46)–(2.47). Also, let $T > 0$ be arbitrarily fixed and k be such that*

$$k \le \min\left\{ \kappa_1, \frac{1}{2c_2 K_1^2 K_5^2(\|v_0\|, |\theta_0|)} \right\} =: \kappa_2(\|v_0\|, |\theta_0|), \tag{4.31}$$

where κ_1 is given by (3.43), $c_2 = 27c_b^4/(32\kappa^2)$ and $K_5(\|v_0\|, |\theta_0|)$ is given in Proposition 4.1. Then we have

$$\|\theta^n\|^2 \le 4^{c_2 K_1^2 K_5^2 T} \left(\|\theta^0\|^2 + \frac{2}{c_2 \kappa K_5^2} \right), \ \forall n = 1, \cdots, N := \lfloor T/k \rfloor. \tag{4.32}$$

Proof. Replacing θ by $2kA_2\theta^n$ in (2.47), we obtain

$$\begin{aligned}\|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 + 2kb_2(v^n, \theta^n, A_2\theta^n) \\ - 2k(v_2^n, A_2\theta^n) + 2\kappa k |A_2\theta^n|^2 = 0.\end{aligned} \tag{4.33}$$

Using property (2.38) of the trilinear form b_2 and recalling (3.30) and (4.17), we have the following bound of the nonlinear term,

$$(4.34)\qquad \begin{aligned} 2kb_2(v^n,\theta^n,A_2\theta^n) &\le 2\,c_b\,k\,|v^n|^{1/2}\|v^n\|^{1/2}\|\theta^n\|^{1/2}|A_2\theta^n|^{3/2} \\ &\le \frac{\kappa}{2}k|A_2\theta^n|^2 + c_2K_1^2K_5^2k\|\theta^n\|^2. \end{aligned}$$

Using the Cauchy–Schwarz inequality and recalling (3.30), we have the following bound

$$(4.35)\qquad \begin{aligned} -2k(v_2^n,A_2\theta^n) &\le 2k|v_2^n||A_2\theta^n| \\ &\le \frac{\kappa}{2}k|A_2\theta^n|^2 + \frac{2}{\kappa}K_1^2k. \end{aligned}$$

Relations (4.33)–(4.35) imply

$$(4.36)\qquad \begin{aligned} \|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n-\theta^{n-1}\|^2 + \kappa k|A_2\theta^n|^2 \\ \le c_2K_1^2K_5^2k\|\theta^n\|^2 + \frac{2}{\kappa}K_1^2k, \end{aligned}$$

from which we obtain

$$(4.37)\qquad \|\theta^n\|^2 \le \frac{1}{\alpha}\|\theta^{n-1}\|^2 + \frac{2}{\kappa\alpha}K_1^2k,$$

where

$$(4.38)\qquad \alpha = 1 - c_2K_1^2K_5^2k.$$

Using recursively (4.37), we find

$$(4.39)\qquad \|\theta^n\|^2 \le (1-c_2K_1^2K_5^2k)^{-n}\left(\|\theta^0\|^2 + \frac{2}{c_2\kappa K_5^2}\right).$$

Since

$$1-x \ge 4^{-x}, \quad 0<x\le \frac{1}{2},$$

and, by hypothesis, $c_2K_1^2K_5^2k \le 1/2$, conclusion (4.32) follows immediately. This completes the proof of Lemma 4.5. □

In order to derive an upper bound on $\|\theta^n\|$, $n\ge N$, we will need the following version of the discrete uniform Gronwall lemma, slightly different from Lemma 4.4:

Lemma 4.6. *We are given $k>0$, positive integers n_0,n_1 and positive sequences ξ_n, η_n, ζ_n such that*

$$(4.40)\qquad k\eta_n < \frac{1}{2}, \qquad \text{for } n\ge n_0,$$

$$(4.41)\qquad (1-k\eta_n)\xi_n \le \xi_{n-1} + k\zeta_n, \qquad \text{for } n\ge n_0.$$

Assume also that

$$(4.42)\qquad \begin{aligned} &k\sum_{n=k_0}^{k_0+n_1}\eta_n \le a_1(n_0,n_1), \qquad k\sum_{n=k_0}^{k_0+n_1}\zeta_n \le a_2(n_0,n_1), \\ &k\sum_{n=k_0}^{k_0+n_1}\xi_n \le a_3(n_0,n_1), \end{aligned}$$

for any $k_0\ge n_0$. We then have,

$$(4.43)\qquad \xi_n \le \left(\frac{a_3(n_0,n_1)}{kn_1} + a_2(n_0,n_1)\right)\mathrm{e}^{4a_1(n_0,n_1)},$$

for any $n\ge n_0+n_1$.

Proof. Let n_3 and n_4 be such that $n_0 \leq n_2 < n_3 \leq n_2 + n_1$. Using recursively (4.41), we derive

$$\xi_{n_2+n_1} \leq \frac{1}{\prod_{n=n_3}^{n_2+n_1}(1-k\eta_n)}\xi_{n_3-1} + +k\sum_{n=n_3}^{n_2+n_1}\frac{1}{\prod_{j=n}^{n_2+n_1}(1-k\eta_j)}\zeta_n. \tag{4.44}$$

Using the fact that $1 - x \geq \mathrm{e}^{-4x}$, $\forall x \in \left(0, \frac{1}{2}\right)$, and recalling assumptions $(4.42)_1$ and $(4.42)_2$, we obtain

$$\xi_{n_2+n_3} \leq (\xi_{n_3-1} + a_2)e^{-4a_1}.$$

Multiplying this inequality by k, summing n_3 from $n_2 + 1$ to $n_2 + n_1$ and using assumption $(4.42)_3$ gives the conclusion (4.43) of the lemma. □

We are now able to derive an upper bound on $\|\theta^n\|$, $n \geq N$. More precisely, we have the following:

