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ON FULLY DISCRETE FINITE ELEMENT SCHEMES FOR EQUATIONS OF MOTION OF KELVIN-VOIGT FLUIDS

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Abstract. In this paper, we study two fully discrete schemes for the equations of motion arising in the Kelvin-Voigt model of viscoelastic fluids. Based on a backward Euler method in time and a finite element method in spatial direction, optimal error estimates which exhibit the exponential decay property in time are derived. In the later part of this article, a second order two step backward difference scheme is applied for temporal discretization and again exponential decay in time for the discrete solution is discussed. Finally, a *priori* error estimates are derived and results on numerical experiments conforming theoretical results are established.

Key words. Viscoelastic fluids, Kelvin-Voigt model, a *priori* bounds, backward Euler method, second order backward difference scheme, optimal error estimates.

1. Introduction

In this article, we discuss the convergence of the backward Euler method and the second order backward difference scheme for the following system of equations of motion arising in the Kelvin-Voigt fluids (see [18]):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \kappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad x\in\Omega,\ t>0, \tag{1.1}$$

and incompressibility condition

$$\nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Omega,\, t>0, \tag{1.2}$$

with initial and boundary conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega,\ t\geq 0, \tag{1.3}$$

where, Ω is a bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$. Here $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ represents the velocity vector, $p = p(\mathbf{x},t)$ the pressure and $\nu > 0$, the kinematic coefficient of viscosity. Moreover, the velocity of the fluid, after instantaneous removal of the stress, does not vanish instantaneously but dies out like $exp(\kappa^{-1}t)$ (see [18]), where κ is the retardation parameter. For details of the physical background and its mathematical modeling, we refer to [6]-[7] and [9]. Throughout this paper, we assume that the right hand side function $\mathbf{f} = 0$. In fact, assuming conservative force, the function $\mathbf{f}$ can be absorbed in the pressure term. Based on the analysis of Ladyzenskaya [16] for the solvability of the Navier Stokes equations, Oskolkov [17, 18], has proved the global existence of a unique 'almost' classical solution in finite time interval for the initial and boundary value problem (1.1)-(1.3). The investigations on solvability are further continued by him and his collaborators, see [20] and [21] and they have discussed the existence and uniqueness results on the entire semiaxis $\mathbb{R}^+$ in time.
For the related literature on the time discretization of equations of motion arising in the viscoelastic model of Oldroyd type see [2], [12], [23] and [25]-[28]. Interestingly, there is hardly any work devoted to the time discretization of (1.1)-(1.3). For the earlier results on the numerical approximations to the solutions of the problem

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(1.1)-(1.3), we refer to [3] and [19]. Under the condition that the solution is asymptotically stable as $t \to \infty$, the authors of [19] have established the convergence of spectral Galerkin approximations for the semi axis $t \geq 0$. Recently, Bajpai et al. [3] have applied finite element methods to discretize the spatial variables and derived optimal error bounds for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$- norm. In [3] and [19], only semidiscrete approximations for (1.1)-(1.3) are discussed, keeping the time variable continuous. In this article, we have discussed both backward Euler method and two step backward difference scheme for the time discretization and have derived optimal error estimates. We have also discussed briefly, the proof of linearized backward Euler method applied to (1.1)-(1.3) for time discretization. More precisely, we have

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k) \quad j = 0,\, 1,$$

and

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the backward Euler or linearized backward Euler method.
In the later part of this article, we have proved the following result for a second order backward difference scheme:

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k^2) \quad j = 0,\, 1,$$

and

$$\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k^{2-\gamma}),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the second order backward difference scheme and

$$\gamma = \begin{cases} 0 & \text{if } n \geq 2; \\ 1 & \text{if } n = 1. \end{cases}$$

The remaining part of this paper is organized as follows. In Section **2**, we discuss the preliminaries. In Section **3**, we derive *a priori* bounds for the semidiscrete solutions and present some spatial error estimates required for error analysis. In Section **4**, we obtain a *priori* bounds for the discrete solution and prove the existence and uniqueness of the discrete solution. In Section **5**, we establish the error estimates for the velocity and pressure of the backward Euler method. Section **6** deals with the error estimates for velocity and pressure using the second order backward difference scheme. In Section **7**, we provide some numerical results to confirm our theoretical results.

2. Preliminaries

For the mathematical formulation of (1.1)-(1.3), we denote $\mathbb{R}^d,\ (d = 2, 3)$-valued function spaces using boldface letters. That is,

$$\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \ \text{ and } \ \mathbf{H}^m = (H^m(\Omega))^d,$$

where $L^2(\Omega)$ is the space of square integrable functions defined in Ω. The space $L^2(\Omega)$ is a Hilbert space endowed with the usual scalar product $(\phi, \psi) = \int_\Omega \phi(x)\psi(x)\, dx$ and the associated norm $\|\phi\| = \left(\int_\Omega |\phi(x)|^2\, dx\right)^{1/2}$. Further, $H^m(\Omega)$

is the standard Hilbert Sobolev space of order $m \in \mathbb{N}^+$ with norm $\|\phi\|_m = \left(\sum_{|\alpha|\le m}\int_\Omega |D^\alpha \phi|^2\, dx\right)^{1/2}$. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^{d} (\partial_j v_i, \partial_j v_i)\right)^{1/2} = \left(\sum_{i=1}^{d} (\nabla v_i, \nabla v_i)\right)^{1/2}.$$

We also use the following spaces of the vector valued functions:

$$\begin{aligned} \mathbf{J}_1 &= \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0\}, \\ \mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla \cdot \boldsymbol{\phi} &= 0 \ \text{ in } \ \Omega, \ \ \boldsymbol{\phi}\cdot \mathbf{n}|_{\partial\Omega} = 0 \ \text{ holds weakly}\}, \end{aligned}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [24]. Let $H^m/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of H^m differing by constants, with norm $\|p\|_{H^m/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|p + c\|_m$. Let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$.

We need further assumptions, that is,

(**A1**). For $\mathbf{g} \in \mathbf{L^2}$, let $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/\mathbb{R}\}$ be the unique pair of solution to the steady state Stokes problem, see [24],

$$\begin{gathered} -\Delta \mathbf{v} + \nabla q = \mathbf{g}, \\ \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega, \qquad \mathbf{v}|_{\partial\Omega} = 0 \end{gathered}$$

satisfying the following regularity result:

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \le C\|\mathbf{g}\|. \tag{2.1}$$

Setting

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$$

as the Stokes operator, (**A1**) implies

$$\|\mathbf{v}\|_2 \le C\|\tilde{\Delta}\mathbf{v}\| \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.2}$$

It is easy to show that

$$\begin{aligned} \|\mathbf{v}\|^2 &\le \lambda_1^{-1}\|\nabla\mathbf{v}\|^2 \ \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ \|\nabla\mathbf{v}\|^2 &\le \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \ \ \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \end{aligned} \tag{2.3}$$

where λ_1^{-1} is a positive constant depending on the domain Ω. In fact, this is known as Poincaré inequality with λ_1^{-1} as the best possible positive constant.

(**A2**). There exists a positive constant M, such that the initial velocity $\mathbf{u}_0$ satisfies

$$\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1 \quad \text{with} \quad \|\mathbf{u}_0\|_2 \le M.$$

Moreover, we define a bilinear form $a(\cdot,\cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1$ by

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla\mathbf{v}, \nabla\boldsymbol{\phi}) \quad \forall \mathbf{v},\ \boldsymbol{\phi} \in \mathbf{H}_0^1, \tag{2.4}$$

and a trilinear form $b(\cdot,\cdot,\cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1 \times \mathbf{H}_0^1$ by

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v}\cdot\nabla\mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v}\cdot\nabla\boldsymbol{\phi}, \mathbf{w}) \ \ \forall \mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1. \tag{2.5}$$

With the help of above notations, the variational formulation of problem (1.1)-(1.3) with $f=0$ is defined as follows: Find $\mathbf{u}(t) \in \mathbf{J}_1$ such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa\, a(\mathbf{u}_t, \boldsymbol{\phi}) + \nu\, a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1 \;\; t > 0, \qquad (2.6)$$
$$\mathbf{u}(0) = \mathbf{u}_0.$$

3. Finite Element Approximation

Let $h > 0$ be a discretization parameter. Further, let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively. Assume that the subspace $\mathbf{H}_h$ and L_h satisfy the following approximation properties:

(**B1**). For each $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$, there exist approximations $i_h\mathbf{w} \in \mathbf{J_h}$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \leq K_0 h^2 \|\mathbf{w}\|_2, \quad \|q - j_h q\|_{L^2/\mathbb{R}} \leq K_0 h \|q\|_{H^1/\mathbb{R}}.$$

We define the subspace $\mathbf{J}_h$ of $\mathbf{H}_h$ as follows:

$$\mathbf{J}_h = \{\mathbf{v}_h \in \mathbf{H}_h : (\chi_h, \nabla\cdot\mathbf{v}_h) = 0 \quad \forall \chi_h \in L_h\}.$$

Note that, the space $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. The discrete analogue of the weak formulation (2.6) is as follows: find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$\begin{aligned} (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) & \\ - (p_h, \nabla\cdot\boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h&, \\ (\nabla\cdot\mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h&. \end{aligned} \qquad (3.1)$$

Equivalently, find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(3.2)\; (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.$$

Once we compute $\mathbf{u}_h(t) \in J_h$, the approximation $p_h(t) \in L_h$ to the pressure $p(t)$ can be found out by solving the following system

$$\begin{aligned} (p_h, \nabla\cdot\boldsymbol{\phi}_h) &= (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) \\ &+ b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h. \end{aligned} \qquad (3.3)$$

For solvability of the systems (3.2) and (3.3), see [3]. Uniqueness is obtained in the quotient space L_h/N_h, where

$$N_h = \{q_h \in L_h : (q_h, \nabla\cdot\boldsymbol{\phi}_h) = 0 \;\; \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

Furthermore, the pair $(\mathbf{H}_h, L_h/N_h)$ satisfies a uniform inf-sup condition:

(**B2**). For every $q_h \in L_h$, there exist a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H_h}$ and a positive constant K_1, independent of h, such that,

$$|(q_h, \nabla\cdot\boldsymbol{\phi}_h)| \geq K_1 \|\nabla\boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

As a consequence of conditions (**B1**), we have the following properties of the L^2 projection $P_h : \mathbf{L^2} \to \mathbf{J_h}$. For $\boldsymbol{\phi} \in \mathbf{J}_1$, we note that, see ([11], [13]),

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla P_h\boldsymbol{\phi}\| \leq Ch\|\nabla\boldsymbol{\phi}\|, \qquad (3.4)$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|. \qquad (3.5)$$

We now define the discrete operator $\Delta_h : \mathbf{H}_h \to \mathbf{H}_h$ through the bilinear form $a(\cdot,\cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{v}_h, \boldsymbol{\phi}_h) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.6}$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h \Delta_h$. Using Sobolev embedding theorems with Sobolev inequalities, it is a routine calculation to derive the following lemma, see page 360 of [14].

Lemma 3.1. *The trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following estimates for all $\boldsymbol{\phi}$, $\boldsymbol{\xi}$, $\chi \in \mathbf{H}_h$:*

$$|b(\boldsymbol{\phi}, \boldsymbol{\xi}, \chi)| \le C\|\nabla \boldsymbol{\phi}\|^{1/2} \|\tilde{\Delta}_h \boldsymbol{\phi}\|^{1/2} \|\nabla \boldsymbol{\xi}\| \, \|\chi\|, \tag{3.7}$$

$$|b(\boldsymbol{\phi}, \boldsymbol{\xi}, \chi)| \le C\|\nabla \boldsymbol{\phi}\| \|\nabla \boldsymbol{\xi}\|^{1/2} \|\tilde{\Delta}_h \boldsymbol{\xi}\|^{1/2} \|\chi\|, \tag{3.8}$$

$$|b(\boldsymbol{\phi}, \boldsymbol{\xi}, \chi)| \le C\|\boldsymbol{\phi}\|^{\frac{1}{2}} \|\nabla \boldsymbol{\phi}\|^{\frac{1}{2}} \|\nabla \boldsymbol{\xi}\| \|\nabla \chi\|. \tag{3.9}$$

Note that, the operator $b(\cdot,\cdot,\cdot)$ preserves the antisymmetric properties of the original nonlinear term, that is,

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h. \tag{3.10}$$

Examples of subspaces $\mathbf{H_h}$ satisfying assumptions (**B1**) and (**B2**) can be found in [4], [5] and [13].

