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NUMERICAL ANALYSIS OF THE FRACTIONAL SEVENTH-ORDER KDV EQUATION USING AN IMPLICIT FULLY DISCRETE LOCAL DISCONTINUOUS GALERKIN METHOD

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Abstract. In this paper an implicit fully discrete local discontinuous Galerkin (LDG) finite element method is applied to solve the time-fractional seventh-order Korteweg-de Vries (sKdV) equation, which is introduced by replacing the integer-order time derivatives with fractional derivatives. We prove that our scheme is unconditional stable and L^2 error estimate for the linear case with the convergence rate $O(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})$ through analysis. Extensive numerical results are provided to demonstrate the performance of the present method.

Key words. Time-fractional partial differential equations; Seventh-order KdV equation; Local discontinuous Galerkin method; Stability; Error estimates.

1. Introduction

Several researchers in fractional calculus mentioned that derivatives of noninteger order are very effective for the description of many physical phenomena such as damping laws, and diffusion process [18, 25]. Some fractional partial differential equations have been solved, such as time-fractional telegraph equation [1], fractional Fokker-Planck equation [5], space-time fractional Schrödinger equation [8, 26], fractional order two point boundary value problem [7], the fractional KdV equation [16], fractional diffusion equation [17, 23], fractional derivative fluid model [9], fractional KdV-Burgers-Kuramoto equation [21] and so on. Machado et al. [14] introduced the recent history of fractional calculus, as for the detailed theory and applications of fractional integrals and derivatives, we can refer to [11, 15, 20] and the references therein. Solving such fractional partial differential by the robust and accurate numerical methods has become popular with their frequent appearance in applied science and engineering.

The KdV type of equations, which were first derived by Korteweg and de Vries (1895) and used to describe weakly nonlinear shallow water waves, have emerged as an important class of nonlinear evolution equation and are often used in pratical applications. The seventh-order KdV (sKdV) equation was first introduced by Pomeau et. al [19] in order to discuss the structural stability of the KdV equation under singular perturbation. Some methods [6, 13] have been used to handle the integer-order equations, however, to the best of our knowledge, the study of the fractional sKdv equations has not been widespread. In this paper, we consider the following generalized time-fractional sKdv equation

$$
\begin{aligned}
&D_t^\alpha u(x,t) + g(u)_x + u_{3x} - u_{5x} + \lambda u_{7x} = 0, \\
&u(x,0) = u_0(x),
\end{aligned}
\tag{1.1}
$$

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where λ is anonzero constant. $0 < \alpha \leq 1$ is a parameter describing the order of the fractional time. We do not pay attention to boundary condition in this paper; hence the solution is considered to be either periodic or compactly supported.

The time fractional derivative in the equation (1.1), uses the Caputo fractional partial derivative of order α, defined as [18]

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}} & if\ 0 < \alpha < 1, \\ \frac{\partial u(x,t)}{\partial t} & if\ \alpha = 1, \end{cases} \tag{1.2}$$

here $\Gamma(\cdot)$ is the Gamma function.

The discontinuous Galerkin finite element method is a very attractive method for partial differential equations because it is naturally formulated for any order of accuracy in each element, flexible and efficient in terms of mesh and shape functions. The purpose of the present paper is to solve and analyze time-fractional sKdV equation by introducing an implicit fully discrete local discontinuous Galerkin method. This development is based on the extensive work on DG for problems founded in classic calculus [10, 22, 24, 27]. We prove that our scheme is unconditionally stable and give an error estimate for the linear case.

The remains of this paper are organized as follows. In the next section, we introduce some basic notations and mathematical preliminaries. Then, in Section 3, we discuss the LDG scheme for the fractional equation (1.1), and prove that the scheme is unconditionally stable, and the numerical solution is convergent. Numerical experiments to illustrate the accuracy and capability of the method are given in Section 4. Finally, in Section 5, concluding remarks are provided.

2. Notations and auxiliary results

2.1. Notations. First, the domain Ω is partitioned into elements $\Omega = \bigcup_j I_j$ with a spatial grid $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$. $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \cdots N$. The cell lengths $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \leq j \leq N$, and $h = \max_{1\leq j\leq N} \Delta x_j$. The solution of the numerical scheme is denoted by u_h^n which belongs to the finite element space V_h^k:

$$V_h^k = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \cdots N\},$$

$P^k(I_j)$ denotes the set of all polynomials of degree at most k on I_j.