Lemma 4.7. *Let $\{v_0, \theta_0\} \in V$ and $\{v^n, \theta^n\}$ be the solution of the numerical scheme* (2.46)–(2.47)*. Also, let $T > 0$ be arbitrarily fixed and k be such that*

$$k \leq \min\left\{\kappa_2(\|v_0\|, |\theta_0|), \frac{T}{2}\right\} =: \kappa_3(\|v_0\|, |\theta_0|), \tag{4.45}$$

where $\kappa_2(\cdot,\cdot)$ is given in Lemma 4.5. Then there exists $M_2 = M_2(\|v_0\|, |\theta_0|, T)$, given in (4.48) *below, such that*

$$\|\theta^n\| \leq M_2(\|v_0\|, |\theta_0|, T),\ \forall n \geq N := \lfloor T/k \rfloor. \tag{4.46}$$

Proof. We apply Lemma 4.6 to (4.36), which we rewrite as

$$\begin{aligned}(1 - c_2K_1^2K_5^2k)\|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 + \kappa k|A_2\theta^n|^2 \\ \leq \frac{2}{\kappa}K_1^2k.\end{aligned} \tag{4.47}$$

We set $\xi_n = \|\theta^n\|^2$, $\eta_n = c_2K_1^2K_5^2$, $\zeta_n = \frac{2}{\kappa}K_1^2$, $n_0 = 1$, $n_1 = N - 1$ and for $k_0 \geq 1$ we compute:

$$k\sum_{n=k_0}^{k_0+n_1}\eta_n = k\sum_{n=k_0}^{k_0+n_1}c_2K_1^2K_5^2 \leq c_2K_1^2K_5^2T,$$

$$k\sum_{n=k_0}^{k_0+n_1}\zeta_n = k\sum_{n=k_0}^{k_0+n_1}\frac{2}{\kappa}K_1^2 \leq \frac{2}{\kappa}K_1^2T,$$

$$k\sum_{n=k_0}^{k_0+n_1}\xi_n = k\sum_{n=k_0}^{k_0+n_1}\|\theta^n\|^2 \leq \frac{1}{\kappa}\left(M_1^2 + \frac{K_1^2}{\kappa}T\right)\text{ (by (3.32))}.$$

Then Lemma 4.6 implies

$$\begin{aligned}\|\theta^n\|^2 &\leq \frac{2}{\kappa}\left(\frac{M_1^2}{T} + \frac{K_1^2}{\kappa} + K_1^2T\right)e^{4c_2K_1^2K_5^2T} \\ &:= M_2^2(\|v_0\|, |\theta_0|, T),\ \forall n \geq N.\end{aligned} \tag{4.48}$$

Thus, the lemma is proved. □

Combining Lemma 4.5 and Lemma 4.7, we obtain that θ^n are uniformly bounded in V, for all $n \geq 0$. More precisely, we have

Proposition 4.2. *Let* $\{v_0, \theta_0\} \in V$ *and* $\{v^n, \theta^n\}$ *be the solution of the numerical scheme* (2.46)–(2.47). *Also, let* $T > 0$ *be arbitrarily fixed and* k *be such that* $k \leq \kappa_3(\|v_0\|, |\theta_0|)$, *where* $\kappa_3(\cdot, \cdot)$ *is given in Lemma 4.7. Then there exists* $M_3 = M_3(\|v_0\|, \|\theta_0\|)$, *such that*

$$\|\theta^n\| \leq M_3(\|v_0\|, \|\theta_0\|), \forall n \geq 0. \tag{4.49}$$

Proof. Taking

$$M_3(\|v_0\|, \|\theta_0\|) = \max\left\{4^{c_2 K_1^2 K_5^2 T}\left(\|\theta_0\|^2 + \frac{2}{c_2 \kappa K_5^2}\right), M_2(\|v_0\|, |\theta_0|, T)\right\},$$

Lemmas 4.5 and 4.7 give conclusion (4.49) of the proposition. □

Corollary 4.1. *Under the assumptions of Proposition 4.2, we also have*

$$\begin{aligned}\sum_{n=i}^{m} \|\theta^n - \theta^{n-1}\|^2 \leq & M_3^2 + c_2 K_1^2 K_5^2 M_3^2 k(m-n+1) \\ & + \frac{2}{\kappa} K_1^2 k(m-n+1), \quad \forall\, i = 1, \cdots, m.\end{aligned} \tag{4.50}$$

Proof. Taking the sum of (4.36) with n from i to m and using (4.49) gives conclusion (4.50) of the corollary right away. □

With the notation $\|\{v_0, \theta_0\}\| = \|v_0\| + \|\theta_0\|$, Proposition 4.1 and Proposition 4.2 can be combined to obtain the following theorem, which is one of our main results:

Theorem 4.1. *Let* $\{v_0, \theta_0\} \in V$ *and* $\{v^n, \theta^n\}$ *be the solution of the numerical scheme* (2.46)–(2.47). *Then there exists a positive function* $\kappa_4(\cdot)$, *depending decreasingly of its argument, and a positive function* $K_6(\cdot)$, *depending increasingly of its argument, such that if*

$$k \leq \kappa_4(\|\{v_0, \theta_0\}\|), \tag{4.51}$$

then

$$\|\{v^n, \theta^n\}\| \leq K_6(\|\{v_0, \theta_0\}\|), \forall n \geq 0. \tag{4.52}$$

5. Convergence of Attractors

In this section we address the issue of the convergence of the attractors generated by the discrete system (2.45)–(2.47) to the attractor generated by the continuous system (2.11)–(2.18). Whereas for the continuous system (2.11)–(2.18) one can prove both the existence and uniqueness of the solution (see, e.g., [15])–and, therefore, define a global attractor–, for the discrete system (2.45)–(2.47) one can prove (using Theorem 4.1) the uniqueness of the solution provided that $k \leq \kappa(\|u_0\|)$, for some $\kappa(\|u_0\|) > 0$. Since the time restriction depends on the initial data, one cannot define a single-valued attractor in the classical sense, and this is why we need to use the attractor theory for the so-called multi-valued mappings. Multi-valued dynamical systems have been investigated by many authors (see, e.g., [1], [2], [4], [11], [12], [13]), but in this article we use the tools developed in [5] to study the convergence of the discrete (multi-valued) attractors to the continuous (single-valued) attractor. For convenience, we recall those results in Subsection 5.1, and then we apply them to the thermohydraulics equations in Subsection 5.2.