Below, we derive some *a priori* estimates for the discrete solution $\mathbf{u}_h$ of (3.2) analogous to those known for continuous solution $\mathbf{u}$ of (2.6) (see [3]).

Lemma 3.2. *Let $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ and $\mathbf{u}_{0h} = P_h \mathbf{u}_0$, and the assumptions* (**A1**)–(**A2**) *hold true. Then, the solution $\mathbf{u}_h$ of (3.2) satisfies*

$$\|\mathbf{u}_h(t)\|^2 + \kappa\|\nabla \mathbf{u}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_h(t)\|^2$$
$$+ \beta e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|\nabla \mathbf{u}_h(s)\|^2 + \|\tilde{\Delta}_h \mathbf{u}_h(s)\|^2)\, ds \le C(\kappa, \nu, \alpha, \lambda_1, M) e^{-2\alpha t} \quad t > 0,$$

where $\beta = \nu - 2\alpha(\lambda_1^{-1} + \kappa) > 0$.

Proof. Setting $\hat{\mathbf{u}}_h(t) = e^{\alpha t} \mathbf{u}_h(t)$ for some $\alpha \ge 0$, we rewrite (3.2) as

$$\begin{aligned}(\hat{\mathbf{u}}_{ht}, \boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) + \kappa(\nabla \hat{\mathbf{u}}_{ht}, \nabla \boldsymbol{\phi}_h) - \kappa\alpha(\nabla \hat{\mathbf{u}}_h, \nabla \boldsymbol{\phi}_h)& \\ + \nu(\nabla \hat{\mathbf{u}}_h, \nabla \boldsymbol{\phi}_h) + e^{-\alpha t} b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.&\end{aligned} \tag{3.11}$$

Choose $\boldsymbol{\phi}_h = \hat{\mathbf{u}}_h$ in (3.11). Using (3.10), $b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h) = 0$ and from (2.3), we find that

$$\frac{d}{dt}(\|\hat{\mathbf{u}}_h\|^2 + \kappa\|\nabla \hat{\mathbf{u}}_h\|^2) + 2\big(\nu - \alpha\big(\kappa + \frac{1}{\lambda_1}\big)\big)\|\nabla \hat{\mathbf{u}}_h\|^2 \le 0. \tag{3.12}$$

Integrate (3.12) with respect to time from 0 to t to obtain

$$\begin{aligned}\|\mathbf{u}_h\|^2 + \kappa\|\nabla \mathbf{u}_h\|^2 + 2\beta e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla \mathbf{u}_h(s)\|^2 ds& \\ \le e^{-2\alpha t}(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2).&\end{aligned} \tag{3.13}$$

Using the discrete Stokes operator $\tilde{\Delta}_h$, we rewrite (3.11) as

$$\begin{aligned}(\hat{\mathbf{u}}_{ht}, \boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) - \kappa(\tilde{\Delta}_h \hat{\mathbf{u}}_{ht}, \boldsymbol{\phi}_h) + \kappa\alpha(\tilde{\Delta}_h \hat{\mathbf{u}}_h, \boldsymbol{\phi}_h)& \\ - \nu(\tilde{\Delta}_h \hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) = -e^{-\alpha t} b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \boldsymbol{\phi}_h).&\end{aligned} \tag{3.14}$$

We note that $-(\hat{\mathbf{u}}_{ht}, \tilde{\Delta}_h \hat{\mathbf{u}}_h) = \frac{1}{2}\frac{d}{dt}\|\nabla \hat{\mathbf{u}}_h\|^2$. With $\boldsymbol{\phi}_h = -\tilde{\Delta}_h \hat{\mathbf{u}}_h$, (3.14) becomes

$$\frac{d}{dt}(\|\nabla \hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2) + 2(\nu - \kappa\alpha)\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2 = 2\alpha\|\nabla \hat{\mathbf{u}}_h\|^2 + 2e^{-\alpha t} b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \tilde{\Delta}_h \hat{\mathbf{u}}_h). \tag{3.15}$$

To estimate the nonlinear term on the right hand side of (3.15), a use of (3.7) yields

$$|I| = |e^{-\alpha t} b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \tilde{\Delta}_h \hat{\mathbf{u}}_h)| \le C\|\nabla \hat{\mathbf{u}}_h\|^{\frac{3}{2}} \|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^{\frac{3}{2}}. \tag{3.16}$$

Applying Young's inequality $ab \le \frac{a^p}{p\epsilon^{p/q}} + \frac{\epsilon b^q}{q}$, $a, b \ge 0$, $\epsilon > 0$ with $p = 4$ and $q = \frac{4}{3}$, we obtain

$$|I| \le C\frac{\|\nabla \hat{\mathbf{u}}_h\|^6}{4\epsilon^3} + \frac{3\epsilon}{4}\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2. \tag{3.17}$$

Choosing $\epsilon = \frac{4\nu}{3}$, we find that

$$|I| \le \frac{C}{4}\left(\frac{3}{4\nu}\right)^3 \|\nabla \hat{\mathbf{u}}_h\|^6 + \nu\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2. \tag{3.18}$$

Substitute (3.18) in (3.15) to arrive at

$$\frac{d}{dt}(\|\nabla \hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2) + (\nu - 2\alpha\kappa)\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2 \le C(\nu)\|\nabla \hat{\mathbf{u}}_h\|^6 + 2\alpha\|\nabla \hat{\mathbf{u}}_h\|^2. \tag{3.19}$$

An integration of (3.19) with respect to time from 0 to t yields

$$\begin{aligned} \|\nabla \hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h \hat{\mathbf{u}}_h\|^2 + \beta\int_0^t \|\tilde{\Delta}_h \hat{\mathbf{u}}_h(s)\|^2 ds &\le \|\nabla \mathbf{u}_{0h}\|^2 \\ &+ \kappa\|\tilde{\Delta}_h \mathbf{u}_{0h}\|^2 + C(\nu,\alpha)\int_0^t \big(\|\nabla \hat{\mathbf{u}}_h(s)\|^6 ds + \|\nabla \hat{\mathbf{u}}_h(s)\|^2\big) ds. \end{aligned} \tag{3.20}$$

Using (3.13), we bound

$$\begin{aligned} \int_0^t \|\nabla \hat{\mathbf{u}}_h(s)\|^6 ds &= \int_0^t \|\nabla \hat{\mathbf{u}}_h(s)\|^4 \|\nabla \hat{\mathbf{u}}_h(s)\|^2 ds \\ &\le C(\kappa)(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2)^2 \int_0^t \|\nabla \hat{\mathbf{u}}_h(s)\|^2 ds \\ &\le C(\kappa,\nu,\alpha,\lambda_1)(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2)^3. \end{aligned} \tag{3.21}$$

Substitute (3.21) and (3.13) in (3.20) and use stability properties of P_h to obtain

$$\begin{aligned} \|\nabla \mathbf{u}_h\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_h\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h \mathbf{u}_h(s)\|^2 ds &\le \Big(\|\nabla \mathbf{u}_{0h}\|^2 \\ &+ \kappa\|\tilde{\Delta}_h \mathbf{u}_{0h}\|^2 + C(\kappa,\nu,\alpha,\lambda_1)\big(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2\big)^3 \\ &+ C(\kappa,\nu,\alpha,\lambda_1)\big(\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2\big)\Big) e^{-2\alpha t} \\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)e^{-2\alpha t}. \end{aligned} \tag{3.22}$$

Combine (3.13) with (3.22) to complete the rest of the proof. □

In the following three lemmas, we derive *a priori* estimates involving time derivatives of the semi-discrete solution.

Lemma 3.3. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$, *such that for all* $t > 0$,

$$\|\mathbf{u}_{ht}(t)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}(s)\|^2)ds \leq Ce^{-2\alpha t}.$$

Proof. Substituting $\boldsymbol{\phi}_h = \mathbf{u}_{ht}$ in (3.2), we obtain

$$\begin{aligned}\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla\mathbf{u}_{ht}\|^2 &= -\nu(\nabla\mathbf{u}_h, \nabla\mathbf{u}_{ht}) - b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_{ht}) \\ &= I_1 + I_2, \text{ say.}\end{aligned} \tag{3.23}$$

To estimate $|I_1|$, we apply Cauchy-Schwarz's inequality and Young's inequality to arrive at

$$|I_1| \leq \frac{\nu}{2\epsilon}\|\nabla\mathbf{u}_h\|^2 + \frac{\epsilon}{2}\|\nabla\mathbf{u}_{ht}\|^2. \tag{3.24}$$

Choose $\epsilon = \kappa$ in (3.24) to obtain

$$|I_1| \leq C(\nu, \kappa)\|\nabla\mathbf{u}_h\|^2 + \frac{\kappa}{2}\|\nabla\mathbf{u}_{ht}\|^2. \tag{3.25}$$

An application of (3.7) and Young's inequality yields

$$\begin{aligned}|I_2| &\leq C\|\nabla\mathbf{u}_h\|^{\frac{1}{2}}\|\tilde{\Delta}_h\mathbf{u}_h\|^{\frac{1}{2}}\|\nabla\mathbf{u}_h\|\|\mathbf{u}_{ht}\| \\ &\leq C\|\nabla\mathbf{u}_h\|^3\|\tilde{\Delta}_h\mathbf{u}_h\| + \frac{1}{2}\|\mathbf{u}_{ht}\|^2.\end{aligned} \tag{3.26}$$

A use of (3.25), (3.26) and Lemma 3.2 in (3.23) yields

$$\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla\mathbf{u}_{ht}\|^2 \leq C(\kappa, \nu, \alpha, \lambda_1, M)e^{-2\alpha t}. \tag{3.27}$$

Next, substituting $\boldsymbol{\phi}_h = e^{2\alpha t}\mathbf{u}_{ht}$ in (3.2), we arrive at

$$e^{2\alpha t}(\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla\mathbf{u}_{ht}\|^2) = -\nu e^{2\alpha t}a(\mathbf{u}_h, \mathbf{u}_{ht}) - e^{2\alpha t}b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_{ht}). \tag{3.28}$$

Using Cauchy-Schwarz's inequality, (3.9), (2.3), Young's inequality and integrating from 0 to t with respect to time, we obtain

$$\begin{aligned}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}(s)\|^2)ds \leq C(\kappa, \nu, \lambda_1)\big(\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_h(s)\|^2 \\ + \|\nabla\mathbf{u}_h(s)\|^4)ds\big).\end{aligned} \tag{3.29}$$

A use of Lemma 3.2 to bound

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_h(s)\|^4 ds &\leq C(\kappa, \nu, \alpha, \lambda_1, M)e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_h(s)\|^2 ds \\ &\leq C(\kappa, \nu, \alpha, \lambda_1, M)e^{-2\alpha t}.\end{aligned} \tag{3.30}$$

An application of (3.30) and Lemma 3.2 in (3.29) yields

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{ht}(s)\|^2)ds \leq C(\kappa, \nu, \alpha, \lambda_1, M). \tag{3.31}$$