For a function $u_h^n \in V_h^k$, We denote the limits at the points $\{x_{j+\frac{1}{2}}\}$ by

$$(u_h^n)^{\pm}_{j+\frac{1}{2}} = \lim_{x \to x^{\pm}_{j+\frac{1}{2}}} u_h^n,$$

$(u_h^n)^-_{j+\frac{1}{2}}$ and $(u_h^n)^+_{j+\frac{1}{2}}$ refer to the value of u_h^n at $x_{j+\frac{1}{2}}$ from the left cell I_j and the right cell I_{j+1}, respectively. The jump $(u_h^n)^+_{j+\frac{1}{2}} - (u_h^n)^-_{j+\frac{1}{2}}$ by $[u_h^n]_{j+\frac{1}{2}}$. The jump will be zero for a continuous function.

2.2. Numerical flux. Consider a scalar conservation law given in differential form

$$\phi_t + g(\phi)_x = 0, \tag{2.1}$$

where $g(\phi)$ is called the flux function. Numerically, $g(\phi)$ should be expressed by a suitable choice at the interface. For discontinuous Galerkin spatial discretization, $g(\phi)$ is approximated by the numerical form at the discontinuous point $x_{j+\frac{1}{2}}$. In this paper, the flux $\widehat{g}(\phi^-, \phi^+)$ will be used to denote the numerical flux, which is related to the discontinuous Galerkin spatial discretization. $\widehat{g}(\phi^-, \phi^+)$ is a monotone numerical flux, which is dependent on the two values of the function ϕ at the discontinuous point $x_{j+\frac{1}{2}}$, and satisfies the following conditions:

(i) it is locally Lipschitz continuous, so it is bounded when $\phi^{\pm}$ are in a bounded interval;

(ii) it is consistent with the flux $g(\phi)$, i.e., $\widehat{g}(\phi, \phi) = g(\phi)$;

(iii) it is a nondecreasing function of its first argument, and a nonincreasing function of its second argument.

2.3. Projections. In order to give a proof of error estimates, two projections will be used for one dimension case $[a, b]$, which denoted by $\mathcal{P}$ and $\mathcal{P}^{\pm}$, i.e., for each j,

$$\int_{I_j} (\mathcal{P}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^k(I_j), \tag{2.2}$$

$$\begin{aligned} &\int_{I_j} (\mathcal{P}^+\omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j), \\ &\mathcal{P}^+\omega(x^+_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}}), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\int_{I_j} (\mathcal{P}^-\omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j), \\ &\mathcal{P}^-\omega(x^-_{j+\frac{1}{2}}) = \omega(x_{j+\frac{1}{2}}). \end{aligned} \tag{2.4}$$

Using standard approximation theory, the projections satisfy the following inequality [2, 3]

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\tau_h} \le Ch^{k+1}, \tag{2.5}$$

where $\omega^e = \mathcal{P}\omega - \omega$ or $\omega^e = \mathcal{P}^{\pm}\omega - \omega$. τ_h denotes the set of boundary points of all elements I_j. From here on, let C denote a positive constant depending on u and its derivatives but independent of h, which may have a different value in each occurrence, and let $\|\cdot\|_D$ denote L^2 norm in D. If $D = \Omega$, we drop D.

3. Fully discrete LDG scheme

In the following, we shall introduce the numerical scheme for the solution of equation (1.1). Let $\Delta t = T/M$ be the time mesh-size and M a positive integer, $t_n = n\Delta t, n = 0, 1, \cdots, M$ be mesh point. An approximation to the time fractional derivative (1.2) can be obtained by a simple quadrature formula [12]

$$D_t^\alpha u(x, t_n) = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_i \frac{u(x, t_{n-i}) - u(x, t_{n-i-1})}{\Delta t} + \gamma^n(x), \tag{3.1}$$

where $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$, $\gamma^n(x) \le C(\Delta t)^{2-\alpha}$, C is dependent on u, T, α.