5.1. Attractors for multi-valued mappings. Throughout this subsection, we consider $(H, |\cdot|)$ to be a Hilbert space and $\mathbb{T}$ to be either $\mathbb{R}^+ = [0, \infty)$ or $\mathbb{N}$.

Definition 5.1. *A one-parameter family of set-valued maps $S(t) : 2^H \to 2^H$ is a* ***multi-valued semigroup*** *(m-semigroup) if it satisfies the following properties:*

(S.1) $S(0) = I_{2^H}$ *(identity in 2^H);*
(S.2) $S(t+s) = S(t)S(s)$, *for all $t, s \in \mathbb{T}$.*

Moreover, the m-semigroup is said to be ***closed*** *if $S(t)$ is a closed map for every $t \in \mathbb{T}$, meaning that if $x_n \to x$ in H and $y_n \in S(t)x_n$ is such that $y_n \to y$ in H, then $y \in S(t)x$. (To simplify the notation, hereafter we have written $S(t)x$ in place of $S(t)\{x\}$.)*

Definition 5.2. *The* ***positive orbit*** *of $\mathcal{B}$, starting at $t \in \mathbb{T}$, is the set*

$$\gamma_t(\mathcal{B}) = \bigcup_{\tau \geq t} S(\tau)\mathcal{B},$$

where

$$S(t)\mathcal{B} = \bigcup_{x \in \mathcal{B}} S(t)x.$$

Definition 5.3. *For any $\mathcal{B} \in 2^H$, the set*

$$\omega(\mathcal{B}) = \bigcap_{t \in \mathbb{T}} \overline{\gamma_t(\mathcal{B})}$$

is called the ***ω-limit set*** *of $\mathcal{B}$.*

Definition 5.4. *A nonempty set $\mathcal{B} \in 2^H$ is* ***invariant*** *for $S(t)$ if*

$$S(t)\mathcal{B} = \mathcal{B}, \qquad \forall t \in \mathbb{T}.$$

Definition 5.5. *A set $\mathcal{B}_0 \in 2^H$ is an* ***absorbing set*** *for the m-semigroup $S(t)$ if for every bounded set $\mathcal{B} \in 2^H$ there exists $t_{\mathcal{B}} \in \mathbb{T}$ such that*

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \qquad \forall t \geq t_{\mathcal{B}}.$$

Definition 5.6. *A nonempty set $\mathcal{C} \in 2^H$ is* ***attracting*** *if for every bounded set $\mathcal{B}$ we have*

$$\lim_{t \to \infty} \operatorname{dist}(S(t)\mathcal{B}, \mathcal{C}) = 0,$$

where $\operatorname{dist}(\cdot, \cdot)$ *is the* ***Hausdorff semidistance****, defined as*

$$\operatorname{dist}(\mathcal{B}, \mathcal{C}) = \sup_{b \in \mathcal{B}} \inf_{c \in \mathcal{C}} |b - c|, \forall\, \mathcal{B}, \mathcal{C} \subset H. \tag{5.1}$$

Definition 5.7. *A nonempty compact set $\mathcal{A} \in 2^X$ is said to be the* ***global attractor*** *of $S(t)$ if $\mathcal{A}$ is an invariant attracting set.*

Remark 5.1. *The global attractor, if it exists, is necessarily unique. Moreover, it enjoys the following maximality and minimality properties:*

(i) *if $\tilde{\mathcal{A}}$ is a bounded invariant set, then $\mathcal{A} \supset \tilde{\mathcal{A}}$;*
(ii) *if $\tilde{\mathcal{A}}$ is a closed attracting set, then $\mathcal{A} \subset \tilde{\mathcal{A}}$.*

Definition 5.8. *Given a bounded set $\mathcal{B} \in 2^H$, the* ***Kuratowski measure of noncompactness*** *$\alpha(\mathcal{B})$ of $\mathcal{B}$ is defined as*

$$\alpha(\mathcal{B}) = \inf\big\{\delta : \mathcal{B} \text{ has a finite cover by balls of } X \text{ of diameter less than } \delta\big\}.$$

We note that $\alpha(\mathcal{B}) = 0$ if and only if $\overline{\mathcal{B}}$ is compact.

The following theorem, whose proof can be found in [5], gives conditions under which a global attractor exists.

Theorem 5.1. *Suppose that the closed m-semigroup $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0 \in 2^H$ and*

$$\lim_{t\to\infty} \alpha(S(t)\mathcal{B}_0) = 0. \tag{5.2}$$

Then $\omega(\mathcal{B}_0)$ is the global attractor of $S(t)$.

For the purpose of this article, we need to introduce the notion of *discrete m-semigroups.* More precisely, we have the following:

Definition 5.9. *Given a set-valued map $S : 2^H \to 2^H$, we define a **discrete m-semigroup**by*

$$S(n) = S^n, \qquad \forall n \in \mathbb{N},$$

and we will denote it by $\{S\}_{n\in\mathbb{N}}$ (instead of $\{S^n\}_{n\in\mathbb{N}}$).

Remark 5.2. *Given two nonempty sets $\mathcal{B}, \mathcal{C} \in 2^H$, we write*

$$\mathcal{B} - \mathcal{C} = \{b - c : \, b \in \mathcal{B}, \, c \in \mathcal{C}\} \qquad \text{and} \qquad |\mathcal{B}| = \sup_{b\in\mathcal{B}} |b|.$$

In order to prove the convergence of the attractors generated by the discrete system (2.45)–(2.47) to the attractor generated by the continuous system (2.11)–(2.18) we will use the following result, whose proof can be found in [5]; see also [21], [19].