A combination of (3.27) and (3.31) would lead us to the desired result. □

Lemma 3.4. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\kappa, \nu, \alpha, \lambda_1, M)$ *such that for all* $t > 0$,

$$\|\mathbf{u}_{htt}(t)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \leq Ce^{-2\alpha t}.$$

Proof. Differentiation of (3.2) with respect to time yields

$$(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + b(\mathbf{u}_{ht}, \mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) = 0 \ \forall \phi_h \in \mathbf{J}_h \ \ t > 0. \tag{3.32}$$

Substitute $\boldsymbol{\phi}_h = \mathbf{u}_{htt}$ in (3.32) to obtain

$$\|\mathbf{u}_{htt}\|^2 + \kappa\|\nabla\mathbf{u}_{htt}\|^2 = -\nu a(\mathbf{u}_{ht}, \mathbf{u}_{htt}) - b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt}) - b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{u}_{htt}). \tag{3.33}$$

An application of Cauchy-Schwarz's inequality, Young's inequality, (3.9) and (2.3) yields

$$\|\mathbf{u}_{htt}\|^2 + \kappa\|\nabla\mathbf{u}_{htt}\|^2 \leq C(\kappa, \nu, \lambda_1)\big(\|\nabla\mathbf{u}_{ht}\|^2 + \|\nabla\mathbf{u}_h\|^2\|\nabla\mathbf{u}_{ht}\|^2\big). \tag{3.34}$$

With the help of estimates obtained from Lemma 3.2 and 3.3, we write

$$\|\mathbf{u}_{htt}(t)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(t)\|^2 \leq C(\kappa, \nu, \alpha, \lambda_1, M)e^{-2\alpha t}. \tag{3.35}$$

Multiply (3.34) by $e^{2\alpha t}$ and integrate with respect to time from 0 to t to arrive at

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \leq C(\kappa, \nu, \lambda_1)\bigg(\int_0^t e^{2\alpha s}\big(\|\nabla\mathbf{u}_{ht}(s)\|^2 + \|\nabla\mathbf{u}_h(s)\|^2\|\nabla\mathbf{u}_{ht}(s)\|^2\big)ds\bigg). \tag{3.36}$$

Applying the estimates from Lemmas 3.2 and 3.3, we obtain the desired result, that is,

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{htt}(s)\|^2)ds \leq C(\kappa, \nu, \alpha, \lambda_1, M). \tag{3.37}$$

A use of (3.35) and (3.37) completes the proof. □

Differentiating (3.32) with respect to time and proceeding as in the proofs of Lemmas 3.3 and 3.4, we arrive at following Lemma:

Lemma 3.5. *Let $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant $C = C(\kappa, \nu, \alpha, \lambda_1, M)$, such that for all $t > 0$,*

$$\|\mathbf{u}_{httt}(t)\|^2 + \kappa\|\nabla\mathbf{u}_{httt}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{httt}(s)\|^2 + \kappa\|\nabla\mathbf{u}_{httt}(s)\|^2)ds \leq Ce^{-2\alpha t}.$$

Before proceeding to the error analysis for time discretization, we recall the following bounds of the error $(\mathbf{u} - \mathbf{u}_h, p - p_h)$ (for a proof see [3]):

Theorem 3.1. *Let assumptions* **(A1)-(A2)** *and* **(B1)-(B2)** *be satisfied and let $\mathbf{u}_{0h} = P_h\mathbf{u}_0$. Then, there exists a positive constant C depending on λ_1, κ, ν, α and M, such that, for all $t > 0$ and for $0 \leq \alpha < \dfrac{\nu\lambda_1}{2\big(1+\lambda_1\kappa\big)}$, the following estimate holds true :*

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| + h\|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\| + h\|(p - p_h)(t)\|_{L^2/N_h} \leq Ch^2e^{-\alpha t}.$$

Remark. For similar semidiscrete error estimates of the viscoelastic model of Oldroyd type, we refer [22].

4. Backward Euler Method

In this section, we consider a backward Euler method for time discretization of the finite element Galerkin approximation (3.1). Let $\{t_n\}_{n=0}^N$ be a uniform partition of $[0,T]$, and $t_n = nk$, with time step $k > 0$. For smooth function $\boldsymbol{\phi}$ defined on $[0,T]$, set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$ and $\bar{\partial}_t \boldsymbol{\phi}^n = \frac{(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1})}{k}$.
Now, the backward Euler method applied to (3.1) determines a sequence of functions $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{H}_h$ and $\{P^n\}_{n\geq 1} \in L_h$ as solutions of the following recursive nonlinear algebraic equations:

$$
\begin{aligned}
(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h)& \\
+ b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = (P^n, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,& \\
(\nabla \cdot \mathbf{U}^n, \chi_h) = 0 \qquad \forall \chi_h \in L_h,& \\
\mathbf{U}^0 = \mathbf{u}_{0h}.&
\end{aligned} \tag{4.1}
$$

Equivalently, for $\boldsymbol{\phi}_h \in \mathbf{J}_h$, we seek $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ such that

$$
\begin{aligned}
(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,& \\
\mathbf{U}^0 = \mathbf{u}_{0h}.&
\end{aligned} \tag{4.2}
$$

Now, to study the issue of the existence and uniqueness of the discrete solutions $\{\mathbf{U}^n\}_{n\geq 1}$, we derive a *priori* bounds for the solution $\{\mathbf{U}^n\}_{n\geq 1}$.

Lemma 4.1. *With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, *choose* k_0 *so that for* $0 < k \leq k_0$

$$
\frac{\nu k \lambda_1}{\kappa\lambda_1 + 1} + 1 > e^{\alpha k}. \tag{4.3}
$$

Then the discrete solution $\mathbf{U}^N$, $N \geq 1$ *of (4.2) satisfies*

$$
(\|\mathbf{U}^N\|^2 + \kappa\|\nabla \mathbf{U}^N\|^2) + 2\beta_1 e^{-2\alpha t_N}\, k\sum_{n=1}^{N} e^{2\alpha t_n}\|\nabla \mathbf{U}^n\|^2 \leq e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla \mathbf{U}^0\|^2),
$$

where

$$
\beta_1 = \left(e^{-\alpha k}\nu - 2\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\right) > 0. \tag{4.4}
$$

Proof. Multiplying (4.2) by $e^{\alpha t_n}$ and setting $\hat{\mathbf{U}}^n = e^{\alpha t_n}\mathbf{U}^n$, we obtain

$$
\begin{aligned}
e^{\alpha t_n}\left((\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h)\right) + \nu a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h)& \\
+ e^{-\alpha t_n} b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.&
\end{aligned} \tag{4.5}
$$

Note that,

$$
e^{\alpha t_n}\bar{\partial}_t \mathbf{U}^n = e^{\alpha k}\bar{\partial}_t \hat{\mathbf{U}}^n - \left(\frac{e^{\alpha k}-1}{k}\right)\hat{\mathbf{U}}^n. \tag{4.6}
$$

Using (4.6) in (4.5) and multiplying the resulting equation by $e^{-\alpha k}$, we obtain

$$
\begin{aligned}
(\bar{\partial}_t \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) - \left(\frac{1-e^{-\alpha k}}{k}\right)(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + e^{-\alpha k}\nu a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h)& \\
- \kappa\left(\frac{1-e^{-\alpha k}}{k}\right) a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) + e^{-\alpha t_{n+1}} b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n, \boldsymbol{\phi}_h) = 0.&
\end{aligned} \tag{4.7}
$$

Note that

$$(\bar{\partial}_t \hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) \geq \frac{1}{2k}(\|\hat{\mathbf{U}}^n\|^2 - \|\hat{\mathbf{U}}^{n-1}\|^2) = \frac{1}{2}\bar{\partial}_t \|\hat{\mathbf{U}}^n\|^2. \tag{4.8}$$

Substituting $\boldsymbol{\phi}_h = \hat{\mathbf{U}}^n$ in (4.7) and using (2.3) along with (3.10) yields

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^n\|^2) + \left(e^{-\alpha k}\nu - (\frac{1-e^{-\alpha k}}{k})\big(\kappa + \frac{1}{\lambda_1}\big)\right)\|\nabla\hat{\mathbf{U}}^n\|^2 \leq 0. \tag{4.9}$$

Note that, the coefficient of the second term on the left hand side is greater than β_1. With $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, choose $k_0 > 0$ such that for $0 < k \leq k_0$

$$\frac{\nu k \lambda_1}{1+\kappa\lambda_1} + 1 > e^{\alpha k}.$$

Then, for $0 < k \leq k_0$, the coefficient β_1 (see (4.4)) of the second term on the left hand side of (4.9) becomes positive. Multiplying (4.9) by $2k$ and summing over $n = 1$ to N, we obtain

$$\|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + 2\beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \leq \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2. \tag{4.10}$$

Divide (4.10) by $e^{2\alpha t_N}$ to complete the rest of the proof. □

Theorem 4.1. *(Brouwer's fixed point theorem)*[15]. *Let $\mathbf{H}$ be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and $\|\cdot\|$. Let $g : \mathbf{H} \to \mathbf{H}$ be a continuous function. If there exists $R > 0$ such that $(g(z), z) > 0$ $\forall z$ with $\|z\| = R$, then there exists $z^* \in \mathbf{H}$ such that $\|z\| \leq R$ and $g(z^*) = 0$.*

Now, we are ready to prove the following existence and uniqueness result.

Theorem 4.2. *Given $\mathbf{U}^{n-1}$, the discrete problem (4.2) has a unique solution $\mathbf{U}^n$, $n \geq 1$.*

Proof. Given $\mathbf{U}^{n-1}$, define a function $\mathbb{F} : \mathbf{J}_h \to \mathbf{J}_h$ for a fixed $'n'$ by

$$\begin{aligned}(\mathbb{F}(\mathbf{v}), \boldsymbol{\phi}_h) &= (\mathbf{v}, \boldsymbol{\phi}_h) + \kappa(\nabla\mathbf{v}, \nabla\boldsymbol{\phi}_h) + k\nu(\nabla\mathbf{v}, \nabla\boldsymbol{\phi}_h) \\ &\quad + k\, b(\mathbf{v}, \mathbf{v}, \boldsymbol{\phi}_h) - (\mathbf{U}^{n-1}, \boldsymbol{\phi}_h) - \kappa(\nabla\mathbf{U}^{n-1}, \nabla\boldsymbol{\phi}_h).\end{aligned} \tag{4.11}$$

Define a norm on $\mathbf{J}_h$ as

$$|||\mathbf{v}||| = (\|\mathbf{v}\|^2 + \kappa\|\nabla\mathbf{v}\|^2)^{\frac{1}{2}}. \tag{4.12}$$

We can easily show that $\mathbb{F}$ is continuous. Now, after substituting $\boldsymbol{\phi}_h = \mathbf{v}$ in (4.11), we use (3.10), (4.12), Cauchy-Schwarz's inequality and Young's inequality to arrive at

$$(\mathbb{F}(\mathbf{v}), \mathbf{v}) \geq (|||\mathbf{v}||| - |||\mathbf{U}^{n-1}|||)|||\mathbf{v}|||.$$

Choose R such that $|||\mathbf{v}||| = R$ and $R - |||\mathbf{U}^{n-1}||| > 0$ and hence,

$$(\mathbb{F}(\mathbf{v}), \mathbf{v}) > 0.$$

A use of Theorem 4.1 would provide us the existence of $\{\mathbf{U}^n\}_{n\geq 1}$.
Now, to prove uniqueness, set $\mathbf{E}^n = \mathbf{U}_1^n - \mathbf{U}_2^n$, where $\mathbf{U}_1^n$ and $\mathbf{U}_2^n$ are the solutions of (4.2).
Note that,

$$\begin{aligned}&(\bar{\partial}_t\mathbf{E}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{E}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{E}^n, \boldsymbol{\phi}_h) \\ &= b(\mathbf{U}_2^n, \mathbf{U}_2^n, \boldsymbol{\phi}_h) - b(\mathbf{U}_1^n, \mathbf{U}_1^n, \boldsymbol{\phi}_h).\end{aligned} \tag{4.13}$$