We know

$$\begin{aligned} &1 = b_0 > b_1 > b_2 > \cdots > b_n > 0, b_n \to 0 (n \to \infty), \\ &\sum_{i=1}^{n} (b_{i-1} - b_i) + b_n = 1. \end{aligned} \tag{3.2}$$

First we rewrite the equation (1.1) as a first-order one

$$\begin{aligned} &D_t^\alpha u + g(u)_x + q^*_x - s_x + \lambda z_x = 0, z - w_x = 0, w - s_x = 0, \\ &s - r_x = 0, r - q_x = 0, q - p_x = 0, q^* - p_x = 0, p - u_x = 0. \end{aligned} \tag{3.3}$$

For convenience, we introduce the following notations

$$\mathbf{C}(\omega, \widehat{\omega}; \eta) = \int_\Omega \omega \eta_x dx - \sum_{j=1}^{N} ((\widehat{\omega}\eta^-)_{j+\frac{1}{2}} - (\widehat{\omega}\eta^+)_{j-\frac{1}{2}}). \tag{3.4}$$

An implicit fully discrete LDG scheme is defined as follows: Find $u_h^n, z_h^n, w_h^n, s_h^n, r_h^n, q_h^n, p_h^n, (q_h^n)^* \in V_h^k$, such that $\forall v, \rho, \psi, \phi, \varphi, \xi, \eta, \theta \in V_h^k$,

$$\begin{aligned}
&\int_\Omega u_h^n v dx - \beta \mathbf{C}(g(u_h^n), \widehat{g(u_h^n)}; v) - \beta \mathbf{C}((q_h^n)^*, \widehat{(q_h^n)^*}; v) \\
&\qquad + \beta \mathbf{C}(s_h^n, \widetilde{s_h^n}; v) - \beta\lambda \mathbf{C}(z_h^n, \widehat{z_h^n}; v) \\
&\qquad = \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega u_h^{n-i} v dx + b_{n-1} \int_\Omega u_h^0 v dx, \\
&\int_\Omega z_h^n \rho dx + \mathbf{C}(w_h^n, \widehat{w_h^n}; \rho) = 0, \int_\Omega w_h^n \psi dx + \mathbf{C}(s_h^n, \widehat{s_h^n}; \psi) = 0, \\
&\int_\Omega s_h^n \phi dx + \mathbf{C}(r_h^n, \widehat{r_h^n}; \phi) = 0, \int_\Omega r_h^n \varphi dx + \mathbf{C}(q_h^n, \widehat{q_h^n}; \varphi) = 0, \\
&\int_\Omega q_h^n \xi dx + \mathbf{C}(p_h^n, \widehat{p_h^n}; \xi) = 0, \int_\Omega p_h^n \eta dx + \mathbf{C}(u_h^n, \widehat{u_h^n}; \eta) = 0, \\
&\int_\Omega (q_h^n)^* \theta dx + \mathbf{C}(p_h^n, \widetilde{p_h^n}; \theta) = 0,
\end{aligned} \tag{3.5}$$

where $\beta = (\Delta t)^\alpha \Gamma(2-\alpha)$. The "hat" terms in (3.5) in the cell boundary from integration by parts are the so-called "numerical fluxes". In order to ensure stability, we can take the following choices simply

$$\begin{aligned}
&\widehat{u_h^n} = (u_h^n)^-, \widehat{(q_h^n)^*} = ((q_h^n)^*)^+, \widetilde{p_h^n} = (p_h^n)^+, \widetilde{s_h^n} = (s_h^n)^+, \widehat{\widehat{p_h^n}} = (p_h^n)^-, \\
&\widehat{p_h^n} = \widehat{\widehat{p_h^n}} + \tau[\lambda w_h^n - r_h^n], \widehat{\widehat{q_h^n}} = (q_h^n)^+, \widehat{q_h^n} = \widehat{\widehat{q_h^n}} + \tau[q_h^n - \lambda s_h^n], \\
&\widehat{z_h^n} = (z_h^n)^+, \widehat{w_h^n} = (w_h^n)^+, \widehat{s_h^n} = (s_h^n)^-, \widehat{r_h^n} = (r_h^n)^+,
\end{aligned} \tag{3.6}$$

where $\tau > 0$. Two dissipation terms in the flux of $\widehat{p_h^n}$ and $\widehat{p_h^n}$ are added in order to get an error estimate by controlling the boundary terms.

Remark 3.1. *In order to obtain the stability result we introduce the term with "$*$". You will find a contradiction if you don't do so while choosing the directions of flux terms.*

The flux $\widehat{g}((u_h^n)^-, (u_h^n)^+)$ is a monotone flux as described in (2.2). Examples of monotone fluxes which are suitable for the local discontinuous Galerkin methods can be found in [4]. For example, one could use the Lax-Friedriches flux, which is given by

$$\widehat{g}^{LF}(w^-, w^+) = \frac{1}{2}(g(w^-) + g(w^+) - \lambda_0(w^+ - w^-)), \ \lambda_0 = \max_w |g'(w)|. \tag{3.7}$$

Now we state the main theoretic analysis.