Theorem 5.2. *Let $S(t)$ be a closed m-semigroup, possessing the global attractor $\mathcal{A}$, and for $\kappa_0 > 0$, let $\{S_k, \, 0 < k \leq \kappa_0\}_{n\in\mathbb{N}}$ be a family of discrete closed m-semigroups, with global attractor $\mathcal{A}_k$. Assume the following:*

(H1) *[Uniform boundedness]: there exists $\kappa_1 \in (0, \kappa_0]$ such that the set*

$$\mathcal{K} = \bigcup_{k\in(0,\kappa_1]} \mathcal{A}_k$$

is bounded in H;

(H2) *[Finite time uniform convergence]: there exists $t_0 \geq 0$ such that for any $T^\star > t_0$,*

$$\lim_{k\to 0} \sup_{x\in\mathcal{A}_k, \, nk\in[t_0,T^\star]} |S_k^n x - S(nk)x| = 0.$$

Then

$$\lim_{k\to 0} \operatorname{dist}(\mathcal{A}_k, \mathcal{A}) = 0,$$

where dist *denotes the Hausdorff semidistance defined in* (5.1)*.*

5.2. Application: The thermohydraulics equations. The system (2.11)–(2.18) possesses a unique solution and thus generates a continuous single-valued dynamical system $S(t) : H \to H$, with global attractor $\mathcal{A}$, bounded in V (see, e.g., [15]). Using Theorem 4.1 one can prove that the discrete system (2.45)–(2.47) has a unique solution provided that $k \leq \kappa(\|u_0\|)$, for some $\kappa(\|u_0\|) > 0$. The dependence of the time step k on the initial data prevents us from defining a single-valued attractor in the classical sense, but this difficulty can be overcome by the theory of the multi-valued attractors. More precisely, in this article we will prove that there exists $\kappa_0 > 0$ such that if $0 < k \leq \kappa_0$, the system (2.45)–(2.47) generates a closed discrete m-semigroup $\{S_k\}_{n\in\mathbb{N}}$, with global attractors $\mathcal{A}_k$, that will converge to $\mathcal{A}$ in the sense of Theorem 5.2.

In order to do that, we define, for $k>0$, the multi-valued map $S_k : 2^H \to 2^H$ as follows: for every $\tilde{u} = \{\tilde{v}, \tilde{\theta}\} \in H$,

$$S_k\tilde{u} = \{u = \{v,\theta\} \in V :\ u \text{ solves (5.3)–(5.4) below with time-step } k\} :$$

$$(v,v') + \nu k((v,v')) + kb_1(v,v,v') - k(e_2\theta, v') = (\tilde{v}, v'),\ \forall v' \in V_1, \tag{5.3}$$

$$(\theta,\theta') + \kappa k((\theta,\theta')) + kb_2(v,\theta,\theta') - k(v_2,\theta') = (\tilde{\theta},\theta'),\ \forall \theta' \in V_2. \tag{5.4}$$

We then have the following:

Theorem 5.3. *The multi-valued map S_k associated with the implicit Euler scheme* (2.45)–(2.47) *generates a closed discrete m-semigroup $\{S_k\}_{n\in\mathbb{N}}$.*

Proof. Since conditions (S.1) and (S.2) are satisfied by definition, we just need to prove that for each $n\in\mathbb{N}$, S_k^n is a closed multi-valued map. For that, we let $n\in\mathbb{N}$ be arbitrarily fixed and, as $j\to\infty$, we let $u_j^0 \to u^0$ in H, where $u_j^0 = \{v_j^0, \theta_j^0\}, u^0 = \{v^0,\theta^0\}$. Also let $u_j^n \in S_k^n u_j^0$ be such that $u_j^n \to u^n$ in H, where $u_j^n = \{v_j^n, \theta_j^n\}, u^n = \{v^n,\theta^n\}$. We need to show that $u^n \in S_k^n u^0$.

Indeed, since $u_j^n \in S_k^n u_j^0$, there exists a sequence $(u_j^0, u_j^1, \ldots, u_j^{n-1}, u_j^n)$, with $u_j^i \in S_k u_j^{i-1}$, such that

$$(v_j^i, v') + \nu k((v_j^i, v')) + kb_1(v_j^i, v_j^i, v') - k(e_2\theta_j^i, v') = (v_j^{i-1}, v'),\ \forall v'\in V_1, \tag{5.5}$$

$$(\theta_j^i, \theta') + \kappa k((\theta_j^i,\theta')) + kb_2(v_j^i,\theta_j^i,\theta') - k((v_j^i)_2,\theta') = (\theta_j^{i-1},\theta'),\ \forall\theta'\in V_2. \tag{5.6}$$

The sequence u_j^0 being convergent in H, it is also bounded in H and thus there exists $M>0$ such that

$$\sup_j |u_j^0|^2 \le M. \tag{5.7}$$

Then Lemmas 3.2 and 3.3 imply that for every $i=1,\ldots,n$, the sequences v_j^i and θ_j^i are bounded in V_1 and V_2, respectively. We therefore have that there exist subsequences still denoted v_j^i and θ_j^i, such that as $j\to\infty$:

$$v_j^i \to v^i, \text{ strongly in } H_1 \text{ and weakly in } V_1, \tag{5.8}$$

$$\theta_j^i \to \theta^i, \text{ strongly in } H_2 \text{ and weakly in } V_2. \tag{5.9}$$

Now, passing to the limit in (5.5)–(5.6), we obtain

$$(v^i, v') + \nu k((v^i,v')) + kb_1(v^i,v^i,v') - k(e_2\theta^i, v') = (v^{i-1}, v'),\ \forall v'\in V_1, \tag{5.10}$$

$$(\theta^i,\theta') + \kappa k((\theta^i,\theta')) + kb_2(v^i,\theta^i,\theta') - k((v^i)_2,\theta') = (\theta^{i-1},\theta'),\ \forall\theta'\in V_2. \tag{5.11}$$

We therefore obtain that $u^i \in S_k u^{i-1}$, for each $i=1,\ldots,n$, and hence, $u^n \in S_k u^{n-1} \subset S_k^n u^0$. This completes the proof of the theorem. $\square$

In order to prove the existence of the discrete global attractors, we first prove the existence of absorbing sets. More precisely, we have the following:

Proposition 5.1. *There exists $\kappa_5 > 0$, independent of $\{v_0,\theta_0\}, n, k$, such that if $k \in (0,\kappa_5]$ the following holds: there exists a constant $R_1 > 0$ such that for every $R\ge 0$ and $|\{v_0,\theta_0\}| \le R$, there exists $N_1 = N_1(R,k) \ge 0$ such that*

$$\|S_k^n\{v_0,\theta_0\}\| \le R_1, \qquad \forall n \ge N_1. \tag{5.12}$$

Hence, the set

$$\mathcal{B}_1 = \{\{v,\theta\}\in V :\ \|\{v,\theta\}\| \le R_1\}$$

is a V-bounded absorbing set for $\{S_k\}_{n\in\mathbb{N}}$, for $k \in (0, \kappa_5]$.