Using $\boldsymbol{\phi}_h = \hat{\mathbf{E}}^n$ and proceeding as in the derivation of (4.9), we obtain

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{E}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^n\|^2) + \Big(e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{E}}^n\|^2 \leq e^{-\alpha k}e^{\alpha t_n}\Lambda_1^n(\hat{\mathbf{E}}^n), \tag{4.14}$$

where

$$\Lambda_1^n(\hat{\mathbf{E}}^n) = -b(\mathbf{U}_1^n, \mathbf{U}_1^n, \hat{\mathbf{E}}^n) + b(\mathbf{U}_2^n, \mathbf{U}_2^n, \hat{\mathbf{E}}^n).$$

Note that,

$$\begin{aligned} e^{\alpha t_n}\Lambda_1^n(\hat{\mathbf{E}}^n) &= e^{-\alpha t_n}|b(\hat{\mathbf{U}}_1^n, \hat{\mathbf{U}}_1^n, \hat{\mathbf{E}}^n) - b(\hat{\mathbf{U}}_2^n, \hat{\mathbf{U}}_2^n, \hat{\mathbf{E}}^n)| \\ &= e^{-\alpha t_n}|b(\hat{\mathbf{E}}^n, \hat{\mathbf{U}}_1^n, \hat{\mathbf{E}}^n) + b(\hat{\mathbf{U}}_2^n, \hat{\mathbf{E}}^n, \hat{\mathbf{E}}^n)|. \end{aligned} \tag{4.15}$$

A use of (3.10), (3.9) and (2.3) in (4.15) yields

$$\begin{aligned} e^{\alpha t_n}|\Lambda_1^n(\hat{\mathbf{E}}^n)| &\leq Ce^{-\alpha t_n}\|\hat{\mathbf{E}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{E}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}_1^n\|\|\nabla\hat{\mathbf{E}}^n\| \\ &\leq C(\lambda_1)e^{-\alpha t_n}\|\nabla\hat{\mathbf{U}}_1^n\|\|\nabla\hat{\mathbf{E}}^n\|^2. \end{aligned} \tag{4.16}$$

Using (4.16), $\mathbf{E}^0 = 0$, Young's inequality in (4.14), multiplying by $2k$, summing over $n = 1$ to N and applying the bounds of Lemma 4.1, we arrive at

$$\begin{aligned} \|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2 &\leq C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N-1} e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}_1^n\|^2\|\nabla\hat{\mathbf{E}}^n\|^2 \\ &\quad + C(\nu,\lambda_1)ke^{-\alpha k}e^{-2\alpha t_N}\|\nabla\hat{\mathbf{U}}_1^N\|^2\|\nabla\hat{\mathbf{E}}^N\|^2 \\ &\leq C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=0}^{N-1} e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}_1^n\|^2\|\nabla\hat{\mathbf{E}}^n\|^2 \\ &\quad + C(\nu,\lambda_1,\kappa,M)ke^{-\alpha k}(\|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2). \end{aligned} \tag{4.17}$$

Since, $(1 - C(\nu,\lambda_1,\kappa,M)ke^{-\alpha k})$ can be made positive for small k, an application of the discrete Gronwall's Lemma and Lemma 4.1 in (4.17) yields

$$\|\hat{\mathbf{E}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{E}}^N\|^2 \leq 0 \tag{4.18}$$

and this provides the uniqueness of the solutions $\{\mathbf{U}^n\}_{n\geq 1}$. □

5. Error Analysis for Backward Euler Method

In this section, we obtain the $\mathbf{H}^1$ and L^2- norm estimates for the error $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h(t_n) = \mathbf{U}^n - \mathbf{u}_h^n$ and the L^2- norm estimate for the error $\rho^n = P^n - p_h(t_n) = P^n - p_h^n$. The following theorem provides a bound on the error $\mathbf{e}^n$:

Theorem 5.1. *Let $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ and $k_0 > 0$ be such that for $0 < k \leq k_0$, (4.3) is satisfied. For some fixed $h > 0$, let $\mathbf{u}_h(t)$ satisfy (3.2). Then, there exists a positive constant C, independent of k, such that for $n = 1, 2, \cdots, N$*

$$\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + \beta_1 ke^{-2\alpha t_n}\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_i\|^2 \leq Ck^2e^{-2\alpha t_n} \tag{5.1}$$

and

$$\|\bar{\partial}_t\mathbf{e}^n\|^2 + \|\bar{\partial}_t\nabla\mathbf{e}^n\|^2 \leq Ck^2e^{-2\alpha t_n}. \tag{5.2}$$

Proof. Consider (3.2) at $t = t_n$ and subtract it from (4.2) to obtain

$$(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) = (\sigma_1^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_1^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.3}$$

where $\sigma_1^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n$ and $\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h)$. Multiplying (5.3) by $e^{\alpha t_n}$, we arrive at

$$(e^{\alpha t_n} \bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(e^{\alpha t_n} \bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) = (e^{\alpha t_n} \sigma_1^n, \boldsymbol{\phi}_h) + \kappa a(e^{\alpha t_n} \sigma_1^n, \boldsymbol{\phi}_h) + e^{\alpha t_n} \Lambda_h(\boldsymbol{\phi}_h). \tag{5.4}$$

Note that,

$$e^{\alpha t_n} \bar{\partial}_t \mathbf{e}^n = e^{\alpha k} \bar{\partial}_t \hat{\mathbf{e}}^n - \big(\frac{e^{\alpha k} - 1}{k}\big) \hat{\mathbf{e}}^n. \tag{5.5}$$

Using (5.5) in (5.4) and dividing the resulting equation by $e^{\alpha k}$, we obtain

$$\begin{aligned}(\bar{\partial}_t \hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) &+ \kappa a(\bar{\partial}_t \hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - \big(\frac{1 - e^{-\alpha k}}{k}\big)(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) \\ &- \big(\frac{1 - e^{-\alpha k}}{k}\big) \kappa a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) + \nu e^{-\alpha k} a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) = e^{-\alpha k}(e^{\alpha t_n} \sigma_1^n, \boldsymbol{\phi}_h) \\ &+ e^{-\alpha k} \kappa a(e^{\alpha t_n} \sigma_1^n, \boldsymbol{\phi}_h) + e^{-\alpha k} e^{\alpha t_n} \Lambda_h(\boldsymbol{\phi}_h).\end{aligned} \tag{5.6}$$

Substitute $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ in (5.6). A use of (2.3) yields

$$\begin{aligned}\frac{1}{2} \bar{\partial}_t \big(\|\hat{\mathbf{e}}^n\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^n\|^2\big) &+ \bigg(\nu e^{-\alpha k} - \big(\frac{1 - e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg) \|\nabla \hat{\mathbf{e}}^n\|^2 \\ &= e^{-\alpha k}(e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) + e^{-\alpha k} \kappa a(e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) + e^{-\alpha k} e^{\alpha t_n} \Lambda_h(\hat{\mathbf{e}}^n).\end{aligned} \tag{5.7}$$

On multiplying (5.7) by $2k$ and summing over $n = 1$ to N, we observe that

$$\begin{aligned}\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 &+ 2k \bigg(\nu e^{-\alpha k} - \big(\frac{1 - e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg) \sum_{n=1}^N \|\nabla \hat{\mathbf{e}}^n\|^2 \\ &\le 2k e^{-\alpha k} \sum_{n=1}^N (e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) + 2k e^{-\alpha k} \sum_{n=1}^N \kappa a(e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) \\ &+ 2k e^{-\alpha k} \sum_{n=1}^N e^{\alpha t_n} \Lambda_h(\hat{\mathbf{e}}^n) = I_1^N + I_2^N + I_3^N, \text{ say.}\end{aligned} \tag{5.8}$$

Using Cauchy-Schwarz's inequality, (2.3) and Young's inequality, we estimate I_1^N as:

$$\begin{aligned}|I_1^N| &\le 2k e^{-\alpha k} \sum_{n=1}^N \|e^{\alpha t_n} \sigma_1^n\| \|\hat{\mathbf{e}}^n\| \\ &\le C(\nu, \lambda_1) k e^{-\alpha k} \sum_{n=1}^N \|e^{\alpha t_n} \sigma_1^n\|^2 + \frac{\nu}{3} k e^{-\alpha k} \sum_{n=1}^N \|\nabla \hat{\mathbf{e}}^n\|^2.\end{aligned} \tag{5.9}$$

Now, using the Taylor series expansion of $\mathbf{u}_h$ around t_n in the interval (t_{n-1}, t_n), we observe that

$$\|e^{\alpha t_n} \sigma_1^n\|^2 \le e^{2\alpha t_n} \frac{1}{k^2} \bigg(\int_{t_{n-1}}^{t_n} (t_n - s) \|\mathbf{u}_{htt}(s)\| ds\bigg)^2. \tag{5.10}$$

An application of Cauchy-Schwarz's inequality in (5.10) yields

$$\begin{aligned}\|e^{\alpha t_n}\sigma_1^n\|^2 &\leq \frac{1}{k^2}\bigg(\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|^2 ds\bigg)\bigg(\int_{t_{n-1}}^{t_n}(t_n-s)^2 ds\bigg)\\ &= \frac{k}{3}\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}(t)\|^2\,dt \qquad (5.11)\end{aligned}$$

and hence, using (5.11) and Lemma 3.4, we write

$$\begin{aligned}k\sum_{n=1}^N\|e^{\alpha t_n}\sigma_1^n\|^2 &\leq \frac{k^2}{3}\sum_{n=1}^N\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|^2\,ds\\ &= \frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^N\int_{t_{n-1}}^{t_n} e^{2\alpha t_{n-1}}\|\mathbf{u}_{htt}(s)\|^2\,ds\\ &\leq \frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^N\int_{t_{n-1}}^{t_n} e^{2\alpha s}\|\mathbf{u}_{htt}(s)\|^2\,ds\\ &= \frac{k^2}{3}e^{2\alpha k}\int_0^{t_N} e^{2\alpha s}\|\mathbf{u}_{htt}(s)\|^2\,ds\\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 e^{-2\alpha t_{N-1}}. \qquad (5.12)\end{aligned}$$

Similarly, we obtain

$$k\sum_{n=1}^N\|e^{\alpha t_n}\nabla\sigma_1^n\|^2 \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 e^{-2\alpha t_{N-1}}. \qquad (5.13)$$

Using (5.12) in (5.9), we find that

$$|I_1^N| \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 + \frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^N\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (5.14)$$

Following the similar steps as for bounding $|I_1^N|$ and using (5.13), we obtain

$$|I_2^N| \leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 + \frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^N\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (5.15)$$

To estimate I_3^N, we note that

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &= b(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h) - b(\mathbf{U}^n-\mathbf{u}_h^n,\mathbf{U}^n,\boldsymbol{\phi}_h) - b(\mathbf{u}_h^n,\mathbf{U}^n,\boldsymbol{\phi}_h)\\ &= -b(\mathbf{u}_h^n,\mathbf{e}^n,\boldsymbol{\phi}_h) - b(\mathbf{e}^n,\mathbf{U}^n,\boldsymbol{\phi}_h). \qquad (5.16)\end{aligned}$$

Hence, we find that

$$e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| = e^{-\alpha t_n}\Big| - b(\hat{\mathbf{e}}^n,\hat{\mathbf{U}}^n,\hat{\mathbf{e}}^n)\Big|. \qquad (5.17)$$