Theorem 3.1. *For periodic or compactly supported boundary conditions, Our fully-discrete LDG scheme (3.5) with flux (3.6) is unconditionally stable, and the* *numerical solution satisfies*

$$\|u_h^n\|^2+\beta\sum_{j=1}^{N}([p_h^n]^2+[q_h^n]^2+\lambda[r_h^n]^2)_{j-\frac{1}{2}}+2\beta\tau\sum_{j=1}^{N}([q_h^n-\lambda s_h^n]^2 + [-r_h^n+\lambda w_h^n]^2)_{j-\frac{1}{2}}\le\|u_h^0\|^2,\ n=1,2\cdots,M, \tag{3.8}$$

where τ ia a positive constant.

Proof. Taking the test functions $v=u_h^n,\eta=-\beta(q_h^n)^*+\beta s_h^n-\beta\lambda z_h^n,\varphi=\beta q_h^n-\beta\lambda s_h^n,\theta=\beta p_h^n,\xi=-\beta r_h^n+\beta\lambda w_h^n,\phi=-\beta p_h^n+\beta\lambda r_h^n\rho=\beta\lambda p_h^n,\psi=-\beta\lambda q_h^n$ in scheme (3.5), and with the fluxes choice (3.6), we obtain

$$\begin{aligned}
&\|u_h^n\|^2+\beta\widetilde{G}(u_h^n)+\frac{\beta}{2}\sum_{j=1}^{N}([p_h^n]^2+[q_h^n]^2+\lambda[r_h^n]^2)_{j-\frac{1}{2}}\\
&\quad+\beta\tau\sum_{j=1}^{N}[q_h^n-\lambda s_h^n]^2+[-r_h^n+\lambda w_h^n]^2)_{j-\frac{1}{2}}\\
&\quad+\beta\sum_{j=1}^{N}(\Psi(u_h^n,(q_h^n)^*,s_h^n,p_h^n,r_h^n,z_h^n,w_h^n,q_h^n)_{j+\frac{1}{2}}\\
&\quad-\Psi(u_h^n,(q_h^n)^*,s_h^n,p_h^n,r_h^n,z_h^n,w_h^n,q_h^n)_{j-\frac{1}{2}}\\
&\quad+\Theta(u_h^n,(q_h^n)^*,s_h^n,p_h^n,r_h^n,z_h^n,w_h^n,q_h^n)_{j-\frac{1}{2}})\\
&=\sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega u_h^{n-i}u_h^n dx+b_{n-1}\int_\Omega u_h^0u_h^n dx,
\end{aligned} \tag{3.9}$$

where

$$\widetilde{G}(u_h^n)=-\int_\Omega g(u_h^n)(u_h^n)_x dx+\sum_{j=1}^{N}((\widehat{g}(u_h^n)^-)_{j+\frac{1}{2}}-(\widehat{g}(u_h^n)^+)_{j-\frac{1}{2}}),$$

$$\begin{aligned}
\Psi(u_h^n,&(q_h^n)^*,s_h^n,p_h^n,r_h^n,z_h^n,w_h^n,q_h^n)=-((q_h^n)^*)^-(u_h^n)^-+\widehat{(q_h^n)^*}(u_h^n)^-+\widehat{u_h^n}((q_h^n)^*)^-\\
&+(s_h^n)^-(u_h^n)^--\widetilde{s_h^n}(u_h^n)^--\widehat{u_h^n}(s_h^n)^--(p_h^n)^-(r_h^n)^-+\widehat{p_h^n}(r_h^n)^-+\widehat{r_h^n}(p_h^n)^-\\
&+\lambda(-(z_h^n)^-(u_h^n)^-+\widehat{z_h^n}(u_h^n)^-+\widehat{u_h^n}(z_h^n)^-+(p_h^n)^-(w_h^n)^--\widehat{p_h^n}(w_h^n)^--\widehat{w_h^n}(p_h^n)^-)\\
&+\lambda(-(s_h^n)^-(q_h^n)^-+\widehat{s_h^n}(q_h^n)^-+\widehat{q_h^n}(s_h^n)^-),
\end{aligned}$$