Proof. Let κ_1 be as in Corollary 3.2 and let $k \le \min\{1, \kappa_1\}$. Also, let $R \ge 0$ and $|\{v_0, \theta_0\}| \le R$. Then, by Corollary 3.2, there exists $N_0 = N_0(R, k) \ge 0$ such that

$$|\{v^n, \theta^n\}| \le \rho_0, \qquad \forall n \ge N_0. \tag{5.13}$$

Let $m := N_0 + \left\lfloor \frac{1}{k} \right\rfloor$. Then equations (3.31) and (3.32) imply

$$\nu k \sum_{j=N_0+1}^{m} \|v^j\|^2 \le \rho_0^2 + \frac{1}{\nu}\rho_0^2(m - N_0)k, \tag{5.14}$$

$$\kappa k \sum_{j=N_0+1}^{m} \|\theta^j\|^2 \le \rho_0^2 + \frac{1}{\kappa}\rho_0^2(m - N_0)k. \tag{5.15}$$

Adding the above relations we obtain

$$k\left(\sum_{j=N_0+1}^{m} (\nu\|v^j\|^2 + \kappa\|\theta^j\|^2)\right) \le \rho_0^2\left(2 + \frac{1}{\nu}(m - N_0)k + \frac{1}{\kappa}(m - N_0)k\right). \tag{5.16}$$

Assuming that for every $j \in \{N_0 + 1, \cdots, m\}$

$$(\nu\|v^j\|^2 + \kappa\|\theta^j\|^2) \ge \frac{\rho_0^2}{k(m - N_0)}\left(2 + \frac{1}{\nu}(m - N_0)k + \frac{1}{\kappa}(m - N_0)k\right),$$

we obtain

$$k\left(\sum_{j=N_0+1}^{m} (\nu\|v^j\|^2 + \kappa\|\theta^j\|^2)\right) \ge \rho_0^2\left(2 + \frac{1}{\nu}(m - N_0)k + \frac{1}{\kappa}(m - N_0)k\right), \tag{5.17}$$

which contradicts (5.16). Hence there exists $l \in \{N_0 + 1, \cdots, m\}$ such that

$$\begin{aligned}(\nu\|v^l\|^2 + \kappa\|\theta^l\|^2) &\le \frac{\rho_0^2}{k(m - N_0)}\left(2 + \frac{1}{\nu}(m - N_0)k + \frac{1}{\kappa}(m - N_0)k\right)\\ &\le 2\rho_0^2\left(2 + \frac{1}{\nu} + \frac{1}{\kappa}\right).\end{aligned} \tag{5.18}$$

We, therefore, have

$$\|\{v^l, \theta^l\}\|^2 \le 2\rho_0^2\left(2 + \frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\frac{1}{\nu} + \frac{1}{\kappa}\right) =: R_*^2. \tag{5.19}$$

Applying Theorem 4.1 with initial data $\{v^l, \theta^l\}$ we obtain that there exists $\kappa_4(\|\{v^l, \theta^l\}\|)$ and $K_6(\|\{v^l, \theta^l\}\|)$ such that if $k \le \kappa_4(\|\{v^l, \theta^l\}\|)$, then

$$\|\{v^n, \theta^n\}\| \le K_6(\|\{v^l, \theta^l\}\|), \forall n \ge l. \tag{5.20}$$

Recalling (5.19) and the fact that $\kappa_4(\cdot)$ and $K_6(\cdot)$ are, respectively, decreasing and increasing functions of their arguments, (5.20) yields

$$\|\{v^n, \theta^n\}\| \le K_6(R_*) =: R_1, \forall n \ge N_1 = N_1(R, k) := N_0 + \left\lfloor \frac{1}{k} \right\rfloor, \tag{5.21}$$

provided that $k \le \kappa_5$, where

$$\kappa_5 = \min\{1, \kappa_1, \kappa_4(R_*)\}. \tag{5.22}$$

This completes the proof of Proposition 5.1. □

We are now in a position to prove the existence of the discrete global attractors. More precisely, we have the following:

Proposition 5.2. *For every $k \in (0, \kappa_5]$, there exists the global attractor $\mathcal{A}_k$ of the m-semigroup $\{S_k\}_{n\in\mathbb{N}}$.*

Proof. Let $\mathcal{B}_0 = B_H(0, \rho_0)$ be the bounded absorbing set given in Corollary 3.2. Then Proposition 5.1 implies that $S_k^n \mathcal{B}_0$ is bounded in V, for all $n \geq N_1(\rho_0, k)$. Since V is compactly embedded in H, we obtain that $S_k^n \mathcal{B}_0$ is relatively compact in H and, thus, $\alpha(S_k^n \mathcal{B}_0) = 0$, for all $n \geq N_1(\rho_0, k)$. Condition (5.2) of Theorem 5.1 is therefore satisfied and then the existence of the discrete global attractor $\mathcal{A}_k$ follows right away. □

Remark 5.3. *Since the global attractor $\mathcal{A}_k$ is the smallest closed attracting set of H, Proposition 5.1 implies*

$$\mathcal{A}_k \subset \mathcal{B}_1, \forall k \in (0, \kappa_5], \tag{5.23}$$

and thus

$$\bigcup_{k\in(0,\kappa_5]} \mathcal{A}_k \subset \mathcal{B}_1. \tag{5.24}$$

Let us recall that our goal is to prove, using Theorem 5.2, that the discrete global attractors $\mathcal{A}_k$ converge to the continuous global attractor $\mathcal{A}$. Thanks to (5.24), condition (H1) of Theorem 5.2 holds true. There remains to prove the finite time uniform convergence required by (H2). In order to do that, we define, for any $k > 0$ and for any function ψ, the following:

$$\psi_k(t) = \psi^n, \quad t \in [(n-1)k, nk), \tag{5.25}$$

$$\tilde{\psi}_k(t) = \psi^n + \frac{t - nk}{k}(\psi^n - \psi^{n-1}), \quad t \in [(n-1)k, nk). \tag{5.26}$$