The first term of (5.16) vanish because of (3.10). A use of the generalized Holder's inequality and Sobolev's embedding theorems in (5.17) yields

$$\begin{aligned}e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| &\leq Ce^{-\alpha t_n}\|\hat{\mathbf{e}}^n\|^{\frac12}\|\nabla\hat{\mathbf{e}}^n\|^{\frac12}\|\nabla\hat{\mathbf{U}}^n\|\|\hat{\mathbf{e}}^n\|^{\frac12}\|\nabla\hat{\mathbf{e}}^n\|^{\frac12}\\ &\leq Ce^{-\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|\|\hat{\mathbf{e}}^n\|\|\nabla\hat{\mathbf{e}}^n\|. \qquad (5.18)\end{aligned}$$

Using Young's inequality, we arrive at

$$|I_3^N| \le C(\nu)\sum_{n=1}^{N-1} ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\,\|\hat{\mathbf{e}}^n\|^2 + C(\nu)ke^{-\alpha k}e^{-2\alpha t_N}\|\nabla\hat{\mathbf{U}}^N\|^2\,\|\hat{\mathbf{e}}^N\|^2$$

$$+\frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.19}$$

An application of Lemma 4.1 to estimate the second term on the right hand side of (5.19) yields

$$|I_3^N| \le C(\nu)\sum_{n=1}^{N-1} ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\,\|\hat{\mathbf{e}}^n\|^2 + C(\nu,M)ke^{-\alpha k}e^{-2\alpha t_N}\,\|\hat{\mathbf{e}}^N\|^2$$

$$+\frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.20}$$

A use of (5.14), (5.15) and (5.20) in (5.8) with $\mathbf{e}^0=0$ yields

$$\begin{aligned}\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2 \quad &+\beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\\ &+C(\nu)ke^{-\alpha k}\sum_{n=0}^{N-1}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2\\ &+C(\nu,M)ke^{-\alpha k}(\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2).\end{aligned} \tag{5.21}$$

Now choose $k_0>0$ such that for $0<k<k_0$, $(1-C(\nu,M)ke^{-\alpha k})>0$ and (4.3) is satisfied. Then, an application of the discrete Gronwall's Lemma yields

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+\beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\times \exp\Big(k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}^n\|^2\Big). \tag{5.22}$$

With the help of Lemma 4.1, we bound

$$k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M). \tag{5.23}$$

Using (5.23) in (5.22), we arrive at

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+\beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2. \tag{5.24}$$

For $0<k\le k_0$, the coefficient of the third term on the left-hand side of (5.24), becomes positive. Dividing (5.24) by $e^{2\alpha t_N}$, we obtain (5.1).
Next, we take $\boldsymbol{\phi}_h=\bar{\partial}_t\hat{\mathbf{e}}^n$ in (5.6) and obtain

$$\begin{aligned}\|\bar{\partial}_t\hat{\mathbf{e}}^n\|^2+\kappa\|\bar{\partial}_t\nabla\hat{\mathbf{e}}^n\|^2 &= \big(\frac{1-e^{-\alpha k}}{k}\big)(\hat{\mathbf{e}}^n,\bar{\partial}_t\hat{\mathbf{e}}^n)\\ &-\big(\nu e^{-\alpha k}-\kappa\big(\frac{1-e^{-\alpha k}}{k}\big)\big)a(\hat{\mathbf{e}}^n,\bar{\partial}_t\hat{\mathbf{e}}^n)+e^{-\alpha k}(e^{\alpha t_n}\sigma_1^n,\bar{\partial}_t\hat{\mathbf{e}}^n)\\ &+e^{-\alpha k}\kappa\ a(e^{\alpha t_n}\sigma_1^n,\bar{\partial}_t\hat{\mathbf{e}}^n)+e^{-\alpha k}e^{\alpha t_n}\Lambda_h(\bar{\partial}_t\hat{\mathbf{e}}^n).\end{aligned} \tag{5.25}$$

Using (5.16), (3.7), (3.9) and (2.3), we observe that

$$e^{\alpha t_n}|\Lambda_h(\boldsymbol{\phi}_h)| = e^{-\alpha t_n}|b(\hat{\mathbf{u}}_h^n,\hat{\mathbf{e}}^n,\boldsymbol{\phi}_h)+b(\hat{\mathbf{e}}^n,\hat{\mathbf{U}}^n,\boldsymbol{\phi}_h)|$$
$$\leq C(\lambda_1)e^{-\alpha t_n}\bigg(\|\nabla\hat{\mathbf{u}}_h^n\|^{\frac12}\|\tilde{\Delta}_h\hat{\mathbf{u}}_h^n\|^{\frac12}+\|\nabla\hat{\mathbf{U}}^n\|\bigg)\|\nabla\hat{\mathbf{e}}^n\|\|\nabla\boldsymbol{\phi}_h\|.$$

With the help of Lemmas 3.2 and 4.1, we bound

$$e^{\alpha t_n}|\Lambda_h(\boldsymbol{\phi}_h)| \leq C(\kappa,\nu,\alpha,\lambda_1,M)\|\nabla\hat{\mathbf{e}}^n\|\|\nabla\boldsymbol{\phi}_h\|. \tag{5.26}$$

A use of Cauchy-Schwarz's inequality, Young's inequality, (2.3) and (5.26) in (5.25) yields

$$\|\bar{\partial}_t\hat{\mathbf{e}}^n\|^2+\kappa\|\bar{\partial}_t\nabla\hat{\mathbf{e}}^n\|^2 \leq C(\kappa,\alpha,\nu,\lambda_1,M)\bigg(\|\nabla\hat{\mathbf{e}}^n\|^2+\|e^{\alpha t_n}\nabla\sigma_1^n\|^2\bigg). \tag{5.27}$$

To estimate the second term in the right hand side of (5.27), we note from (5.11) and Lemma 3.4 that

$$\begin{aligned}\|e^{\alpha t_n}\nabla\sigma_1^n\|^2 &\leq \frac{k}{3}\int_{t_{n-1}}^{t_n} e^{2\alpha t_n}\|\nabla\mathbf{u}_{htt}(t)\|^2\,dt\\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)ke^{2\alpha t_n}\int_{t_{n-1}}^{t_n} e^{-2\alpha s}ds\\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2e^{2\alpha k^*},\end{aligned} \tag{5.28}$$

for $k^*\in(0,k)$. In view of (5.1) and (5.28), (5.27) implies (5.2). This completes the rest of the proof. $\square$.

It remains to prove the error estimate for the pressure P^n. Consider (3.1) at $t=t_n$ and subtract it from (4.1) to obtain

$$\begin{aligned}(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) &= (\bar{\partial}_t\mathbf{e}^n,\boldsymbol{\phi}_h)+\kappa a(\bar{\partial}_t\mathbf{e}^n,\boldsymbol{\phi}_h)+\nu a(\mathbf{e}^n,\boldsymbol{\phi}_h)\\ &- (\sigma_1^n,\boldsymbol{\phi}_h)-\kappa a(\sigma_1^n,\boldsymbol{\phi}_h)-\Lambda_h(\boldsymbol{\phi}_h).\end{aligned}$$

Using Cauchy-Schwarz's inequality, (2.3) and (5.26), we obtain

$$(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) \leq C(\kappa,\nu,\lambda_1)\big(\|\bar{\partial}_t\nabla\mathbf{e}^n\|+\|\nabla\mathbf{e}^n\|+\|\nabla\sigma_1^n\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{5.29}$$

A use of Theorem 5.1 and (5.28) in (5.29) would lead us to the desired result, that is

$$\|\boldsymbol{\rho}^n\| \leq C(\kappa,\nu,\alpha,\lambda_1,M)k\;e^{-\alpha t_n}. \tag{5.30}$$

Remark 1. *Note that in the estimate of I_3^N, that is, the estimate (5.20), we have used Lemma 4.1 to bound only $\|\hat{\mathbf{U}}^N\|$ for the second term on the right hand side of (5.20). But we could have bounded $\|\hat{\mathbf{U}}^n\|$, $n=1,\cdots,N-1$ using again Lemma 4.1, but that would have resulted in exponential dependence of CT in the final estimate.*

The backward Euler method applied to (3.1) gives rise to a nonlinear system at $t=t_n$. Here, we introduce a linearized version of this method which solves a system of linear equations at each time step.

The linearized backward Euler method is as follows: find a sequence of functions $\{\mathbf{U}^n\}_{n\geq1}\in\mathbf{H}_h$ and $\{P^n\}_{n\geq1}\in L_h$ as solutions of the following recursive linear algebraic equations:

$$
\begin{aligned}
&(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) \\
&+ b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) = (P^n, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\
&(\nabla \cdot \mathbf{U}^n, \chi_h) = 0 \qquad \forall \chi_h \in L_h, \\
&\mathbf{U}^0 = \mathbf{u}_{0h}.
\end{aligned} \tag{5.31}
$$

Equivalently, for $\boldsymbol{\phi}_h \in \mathbf{J}_h$, we seek $\{\mathbf{U}^n\}_{n \geq 1} \in \mathbf{J}_h$ such that

$$
\begin{aligned}
&(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) \\
&+ b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \\
&\mathbf{U}^0 = \mathbf{u}_{0h}.
\end{aligned} \tag{5.32}
$$

The proof for the linearized backward Euler method proceeds along the same lines as in the derivation of Theorem 5.1. Here, the equation in error $\mathbf{e}^n$ is:

$$
\begin{aligned}
(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) &= (\sigma_1^n, \boldsymbol{\phi}_h) \\
&+ \kappa a(\sigma_1^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,
\end{aligned} \tag{5.33}
$$

where $\sigma_1^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n$ and $\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)$. Note that, the difference here is only in the nonlinear term.
Again, with the help of similar applications as in the proof of Theorem 5.1, we arrive at

$$
\begin{aligned}
\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 &+ 2\Big(\nu e^{-\alpha k} - \big(\frac{1 - e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big) k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \\
&\leq 2k e^{-\alpha k} \sum_{n=1}^{N} (e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) + 2k e^{-\alpha k} \sum_{n=1}^{N} \kappa a(e^{\alpha t_n} \sigma_1^n, \hat{\mathbf{e}}^n) \\
&+ 2k e^{-\alpha k} \sum_{n=1}^{N} e^{\alpha t_n} \Lambda_h(\hat{\mathbf{e}}^n) = I_1^N + I_2^N + I_3^N, \ \text{say}.
\end{aligned} \tag{5.34}
$$

The first two terms in the right hand side of (5.34) are bounded by (5.14) and (5.15). Hence, we need to estimate the third term. In this case, we write

$$
\begin{aligned}
e^{\alpha t_n} |\Lambda_h(\boldsymbol{\phi}_h)| &= e^{\alpha t_n} |b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^{n-1} - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) \\
&- b(\mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)| \\
&= e^{\alpha t_n} |b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) \\
&+ b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{U}^n, \boldsymbol{\phi}_h)| \\
&= e^{\alpha t_n} | - b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) \\
&- b(\mathbf{e}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{n-1}, \mathbf{e}^n, \boldsymbol{\phi}_h)|.
\end{aligned} \tag{5.35}
$$

A use of (3.10) along with (2.3) and (3.9) in (5.35) with $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ yields

$$
\begin{aligned}
e^{\alpha t_n} |\Lambda_h(\hat{\mathbf{e}}^n)| &\leq e^{\alpha t_n} |b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \hat{\mathbf{e}}^n) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \hat{\mathbf{e}}^n)| \\
&\leq C(\lambda_1) e^{\alpha t_n} \big(\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| \|\nabla \mathbf{U}^n\| \|\nabla \hat{\mathbf{e}}^n\| \\
&+ \|\nabla \mathbf{e}^{n-1}\| \|\nabla \mathbf{U}^n\| \|\nabla \hat{\mathbf{e}}^n\|\big).
\end{aligned} \tag{5.36}
$$