With the above notations, equations (2.46) and (2.47) can be rewritten as follows; for $t \in [(n-1)k, nk)$:

(5.27)
$$\left(\frac{\partial \tilde{v}_k(t)}{\partial t}, v\right) + \nu((\tilde{v}_k(t), v)) + b_1(\tilde{v}_k(t), \tilde{v}_k(t), v) = (e_2\tilde{\theta}_k(t), v) + (f_k(t), v), \; \forall v \in V_1,$$

(5.28)
$$\left(\frac{\partial \tilde{\theta}_k(t)}{\partial t}, \theta\right) + \kappa((\tilde{\theta}_k(t), \theta)) + b_2(\tilde{v}_k(t), \tilde{\theta}_k(t), \theta) - (\tilde{v}_k(t))_2, \theta) = (g_k(t), \theta), \; \forall \theta \in V_2,$$

where

$$\begin{aligned}(f_k(t), v) = \nu((\tilde{v}_k(t) - v_k(t), v)) + b_1(\tilde{v}_k(t), \tilde{v}_k(t), v) \\ - b_1(v_k(t), v_k(t), v) - (e_2(\tilde{\theta}_k(t) - \theta_k(t)), v),\end{aligned} \tag{5.29}$$

$$\begin{aligned}(g_k(t), \theta) = \kappa((\tilde{\theta}_k(t) - \theta_k(t), \theta)) + b_2(\tilde{v}_k(t), \tilde{\theta}_k(t), \theta) \\ - b_2(v_k(t), \theta_k(t), \theta) - ((\tilde{v}_k(t) - v_k(t))_2, \theta).\end{aligned} \tag{5.30}$$

Lemma 5.1. *Let $T^* > 0$ be arbitrarily fixed and let $k \leq \kappa_0$, where*

$$\kappa_0 = \min\{\kappa_5, \kappa_4(R_1)\}, \tag{5.31}$$

with κ_5 being given in (5.22) and κ_4 being given in Theorem 4.1. Assume that $\{v_0, \theta_0\} \in \mathcal{A}_k$ and let $\{v^n, \theta^n\}$ be the solution of the numerical scheme (2.45)–(2.47). Then there exist $K_7(T^*)$ and $K_8(T^*)$ such that

$$\|f_k\|^2_{L^2(0,T^*;V_1')} \leq kK_7(T^*), \tag{5.32}$$

and

$$\|g_k\|^2_{L^2(0,T^*;V_2')} \le kK_8(T^*). \tag{5.33}$$

Proof. Let us first note that for any $t \in [(n-1)k, nk)$ we have

$$\tilde{\psi}_k(t) - \psi_k(t) = \frac{t-nk}{k}(\psi^n - \psi^{n-1}). \tag{5.34}$$

Also, since $\{v_0, \theta_0\} \in \mathcal{A}_k$, we have that $\|\{v_0, \theta_0\}\| \le R_1$ (by (5.23)) and then Theorem 4.1 implies that for $k \le \kappa_0$,

$$\|\{v^n, \theta^n\}\| \le K_6(R_1),\ \forall n \ge 0. \tag{5.35}$$

Now let $v \in V_1$ be such that $\|v\| \le 1$, and let $t \in [(n-1)k, nk)$ be fixed. Using property (2.31) of the trilinear form b_1, we have

$$\begin{aligned} &|b_1(\tilde{v}_k(t), \tilde{v}_k(t), v) - b_1(v_k(t), v_k(t), v)| \\ &= |b_1(\tilde{v}_k(t) - v_k(t), \tilde{v}_k(t), v) + b_1(v_k(t), \tilde{v}_k(t) - v_k(t), v)| \\ &\le c_b(\|\tilde{v}_k(t) - v_k(t)\|(\|\tilde{v}_k(t)\| + \|v_k(t)\|)\|v\| \\ &\le c\|v^n - v^{n-1}\| \quad (\text{by } (5.34), (5.35) \text{ and } \|v\| \le 1). \end{aligned} \tag{5.36}$$

We also have

$$\nu|((\tilde{v}_k(t) - v_k(t), v))| \le \nu\|v^n - v^{n-1}\|, \tag{5.37}$$

$$|(e_2(\tilde{\theta}_k(t) - \theta_k(t)), v)| \le \|\theta^n - \theta^{n-1}\|. \tag{5.38}$$

Relations (5.36)–(5.38) imply

$$\|f_k(t)\|_{V_1'} \le c(\|v^n - v^{n-1}\| + \|\theta^n - \theta^{n-1}\|), \tag{5.39}$$

and thus, setting $N^* = \lfloor T^\star/k \rfloor$ and recalling that $\|\{v_0, \theta_0\}\| \le R_1$, we obtain

$$\begin{aligned} \|f_k\|^2_{L^2(0,T^*;V_1')} &= \int_0^{T^*} \|f_k(t)\|^2_{V_1'} dt = \sum_{n=1}^{N^*+1} \int_{(n-1)k}^{nk} \|f_k(t)\|^2_{V_1'} dt \\ &\le kK_7(T^*) \quad (\text{by } (5.39), (4.18), (4.50)), \end{aligned} \tag{5.40}$$

which proves (5.32).

Now let $\theta \in V_2$ be such that $\|\theta\| \le 1$, and let $t \in [(n-1)k, nk)$ be fixed. Using property (2.36) of the trilinear form b_2, we have

$$\begin{aligned} &|b_2(\tilde{v}_k(t), \tilde{\theta}_k(t), \theta) - b_2(v_k(t), \theta_k(t), \theta)| \\ &= |b_2(\tilde{v}_k(t) - v_k(t), \tilde{\theta}_k(t), \theta) + b_2(v_k(t), \tilde{\theta}_k(t) - \theta_k(t), \theta)| \\ &\le c_b(\|\tilde{v}_k(t) - v_k(t)\|\|\tilde{\theta}_k(t)\| + \|v_k(t)\|\|\tilde{\theta}_k(t) - \theta_k(t)\|)\|\theta\| \\ &\le c(\|v^n - v^{n-1}\| + \|\theta^n - \theta^{n-1}\|) \quad (\text{by } (5.34), (5.35) \text{ and } \|\theta\| \le 1). \end{aligned} \tag{5.41}$$

We also have

$$\kappa|((\tilde{\theta}_k(t) - \theta_k(t), \theta))| \le \kappa\|\theta^n - \theta^{n-1}\|, \tag{5.42}$$

$$|((\tilde{v}_k(t) - v_k(t))_2, \theta)| \le \|v^n - v^{n-1}\|. \tag{5.43}$$

Relations (5.41)–(5.43) imply

$$\|g_k(t)\|_{V_2'} \le c(\|v^n - v^{n-1}\| + \|\theta^n - \theta^{n-1}\|), \tag{5.44}$$

and thus setting $N^* = \lfloor T^\star/k \rfloor$ and recalling that $\|\{v_0, \theta_0\}\| \le R_1$, we obtain

$$\begin{aligned}\|g_k\|^2_{L^2(0,T^*;V_2')} &= \int_0^{T^*} \|g_k(t)\|^2_{V_2'} dt = \sum_{n=1}^{N^*+1} \int_{(n-1)k}^{nk} \|g_k(t)\|^2_{V_2'} dt \\ &\le kK_8(T^*) \quad \text{(by (5.44), (4.18), (4.50))},\end{aligned} \tag{5.45}$$

which proves (5.33) and the proof of the lemma is complete. □

We are now able to prove that condition (H2) of Theorem 5.2 is satisfied. More precisely, we have the following

Proposition 5.3 (Finite time uniform convergence)**.** *For any $T^* > 0$ we have*

$$\lim_{k\to 0} \sup_{\{v_0,\theta_0\}\in\mathcal{A}_k,\, nk\in[0,T^*]} |S_k^n\{v_0, \theta_0\} - S(nk)\{v_0, \theta_0\}| = 0. \tag{5.46}$$

Proof. Let

$$\xi_k(t) = v(t) - \tilde{v}_k(t), \qquad \eta_k(t) = \theta(t) - \tilde{\theta}_k(t). \tag{5.47}$$

Subtracting (5.27) and (5.28) from (2.11) and (2.12) written in their week form, respectively, we obtain

$$\begin{aligned}&\left(\frac{\partial \xi_k(t)}{\partial t}, v'\right) + \nu((\xi_k(t), v')) + b_1(\xi_k(t), v(t), v') \\ &+ b_1(\tilde{v}_k(t), \xi_k(t), v') = (e_2\eta_k(t), v') - (f_k(t), v'),\ \forall v' \in V_1,\end{aligned} \tag{5.48}$$

$$\begin{aligned}&\left(\frac{\partial \eta_k(t)}{\partial t}, \theta'\right) + \kappa((\eta_k(t), \theta')) + b_2(\xi_k(t), \theta(t), \theta') \\ &+ b_2(\tilde{v}_k(t), \eta_k(t), \theta') - ((\xi_k(t))_2, \theta') = -(g_k(t), \theta'),\ \forall \theta' \in V_2.\end{aligned} \tag{5.49}$$

Replacing v' by $\xi_k(t)$ in (5.48), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\xi_k(t)|^2 + \nu\|\xi_k(t)\|^2 + b_1(\xi_k(t), v(t), \xi_k(t)) \\ &\quad = (e_2\eta_k(t), \xi_k(t)) - (f_k(t), \xi_k(t)).\end{aligned} \tag{5.50}$$

Using property (2.31) of the form b_1, we bound the nonlinear term as

$$\begin{aligned}b_1(\xi_k(t), v(t), \xi_k(t)) &\le c_b|\xi_k(t)|\|\xi_k(t)\|\|v(t)\| \\ &\le \frac{\nu}{6}\|\xi_k(t)\|^2 + \frac{c}{\nu}|\xi_k(t)|^2\|v(t)\|^2.\end{aligned} \tag{5.51}$$

Using the Cauchy–Schwarz inequality, we also have

$$\begin{aligned}|(e_2\eta_k(t), \xi_k(t))| &\le |\eta_k(t)||\xi_k(t)| \\ &\le |\eta_k(t)|\|\xi_k(t)\| \\ &\le \frac{\nu}{6}\|\xi_k(t)\|^2 + \frac{c}{\nu}|\eta_k(t)|^2,\end{aligned} \tag{5.52}$$

$$\begin{aligned}|(f_k(t), \xi_k(t))| &\le \|f_k(t)\|_{V_1'}\|\xi_k(t)\| \\ &\le \frac{\nu}{6}\|\xi_k(t)\|^2 + \frac{c}{\nu}\|f_k(t)\|^2_{V_1'}.\end{aligned} \tag{5.53}$$

Relations (5.50)–(5.53) imply

$$\begin{aligned}\frac{d}{dt}|\xi_k(t)|^2 + \nu\|\xi_k(t)\|^2 \le& \frac{c}{\nu}\|v(t)\|^2|\xi_k(t)|^2 \\ &+ \frac{c}{\nu}|\eta_k(t)|^2 + \frac{c}{\nu}\|f_k(t)\|^2_{V_1'}.\end{aligned} \tag{5.54}$$

Now replacing θ' by $\eta_k(t)$ in (5.49), we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta_k(t)|^2 + \kappa\|\eta_k(t)\|^2 + b_2(\xi_k(t),\theta(t),\eta_k(t))\\ - ((\xi_k(t))_2,\eta_k(t)) = -(g_k(t),\eta_k(t)).\end{aligned} \tag{5.55}$$

Using property (2.36) of the form b_2, we bound the nonlinear term as

$$\begin{aligned}|b_2(\xi_k(t),\theta(t),\eta_k(t))| &\le c_b|\xi_k(t)|^{1/2}\|\xi_k(t)\|^{1/2}\|\theta(t)\||\eta_k(t)|^{1/2}\|\eta_k(t)\|^{1/2}\\ &\le \frac{\nu}{6}\|\xi_k(t)\|^2 + \frac{\kappa}{6}\|\eta_k(t)\|^2\\ &\quad + \frac{c}{\nu}\|\theta(t)\|^2|\xi_k(t)|^2 + \frac{c}{\kappa}\|\theta(t)\|^2|\eta_k(t)|^2.\end{aligned} \tag{5.56}$$