Hence, we observe that

$$
\begin{aligned}
|I_3^N| \quad &\leq 2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| \leq C(\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}\big(e^{\alpha t_n}\|\nabla(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})\| \times \\
&\|\nabla\mathbf{U}^n\|\|\nabla\hat{\mathbf{e}}^n\| + e^{\alpha t_n}\|\nabla\mathbf{U}^n\|\|\nabla\mathbf{e}^{n-1}\|\|\nabla\hat{\mathbf{e}}^n\|\big) = |I_4^N| + |I_5^N|, \text{ say.} \qquad (5.37)
\end{aligned}
$$

Note that, a use of Taylor's series expansion of $u_h(t)$ at t_n in the interval (t_{n-1}, t_n) yields

$$
\|\nabla(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})\| = \|\int_{t_{n-1}}^{t_n}\nabla\mathbf{u}_{ht}(s)ds\|. \tag{5.38}
$$

With the help of Lemma 3.3 and mean value theorem, we observe that

$$
\begin{aligned}
\|\nabla(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})\| &\leq \int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{ht}(s)\|\,ds \leq C(\kappa,\nu,\alpha,\lambda_1,M)\int_{t_{n-1}}^{t_n}e^{-\alpha s}ds \\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M)e^{-\alpha t_n}\frac{1}{\alpha}\left(e^{\alpha k}-1\right) \\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M)ke^{\alpha k^*}, \qquad (5.39)
\end{aligned}
$$

where $k^* \in (0,k)$.
Using Young's inequality, (5.39) and Lemma 4.1, we bound $|I_4^N|$ as

$$
\begin{aligned}
|I_4^N| &\leq C(\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2\|\nabla(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})\|^2 + \frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2\big(k\sum_{n=1}^{N}e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2\big) + \frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\
&\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^2 + \frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (5.40)
\end{aligned}
$$

A use of Young's inequality yields

$$
\begin{aligned}
|I_5^N| &= C(\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\|\nabla\mathbf{U}^n\|\|\nabla\mathbf{e}^{n-1}\|\|\nabla\hat{\mathbf{e}}^n\| \\
&\leq C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}e^{-2\alpha t_{n-1}}\|\nabla\hat{\mathbf{U}}^n\|^2\|\nabla\hat{\mathbf{e}}^{n-1}\|^2 + \frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\
&\leq C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=0}^{N-1}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^{n+1}\|^2\|\nabla\hat{\mathbf{e}}^n\|^2 + \frac{\nu}{6}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \qquad (5.41)
\end{aligned}
$$

Substitute (5.14), (5.15), (5.40) and (5.41) in (5.34). As in the estimate of (5.21), we now apply Gronwall's Lemma to complete the rest of the proof. □

Now a use of Theorems 3.1, 5.1 and (5.30) completes the proof of the following Theorem.

Theorem 5.2. *Under the assumptions of Theorems 3.1 and 5.1, the following holds true:*

$$
\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \leq Ce^{-\alpha t_n}(h^{2-j} + k) \quad j = 0,\, 1
$$

and

$$
\|(p(t_n) - P^n)\| \leq Ce^{-\alpha t_n}(h + k).
$$

6. Second Order Backward Difference Scheme

Since the backward Euler method is only first order accurate, we now try to obtain a second order accuracy by employing a second order backward difference scheme. Setting

$$D_t^{(2)}\mathbf{U}^n = \frac{1}{2k}(3\mathbf{U}^n - 4\mathbf{U}^{n-1} + \mathbf{U}^{n-2}), \tag{6.1}$$

we obtain the second order backward difference applied to (3.1) as follows: find a sequence of functions $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{H}_h$ and $\{P^n\}_{n\geq 1} \in L_h$ as solutions of the following recursive nonlinear algebraic equations:

$$\begin{aligned}(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ = (P^n, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\bar{\partial}_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \kappa a(\partial_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}^1, \mathbf{U}^1, \boldsymbol{\phi}_h) \\ = (P^1, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\nabla\cdot\mathbf{U}^n, \chi_h) = 0 \,\forall \chi_h \in L_h, \\ \mathbf{U}^0 = \mathbf{u}_{0h}.\end{aligned} \tag{6.2}$$

Equivalently, find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ to be the solutions of

$$\begin{aligned}(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = 0 \quad \forall n \geq 2 \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \\ (\bar{\partial}_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \kappa a(\partial_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h) \\ + b(\mathbf{U}^1, \mathbf{U}^1, \boldsymbol{\phi}_h) = 0, \\ \mathbf{U}^0 = \mathbf{u}_{0h}.\end{aligned} \tag{6.3}$$

The results of this section are based on the identity which is obtained by a modification of a similar identity in [1]:

$$\begin{aligned}2e^{2\alpha t_n}(a^n, 3a^n - 4a^{n-1} + a^{n-2}) = \|\hat{a}^n\|^2 - \|\hat{a}^{n-1}\|^2 \\ + (1 - e^{2\alpha k})(\|\hat{a}^n\|^2 + \|\hat{a}^{n-1}\|^2) + \|\delta^2\hat{a}^{n-1}\|^2 \\ + \|2\hat{a}^n - e^{\alpha k}\hat{a}^{n-1}\|^2 - \|2\hat{a}^{n-1} - e^{\alpha k}\hat{a}^{n-2}\|^2,\end{aligned} \tag{6.4}$$

where

$$\delta^2\hat{a}^{n-1} = e^{\alpha k}\hat{a}^n - 2\hat{a}^{n-1} + e^{\alpha k}\hat{a}^{n-2}.$$

Next, we discuss the decay properties for the solution of (6.3).

Lemma 6.1. *With $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, choose k_0 small so that for $0 < k \leq k_0$*

$$\frac{\nu k\lambda_1}{\kappa\lambda_1 + 1} + 1 > e^{2\alpha k}. \tag{6.5}$$

Then, the discrete solution $\mathbf{U}^N$, $N \geq 1$ of (6.3) satisfies the following a priori bound:

$$\begin{aligned}(\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2) + e^{-2\alpha t_N}\; k\sum_{n=1}^{N} e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2 \\ \leq C(\kappa, \nu, \alpha, \lambda_1)e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2).\end{aligned} \tag{6.6}$$

Proof. Multiply (6.3) by $e^{\alpha t_n}$ and substitute $\boldsymbol{\phi}_h = \hat{\mathbf{U}}^n$. Then, using identity (6.4), we obtain

$$\begin{aligned}
&\frac{1}{4}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2+\kappa\|\nabla\hat{\mathbf{U}}^n\|^2)+\nu\|\nabla\hat{\mathbf{U}}^n\|^2+\Big(\frac{1-e^{2\alpha k}}{4k}\Big)\Big(\|\hat{\mathbf{U}}^n\|^2+\kappa\|\nabla\hat{\mathbf{U}}^n\|^2\Big)\\
&+\Big(\frac{1-e^{2\alpha k}}{4k}\Big)\Big(\|\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{n-1}\|^2\Big)+\frac{1}{4k}\|\delta^2\hat{\mathbf{U}}^{n-1}\|^2+\frac{1}{4k}\kappa\|\delta^2\nabla\hat{\mathbf{U}}^{n-1}\|^2\\
&+\frac{1}{4k}\Big((2\hat{\mathbf{U}}^n-e^{\alpha k}\hat{\mathbf{U}}^{n-1})^2-(2\hat{\mathbf{U}}^{n-1}-e^{\alpha k}\hat{\mathbf{U}}^{n-2})^2\Big)\\
&+\frac{\kappa}{4k}\big((2\nabla\hat{\mathbf{U}}^n-e^{\alpha k}\nabla\hat{\mathbf{U}}^{n-1})^2-(2\nabla\hat{\mathbf{U}}^{n-1}-e^{\alpha k}\nabla\hat{\mathbf{U}}^{n-2})^2\big)=0.
\end{aligned}\tag{6.7}$$

Note that, the fifth and sixth terms on the left hand side of (6.7) are non-negative. Therefore, we have dropped these terms. Further, observe that

$$\begin{aligned}
\sum_{n=2}^{N}(\|\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{n-1}\|^2) &= (\|\hat{\mathbf{U}}^1\|^2+\kappa\|\nabla\hat{\mathbf{U}}^1\|^2)-(\|\hat{\mathbf{U}}^N\|^2+\kappa\|\nabla\hat{\mathbf{U}}^N\|^2)\\
&+\sum_{n=2}^{N}(\|\hat{\mathbf{U}}^n\|^2+\kappa\|\nabla\hat{\mathbf{U}}^n\|^2).
\end{aligned}\tag{6.8}$$

Multiplying (6.7) by $4ke^{-2\alpha k}$, summing over $n=2$ to N, using (2.3) and (6.8), we obtain

$$\begin{aligned}
&\|\hat{\mathbf{U}}^N\|^2+\kappa\|\nabla\hat{\mathbf{U}}^N\|^2+k\Big(4\nu e^{-2\alpha k}-2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2\\
&+\|2e^{-\alpha k}\hat{\mathbf{U}}^N-\hat{\mathbf{U}}^{N-1}\|^2+\kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^N-\nabla\hat{\mathbf{U}}^{N-1}\|^2\le\big(\|\hat{\mathbf{U}}^1\|^2\\
&\quad+\kappa\|\nabla\hat{\mathbf{U}}^1\|^2\big)+\big((2e^{-\alpha k}\hat{\mathbf{U}}^1-\mathbf{U}^0)^2+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1-\nabla\mathbf{U}^0)^2\big).
\end{aligned}\tag{6.9}$$

To estimate the first term on the right hand side, we choose $n=1$ in (4.9) to obtain

$$\frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^1\|^2+\kappa\|\nabla\hat{\mathbf{U}}^1\|^2)+\Big(e^{-\alpha k}\nu-\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{U}}^1\|^2\le 0.\tag{6.10}$$

Since $\dfrac{\nu k\lambda_1}{1+\kappa\lambda_1}+1>e^{2\alpha k}$, the coefficient of second term becomes positive. Therefore, we drop this term and obtain

$$\|\hat{\mathbf{U}}^1\|^2+\kappa\|\nabla\hat{\mathbf{U}}^1\|^2\le C(\|\mathbf{U}^0\|^2+\kappa\|\nabla\mathbf{U}^0\|^2).\tag{6.11}$$

Now, we bound the second term on the right hand side of (6.9) by using Cauchy Schwarz's inequality, Young's inequality and (6.11) as follows:

$$(2e^{-\alpha k}\hat{\mathbf{U}}^1-\mathbf{U}^0)^2+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1-\nabla\mathbf{U}^0)^2\le C(\kappa)(\|\mathbf{U}^0\|^2+\|\nabla\mathbf{U}^0\|^2).\tag{6.12}$$

Using (6.11), (6.12) and (6.5) in (6.9), we complete the rest of the proof. □

Remark. Existence of solution to (6.3) can be proved using Theorem 4.1 and Lemma 6.1.

As a consequence of Lemma 6.1, we have the following error estimates.