Using the Cauchy–Schwarz inequality, we also have the following bounds:

$$\begin{aligned}|((\xi_k(t))_2,\eta_k(t))| &\le |\xi_k(t)||\eta_k(t)|\\ &\le \frac{\kappa}{6}\|\eta_k(t)\|^2 + \frac{c}{\kappa}|\xi_k(t)|^2,\end{aligned} \tag{5.57}$$

$$\begin{aligned}|(g_k(t),\eta_k(t))| &\le \|g_k(t)\|_{V_2'}\|\eta_k(t)\|\\ &\le \frac{\kappa}{6}\|\eta_k(t)\|^2 + \frac{c}{\kappa}\|g_k(t)\|^2_{V_2'}.\end{aligned} \tag{5.58}$$

Relations (5.55)–(5.58) imply

$$\begin{aligned}\frac{d}{dt}|\eta_k(t)|^2 + \kappa\|\eta_k(t)\|^2 \le& \frac{\nu}{3}\|\xi_k(t)\|^2 + \frac{c}{\nu}\|\theta(t)\|^2|\xi_k(t)|^2\\ &+ \frac{c}{\kappa}\|\theta(t)\|^2|\eta_k(t)|^2 + \frac{c}{\kappa}|\xi_k(t)|^2\\ &+ \frac{c}{\kappa}\|g_k(t)\|^2_{V_2'}.\end{aligned} \tag{5.59}$$

Adding equations (5.54) and (5.59), we obtain

$$\begin{aligned}&\frac{d}{dt}(|\xi_k(t)|^2 + |\eta_k(t)|^2) + \frac{2}{3}\nu\|\xi(t)\|^2 + \kappa\|\eta(t)\|^2\\ &\le \frac{c}{\nu}\left(\|v(t)\|^2 + \|\theta(t)\|^2 + \frac{\nu}{\kappa}\right)|\xi_k(t)|^2\\ &+ c\left(\frac{1}{\nu} + \frac{1}{\kappa}\|\theta(t)\|^2\right)|\eta_k(t)|^2\\ &+ \frac{c}{\nu}\|f_k(t)\|^2_{V_1'} + \frac{c}{\kappa}\|g_k(t)\|^2_{V_2'}.\end{aligned} \tag{5.60}$$

As shown in [15], the solution $\{v,\theta\}$ of the continuous problem is uniformly bounded in V for all $t \ge 0$. More precisely, we have

$$\sup_{t\ge 0}\ \sup_{\{v_0,\theta_0\}\in\mathcal{B}_1} \|S(t)\{v_0,\theta_0\}\| \le c. \tag{5.61}$$

Thus, inequality (5.60) becomes

$$\begin{aligned}&\frac{d}{dt}(|\xi_k(t)|^2 + |\eta_k(t)|^2) + \frac{2}{3}\nu\|\xi(t)\|^2 + \kappa\|\eta(t)\|^2\\ &\quad\le c(|\xi_k(t)|^2 + |\eta_k(t)|^2) + \frac{c}{\nu}\|f_k(t)\|^2_{V_1'} + \frac{c}{\kappa}\|g_k(t)\|^2_{V_2'}.\end{aligned} \tag{5.62}$$

By Gronwall's lemma and using the fact that $\xi_k(0) = \eta(0) = 0$, we obtain

$$|\xi_k(t)|^2 + |\eta_k(t)|^2 \le ce^{cT^*}(\|f_k\|^2_{L^2(0,T^*;V_1')} + \|g_k\|^2_{L^2(0,T^*;V_2')}), \tag{5.63}$$

and recalling (5.32) and (5.33), we find

$$|\xi_k(t)|^2 + |\eta_k(t)|^2 \le ck, \tag{5.64}$$

for some constant $c = c(T^*) > 0$.

We therefore have,

(5.65)

$$\begin{aligned}
&\lim_{k\to 0} \sup_{\{v_0,\theta_0\}\in\mathcal{A}_k,\, nk\in[0,T^*]} |S_k^n\{v_0,\theta_0\} - S(nk)\{v_0,\theta_0\}| \\
&= \lim_{k\to 0} \sup_{\{v_0,\theta_0\}\in\mathcal{A}_k,\, nk\in[0,T^*]} \sup_{\{v^n,\theta^n\}\in S_k^n\{v_0,\theta_0\}} |\{v^n,\theta^n\} - \{v(nk),\theta(nk)\}| \\
&= \lim_{k\to 0} \sup_{\{v_0,\theta_0\}\in\mathcal{A}_k,\, nk\in[0,T^*]} \sup_{\{v^n,\theta^n\}\in S_k^n\{v_0,\theta_0\}} |\{\tilde{v}_k(nk),\tilde{\theta}_k(nk)\} - \{v(nk),\theta(nk)\}| \\
&= \lim_{k\to 0} \sup_{\{v_0,\theta_0\}\in\mathcal{A}_k,\, nk\in[0,T^*]} \sup_{\{v^n,\theta^n\}\in S_k^n\{v_0,\theta_0\}} |\{\xi_k(nk),\eta_k(nk)\}| = 0,
\end{aligned}$$

which concludes the proof of the lemma. □

We have, therefore, proved that conditions (H1) and (H2) of Theorem 5.2 are both satisfied and thus, the long-term behavior of the semigroup $S(t)$ generated by the continuous thermohydraulics equations (2.11)–(2.12) is approximated by that of the m-semigroups generated by the discrete system (2.45)–(2.47). More precisely, we have the following result concerning the approximation of the attractor; this is our second main result:

Theorem 5.4. *The family of attractors $\{\mathcal{A}_k\}_{k\in(0,\kappa_0]}$ converges, as $k \to 0$, to $\mathcal{A}$, in the following sense:*

$$\lim_{k\to 0} \operatorname{dist}(\mathcal{A}_k, \mathcal{A}) = 0,$$

where dist *denotes the Hausdorff semidistance in H, namely*

$$\operatorname{dist}(\mathcal{A}_k, \mathcal{A}) = \sup_{x_k\in\mathcal{A}_k} \inf_{x\in\mathcal{A}} |x_k - x|.$$

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