Theorem 6.1. *Assume that* $0\le\alpha<\dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and choose* $k_0\ge 0$ *such that for* $0<k\le k_0$,

$$\frac{\nu k\lambda_1}{1+\kappa\lambda_1}+1>e^{2\alpha k}.$$

Let $u_h(t)$ be a solution of (3.2) and $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h(t_n)$, for $n = 1, 2, \cdots, N$. Then, for some positive constant $C = C(\kappa, \nu, \alpha, \lambda_1, M)$, there hold

$$\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + ke^{-2\alpha t_n}\sum_{i=2}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_i\|^2 \le Ck^4 e^{-2\alpha t_n} \tag{6.13}$$

and for $n = 2, \cdots, N$,

$$\|D_t^2\mathbf{e}^n\|^2 + \kappa\|D_t^2\nabla\mathbf{e}^n\|^2 \le Ck^4 e^{-2\alpha t_n}. \tag{6.14}$$

Proof. The proof for error analysis is on the similar lines as that of Theorem 5.1. This time the equation in $\mathbf{e}^n$ for $n \ge 2$ is

$$\begin{aligned}(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h)\\ = (\sigma_2^n, \boldsymbol{\phi}_h) + \kappa a(\sigma_2^n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h),\end{aligned} \tag{6.15}$$

where σ_2^n and $\Lambda_h(\boldsymbol{\phi}_h)$ are defined by

$$\sigma_2^n = \mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n,\ \Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h).$$

Multiply (6.15) by $4ke^{\alpha t_n}$ and substitute $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$. Using identity (6.4), we arrive at

$$\begin{aligned}& k\bar{\partial}_t(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + \|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2 + 4k\nu\|\nabla\hat{\mathbf{e}}^n\|^2\\ &+(1-e^{2\alpha k})(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + (1-e^{2\alpha k})(\|\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^{n-1}\|^2)\\ &+(2\hat{\mathbf{e}}^n - e^{\alpha k}\hat{\mathbf{e}}^{n-1})^2 - (2\hat{\mathbf{e}}^{n-1} - e^{\alpha k}\hat{\mathbf{e}}^{n-2})^2 + \kappa(2\nabla\hat{\mathbf{e}}^n - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-1})^2\\ &-\kappa(2\nabla\hat{\mathbf{e}}^{n-1} - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-2})^2 = 4k(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,\kappa\, a(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n)\\ &+4k\, e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n).\end{aligned} \tag{6.16}$$

Summing (6.16) over $n = 2$ to N, using (6.8), $\mathbf{e}^o = 0$ and dividing by $e^{2\alpha k}$, we obtain

$$\begin{aligned}& \|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2) + (2e^{-\alpha k}\hat{\mathbf{e}}^N - \hat{\mathbf{e}}^{N-1})^2\\ &+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N - \nabla\hat{\mathbf{e}}^{N-1})^2 + k\left(4\nu e^{-2\alpha k} - 2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\\ &\le \|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2 + (2e^{-\alpha k}\hat{\mathbf{e}}^1 - \mathbf{e}^o)^2 + \kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^1 - \nabla\mathbf{e}^o)^2\\ &+4ke^{-2\alpha k}\sum_{n=2}^{N}(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4k\,\kappa e^{-2\alpha k}\sum_{n=2}^{N} a(e^{\alpha t_n}\sigma_2^n, \hat{\mathbf{e}}^n) + 4ke^{-2\alpha k}\sum_{n=2}^{N} e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n)\\ &\le C(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2) + I_1^* + I_2^* + I_3^*,\ \textit{say}.\end{aligned} \tag{6.17}$$

Now, with the help of Cauchy-Schwarz's inequality, (2.3) and Young's inequality, we bound $|I_1^*|$ as:

$$\begin{aligned}|I_1^*| &\le 4ke^{-2\alpha k}(\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2)^{\frac{1}{2}}(\sum_{n=2}^{N}\|\hat{\mathbf{e}}^n\|^2)^{\frac{1}{2}}\\ &\le C(\epsilon, \lambda_1)ke^{-2\alpha k}\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 + \epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2.\end{aligned} \tag{6.18}$$

Using $\|\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n\| \leq \frac{(k)^{\frac{3}{2}}}{\sqrt{2}} \int_{t_{n-2}}^{t_n} \|\mathbf{u}_{httt}\| dt$ ([1]), we note that

$$\|e^{\alpha t_n}\sigma_2^n\|^2 \leq \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|^2\,dt \tag{6.19}$$

and hence, using (6.19) and Lemma 3.5, we obtain

$$\begin{aligned} k\sum_{n=2}^{N}\|e^{\alpha t_n}\sigma_2^n\|^2 &\leq \frac{k^4}{2}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|^2 dt \\ &= \frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t_{n-2}}\|\mathbf{u}_{httt}(t)\|^2 dt \\ &\leq \frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n} e^{2\alpha t}\|\mathbf{u}_{httt}(t)\|^2 dt \\ &\leq k^4 e^{4\alpha k}\int_0^{t_N} e^{2\alpha t}\|\mathbf{u}_{httt}(t)\|^2 dt \\ &\leq C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k}e^{-2\alpha t_N}. \end{aligned} \tag{6.20}$$

Using (6.20) in (6.18), we arrive at

$$|I_1^*| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{6.21}$$

Similarly, with the help of Cauchy-Schwarz's inequality, Young's inequality and Lemma 3.5, we bound

$$|I_2^*| \leq C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4 + \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{6.22}$$

Once again, a use of (5.18) yields

$$\begin{aligned} |I_3^*| \leq C(\epsilon)&\sum_{n=2}^{N} k e^{-2\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2 \\ &+ \epsilon k e^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \end{aligned} \tag{6.23}$$

To bound the first term in the right hand side of (6.17), we choose $n = 1$ in (5.7) and obtain

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2\big) &+ \bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^1\|^2 \\ &= e^{-\alpha k}(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1) + e^{-\alpha k}\kappa a(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1) + e^{-\alpha k}e^{\alpha k}\Lambda_h(\hat{\mathbf{e}}^1). \end{aligned} \tag{6.24}$$

On multiplying (6.24) by $2k$, using Cauchy-Schwarz's inequality, (2.3), Young's inequality appropriately with the estimates (5.20) (for $n = 1$ and $\epsilon = \nu$), we obtain

$$\begin{aligned} \|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2 &+ 2k\bigg(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\bigg)\|\nabla\hat{\mathbf{e}}^1\|^2 \\ &\leq 2ke^{-\alpha k}(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1) + 2ke^{-\alpha k}\kappa a(e^{\alpha k}\sigma_1^1,\hat{\mathbf{e}}^1) + 2ke^{-\alpha k}e^{\alpha k}\Lambda_h(\hat{\mathbf{e}}^1) \\ &\leq Ck^2e^{-2\alpha k}\big(\|e^{\alpha k}\sigma_1^1\|^2 + \kappa\|e^{\alpha k}\nabla\sigma_1^1\|^2\big) + \frac{1}{2}(\|\hat{\mathbf{e}}^1\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^1\|^2) \\ &+ C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2\|\hat{\mathbf{e}}^1\|^2 + \nu ke^{-\alpha k}\|\nabla\hat{\mathbf{e}}^1\|^2 \end{aligned} \tag{6.25}$$

and hence, a use of (5.28) with $n=1$ along with (2.3) yields

$$\|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2+k\Big(\nu e^{-\alpha k}-2\big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{e}}^1\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M,\epsilon)k^4+C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2\|\hat{\mathbf{e}}^1\|^2. \tag{6.26}$$

Using (6.21), (6.22), (6.23) with $\epsilon=\frac{2\nu}{3}$, (6.26), $\mathbf{e}^o=0$ and bounds from Lemma 6.1 in (6.17), we obtain

$$\begin{aligned}
&\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2+\kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2)+(2e^{-\alpha k}\hat{\mathbf{e}}^N-\hat{\mathbf{e}}^{N-1})^2\\
&+\kappa(2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N-\nabla\hat{\mathbf{e}}^{N-1})^2+2k\Big(\nu e^{-2\alpha k}-\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\Big)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)\sum_{n=2}^{N}ke^{-2\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2\\
&+C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{U}}^1\|^2\|\hat{\mathbf{e}}^1\|^2\\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)\sum_{n=0}^{N-1}ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2\\
&+C(\nu)ke^{-2\alpha k}e^{-2\alpha t_N}\|\nabla\hat{\mathbf{U}}^N\|^2\|\hat{\mathbf{e}}^N\|^2\\
&\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4+C(\nu)\sum_{n=0}^{N-1}ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{U}}^n\|^2\|\hat{\mathbf{e}}^n\|^2\\
&+Cke^{-2\alpha k}(\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2).
\end{aligned} \tag{6.27}$$

Choose k_0, so that (6.5) is satisfied and $(1-Cke^{-2\alpha k})>0$ for $0<k\le k_0$. Then, an application of the discrete Gronwall's Lemma yields

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4\times \exp\big(k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{U}}^n\|^2\big). \tag{6.28}$$

The bounds obtained from Lemma 6.1 in (6.28) would lead us to (6.13), that is,

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 \tag{6.29}$$

and this completes the proof of (6.13) for $n\ge 2$. For $n=1$, we use (6.26) along with the bounds in Lemma 4.1. Then, a choice of k such that $(1-Cke^{-\alpha k})>0$ would lead us to the desired result, that is,

$$\|\mathbf{e}^1\|^2+\kappa\|\nabla\mathbf{e}^1\|^2+k\|\nabla\mathbf{e}^1\|^2\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4e^{-2\alpha k}. \tag{6.30}$$

To arrive at the estimates in (6.14), we choose $\boldsymbol{\phi}_h=D_t^{(2)}\mathbf{e}^n$ in (6.15) and obtain

$$\begin{aligned}
\|D_t^{(2)}\mathbf{e}^n\|^2+\kappa\|\nabla D_t^{(2)}\mathbf{e}^n\|^2 &= -\nu a(\mathbf{e}^n,D_t^{(2)}\mathbf{e}^n)+(\sigma_2^n,D_t^{(2)}\mathbf{e}^n)\\
&+\ \kappa a(\sigma_2^n,D_t^{(2)}\mathbf{e}^n)+\Lambda_h(D_t^{(2)}\mathbf{e}^n).
\end{aligned} \tag{6.31}$$

It follows from (5.26) that

$$|\Lambda_h(D_t^{(2)}\mathbf{e}^n)| \le C(\kappa,\nu,\alpha,\lambda_1,M)\|\nabla\mathbf{e}^n\|\|D_t^{(2)}\nabla\mathbf{e}^n\|. \tag{6.32}$$

Using Cauchy-Schwarz's inequality, Young's inequality, (2.3) and (6.32) in (6.31), we arrive at

$$\|D_t^{(2)}\mathbf{e}^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}^n\|^2 \le C(\kappa,\nu,\alpha,\lambda_1,M)\Big(\|\nabla\mathbf{e}^n\|^2 + \|\nabla\sigma_2^n\|^2\Big). \tag{6.33}$$

For the second term on the right hand side of (6.33), we use (6.19), Lemma 3.5 and obtain

$$\begin{aligned}\|e^{\alpha t_n}\nabla\sigma_2^n\|^2 &\le \frac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\|\nabla\mathbf{u}_{htt t}(t)\|^2 dt\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^3\, e^{2\alpha t_n}\int_{t_{n-2}}^{t_n} e^{-2\alpha t}dt\\ &\le C(\kappa,\nu,\alpha,\lambda_1,M)k^4 e^{4\alpha k^*},\end{aligned} \tag{6.34}$$

where $k^* \in (0,k)$. Now with the help of (6.13) and (6.34), (6.33) implies (6.14). This completes the rest of the proof. □

Finally, we obtain the error estimates for the pressure P^n. Consider (3.1) at $t=t_n$ and subtract it from (6.2) to obtain

$$\begin{aligned}(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) &= (D_t^2\mathbf{e}^n,\boldsymbol{\phi}_h) + \kappa a(D_t^2\mathbf{e}^n,\boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n,\boldsymbol{\phi}_h)\\ &- (\sigma_2^n,\boldsymbol{\phi}_h) - \kappa a(\sigma_2^n,\boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h).\end{aligned}$$

With the help of Cauchy-Schwarz's inequality, (2.3) and (5.26), we obtain

$$(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) \le C(\kappa,\nu,\lambda_1)\big(\|D_t^2\nabla\mathbf{e}^n\| + \|\nabla\mathbf{e}^n\| + \|\nabla\sigma_2^n\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{6.35}$$

A use of the Theorem 6.1 and (6.34) in (6.35) yields

$$\|\boldsymbol{\rho}^n\| \le C(\kappa,\nu,\alpha,\lambda_1,M)k^2\ e^{-\alpha t_n} \text{ for } n\ge 2. \tag{6.36}$$

For $n=1$, we use estimates obtained from backward Euler method. Substitute $n=1$ in (5.29) to obtain

$$(\boldsymbol{\rho}^1,\nabla\cdot\boldsymbol{\phi}_h) \le C(\kappa,\nu,\lambda_1)\big(\|\bar\partial_t\nabla\mathbf{e}^1\| + \|\nabla\mathbf{e}^1\| + \|\nabla\sigma_1^1\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{6.37}$$

A use of (5.27) with $n=1$ in (6.37) yields

$$(\boldsymbol{\rho}^1,\nabla\cdot\boldsymbol{\phi}_h) \le C(\kappa,\alpha,\nu,\lambda_1,M)\big(\|\nabla\mathbf{e}^1\| + \|\nabla\sigma_1^1\|\big)\|\nabla\boldsymbol{\phi}_h\|. \tag{6.38}$$

Using bounds obtained from (5.28) (for $n=1$) and (6.30) in (6.38), we arrive at

$$\|\boldsymbol{\rho}^1\| \le C(\kappa,\alpha,\nu,\lambda_1,M)k\ e^{-\alpha t_1}. \tag{6.39}$$

Theorem 6.2. *Under the assumption of Theorems 3.1, 5.1 and 6.1, the following holds true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \le C(h^{2-j} + k^2)e^{-\alpha t_n} \quad j = 0,\,1$$

and

$$\|(p(t_n) - P^n)\| \le Ce^{-\alpha t_n}(h + k^{2-\gamma}),$$

where

$$\gamma = \begin{cases} 0 & \text{if } n\ge 2;\\ 1 & \text{if } n = 1.\end{cases}$$

Proof of Theorem 6.2. A use of Theorems 3.1, 6.1, (6.30), (6.36) and (6.39) would complete the proof. □

7. Numerical Experiments

In this section, we provide a few computational results to support our theoretical estimates for the equations of motion arising in the Kelvin-Voigt fluid (1.1)-(1.3). In example 1, for space discretization, P_2-P_0 mixed finite element space (see [5]) is used: the velocity space consists of continuous piecewise polynomials of degree less than or equal to 2 and the pressure space consists of piecewise constants, that is, we consider the finite dimensional subspaces $\mathbf{V}_h$ and W_h of $\mathbf{H}_0^1$ and L^2 respectively, as:

$$\mathbf{V}_h = \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \big(C(\bar{\Omega})\big)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \tau_h\},$$
$$W_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \tau_h\},$$

where τ_h denotes the triangulation of the domain $\bar{\Omega}$. Below, we discuss the fully discrete finite element formulation of (1.1)-(1.3) using backward Euler method and second order backward difference scheme.

Fully discrete finite element approximation: In this scheme, we discuss the discretization of the time variable by replacing the time derivative by difference quotient. Let Δt be the time step and $\mathbf{U}^n$ be the approximation of $\mathbf{u}(t)$ in $\mathbf{V}_h$ at $t = t_n = n\Delta t$.

The backward Euler approximation to (3.1) can be stated as: given $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$\begin{aligned}(\mathbf{U}^n, \mathbf{v}_h) + (\kappa + \nu\Delta t)\ a(\mathbf{U}^n,\mathbf{v}_h) + \Delta t\ c(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) + \Delta t\ b(\mathbf{v}_h, P^n)\\ = (\mathbf{U}^{n-1}, \mathbf{v}_h) + \kappa a(\mathbf{U}^{n-1},\mathbf{v}_h) + \Delta t\ (\mathbf{f}(t_n), \mathbf{v}_h)\ \ \forall \mathbf{v}_h \in \mathbf{V}_h,\\ b(\mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h.\end{aligned} \tag{7.1}$$

Similarly, the second order backward difference approximation to (3.1) is as follows: given $\mathbf{U}^{n-2}$ and $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$\begin{aligned}&(3\mathbf{U}^n, \mathbf{v}_h) + (\kappa + 2\nu\Delta t)\ a(\mathbf{U}^n, \mathbf{v}_h) + 2\Delta t\ c(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h)\\ &+2\Delta t\ b(\mathbf{v}_h, P^n) = 4(\mathbf{U}^{n-1}, \mathbf{v}_h) + 4\ \kappa a(\mathbf{U}^{n-1}, \mathbf{v}_h) - (\mathbf{U}^{n-2}, \mathbf{v}_h)\\ &-\kappa\ a(\mathbf{U}^{n-2}, \mathbf{v}_h) + \Delta t\ (\mathbf{f}(t_n), \mathbf{v}_h)\ \ \forall \mathbf{v}_h \in \mathbf{V}_h,\\ &\qquad b(\mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h.\end{aligned} \tag{7.2}$$

Now, we approximate the velocity and pressure by

$$\mathbf{U}^n = \sum_{j=1}^{ng} \begin{pmatrix} \mathbf{u}_j^{nx} \\ \mathbf{u}_j^{ny} \end{pmatrix} \boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x}), \quad P^n = \sum_{j=1}^{ne} p_j^n \phi_j^p(\mathbf{x}), \tag{7.3}$$

where $\boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x})$ and $\phi_j^p(\mathbf{x})$ form bases for $\mathbf{V}_h$ and W_h respectively with cardinality ng and ne, respectively. Here, $\mathbf{u}_j^{nx}$ and $\mathbf{u}_j^{ny}$ represent the x and y component of the approximate velocity field, respectively, at time $t = t_n$.

Using (7.3), the basis functions for $\mathbf{V}_h$ and W_h in (7.1) (respectively (7.2)), we obtain nonlinear systems which are solved using Newton's method.

Example 1: In this example, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is

$u_1 = 10e^{-t}x^2(x-1)^2y(y-1)(2y-1),\ \ u_2 = -10e^{-t}y^2(y-1)^2x(x-1)(2x-1),$
$p = e^{-t}y.$

We choose $\nu = 1$, $\kappa = 10^{-2}$ with $\Omega = (0,1) \times (0,1)$ and time $t = [0,1]$. Here, $\bar{\Omega}$ is subdivided into triangles with mesh size h. The theoretical analysis provides a convergence rate of $\mathcal{O}(h^2)$ in $\mathbf{L^2}$-norm, of $\mathcal{O}(h)$ in $\mathbf{H}^1$-norm for velocity and of $\mathcal{O}(h)$ in L^2-norm for pressure. Table 1 gives the numerical errors and convergence rates obtained on successively refined meshes for the first order backward Euler method and Table 2 contains the errors and convergence rates of the second order two step backward difference method. These results agree with the optimal theoretical convergence rates obtained in Theorem 5.2 and 6.2.

TABLE 1. Errors and Convergence rates for backward Euler method with $k = \mathbb{O}(h^2)$.

h	$\lVert\mathbf{u}(t_n) - \mathbf{U}^n\rVert_{\mathbf{L^2}}$	Rate	$\lVert\mathbf{u}(t_n) - \mathbf{U}^n\rVert_{\mathbf{H}^1}$	Rate	$\lVert p(t_n) - P^n\rVert$	Rate
1/2	0.0045761		0.056318		0.136225	
1/4	0.0013220	1.791328	0.024171	1.220311	0.072946	0.901096
1/8	0.0003651	1.856036	0.010997	1.136107	0.037920	0.943847
1/16	0.0000970	1.911519	0.005371	1.033759	0.019201	0.981790

TABLE 2. Errors and Convergence rates for backward difference scheme with $k = \mathbb{O}(h)$.

h	$\lVert\mathbf{u}(t_n) - \mathbf{U}^n\rVert_{\mathbf{L^2}}$	Rate	$\lVert\mathbf{u}(t_n) - \mathbf{U}^n\rVert_{\mathbf{H}^1}$	Rate	$\lVert p(t_n) - P^n\rVert$	Rate
1/2	0.0047034		0.043623		0.135666	
1/4	0.0013240	1.828747	0.019412	1.168111	0.073007	0.893959
1/8	0.0003653	1.857587	0.009233	1.072039	0.037925	0.944881
1/16	0.0000970	1.912022	0.004602	1.004511	0.019201	0.981941

Remark 2. *Note that, under extra regularity assumptions on the exact solution pair, one can obtain better rates of convergence by using higher order finite element space.*

We illustrate this in example 2, by choosing an appropriate right hand side function f. **Example 2:** In this example, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is:

$$
\begin{aligned}
&u_1 = te^{-t^2} sin^2(3\pi x)\ sin(6\pi y),\ u_2 = -te^{-t^2} sin^2(3\pi y)\ sin(6\pi x), \\
&p = te^{-t}\ sin(2\pi x)\ sin(2\pi y).
\end{aligned} \tag{7.4}
$$

We assume that the viscosity of the fluid(ν) is 10^{-2} and the retardation κ is 10^{-4} with $\Omega = (0,1)\times(0,1)$ and time $t = [0,1]$. Here again, $\bar{\Omega}$ is subdivided into triangles with mesh size h. For the problem defined in example 2, we have conducted numerical experiments using P_2-P_1 mixed finite element spaces for space discretization, that is, if we choose

$$
\begin{aligned}
\mathbf{V}_h &= \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \left(C(\bar{\Omega})\right)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \tau_h\}, \\
W_h &= \{q \in L^2(\Omega) \cap C(\bar{\Omega}) : q|_K \in P_1(K), K \in \tau_h\},
\end{aligned}
$$

we obtain $\|\mathbf{u}(t_n) - \mathbf{U}^n\|_j \le \mathcal{O}(h^{3-j})$, $j = 0,1$ and $\|(p(t_n) - P^n)\| \le \mathcal{O}(h^2)$. Since the solution (7.4) has extra regularity, we obtained better order of convergence for velocity and pressure as expected [5]. In Tables 3 and 4, we have shown the convergence rates for backward Euler method and backward difference scheme respectively for $\mathbf{L^2}$ and $\mathbf{H}^1$-norms in velocity and L^2-norm in pressure. In case, we choose $k = \mathcal{O}(h^{3/2})$ for backward Euler method, we observe that convergence rate in comparison with that of the backward difference scheme is lower. Table 5 represents the comparison between the errors obtained from backward Euler method and backward difference scheme with $k = \mathcal{O}(h^{3/2})$.

Table 3. Errors and Convergence rates for backward Euler method with $k = \mathbb{O}(h^3)$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L}^2}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/4	0.480585		13.147142		0.088453	
1/8	0.085185	2.496114	4.135230	1.668709	0.015389	2.522972
1/16	0.007371	3.530504	0.849642	2.283040	0.002566	2.584331
1/32	0.000709	3.377407	0.163177	2.380417	0.000610	2.072467

Table 4. Errors and Convergence rates for backward difference scheme with $k = \mathbb{O}(h^{3/2})$.

h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L}^2}$	Rate	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$	Rate	$\|p(t_n) - P^n\|$	Rate
1/4	0.466215		12.744857		0.085101	
1/8	0.083276	2.485017	4.111723	1.63210	0.015361	2.469883
1/16	0.007224	3.526926	0.850460	2.273426	0.002504	2.616902
1/32	0.000609	3.568176	0.163161	2.381940	0.000596	2.068971

Table 5. Comparison of errors with $k = \mathcal{O}(h^{3/2})$ between the two schemes.

h	BE velocity in $\mathbf{L}^2$-norm	Bd velocity in $\mathbf{L}^2$-norm	BE pressure in $\mathbf{L}^2$-norm	Bd pressure in $\mathbf{L}^2$-norm
1/4	0.514184	0.466215	0.102280	0.085101
1/8	0.084014	0.083276	0.015499	0.015361
1/16	0.008783	0.007224	0.003010	0.002504
1/32	0.001991	0.000609	0.000880	0.000596

Here BE = Backward Euler, Bd = Backward difference.

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