$$\begin{aligned}
\Theta(u_h^n,&(q_h^n)^*,s_h^n,p_h^n,r_h^n,z_h^n,w_h^n,q_h^n)=-((q_h^n)^*)^-(u_h^n)^-+((q_h^n)^*)^+(u_h^n)^+\\
&+\widehat{(q_h^n)^*}(u_h^n)^--\widehat{(q_h^n)^*}(u_h^n)^++\widehat{u_h^n}((q_h^n)^*)^--\widehat{u_h^n}((q_h^n)^*)^+\\
&+(s_h^n)^-(u_h^n)^--(s_h^n)^+(u_h^n)^+-\widetilde{s_h^n}(u_h^n)^-+\widetilde{s_h^n}(u_h^n)^+-\widehat{u_h^n}(s_h^n)^-+\widehat{u_h^n}(s_h^n)^+\\
&-(p_h^n)^-(r_h^n)^-+(p_h^n)^+(r_h^n)^++\widehat{p_h^n}(r_h^n)^--\widehat{p_h^n}(r_h^n)^++\widehat{r_h^n}(p_h^n)^--\widehat{r_h^n}(p_h^n)^+\\
&+\lambda(-(z_h^n)^-(u_h^n)^-+(z_h^n)^+(u_h^n)^++\widehat{z_h^n}(u_h^n)^--\widehat{z_h^n}(u_h^n)^++\widehat{u_h^n}(z_h^n)^--\widehat{u_h^n}(z_h^n)^+)\\
&+\lambda((p_h^n)^-(w_h^n)^--(p_h^n)^+(w_h^n)^+-\widehat{p_h^n}(w_h^n)^-+\widehat{p_h^n}(w_h^n)^+-\widehat{w_h^n}(p_h^n)^-+\widehat{w_h^n}(p_h^n)^+)\\
&+\lambda(-(s_h^n)^-(q_h^n)^-+(s_h^n)^+(q_h^n)^++\widehat{s_h^n}(q_h^n)^--\widehat{s_h^n}(q_h^n)^++\widehat{q_h^n}(s_h^n)^--\widehat{q_h^n}(s_h^n)^+).
\end{aligned}$$

If we take the fluxes (3.6), and after some manual calculation, we can easily obtain

$$\Theta(u_h^n,q_h^n,s_h^n,p_h^n,r_h^n)=0.$$

For the nonlinear term, Let $G(u) = \int^u g(u)du$, and use a mean value theorem, then we can obtain

$$\widetilde{G}(u_h^n) = \sum_{j=1}^{N}(G'(\eta) - \widehat{g})[u_h^n]_{j-\frac{1}{2}} \geq 0, \tag{3.10}$$

where η is a value between $(u_h^n)^-$ and $(u_h^n)^+$. By the monotonicity of flux function $\widehat{g}$, we have the inequality (3.10).

Then based on the equation (3.9), we can get

$$\begin{aligned}
&\|u_h^n\|^2 + \frac{\beta}{2}\sum_{j=1}^{N}([p_h^n]^2 + [q_h^n]^2 + \lambda[r_h^n]^2)_{j-\frac{1}{2}} \\
&\quad + \beta\tau\sum_{j=1}^{N}[q_h^n - \lambda s_h^n]^2 + [-r_h^n + \lambda w_h^n]^2)_{j-\frac{1}{2}} \\
&\quad \leq \sum_{i=1}^{n-1}(b_{i-1} - b_i)\int_\Omega u_h^{n-i}u_h^n dx + b_{n-1}\int_\Omega u_h^0 u_h^n dx \\
&\quad \leq \frac{1}{2}(\sum_{i=1}^{n-1}(b_{i-1} - b_i)\|u_h^{n-i}\| + b_{n-1}\|u_h^0\|)^2 + \frac{1}{2}\|u_h^n\|^2,
\end{aligned}$$

that is

$$\begin{aligned}
&\|u_h^n\|^2 + \beta\sum_{j=1}^{N}([p_h^n]^2 + [q_h^n]^2 + \lambda[r_h^n]^2)_{j-\frac{1}{2}} \\
&\quad + 2\beta\tau\sum_{j=1}^{N}[q_h^n - \lambda s_h^n]^2 + [-r_h^n + \lambda w_h^n]^2)_{j-\frac{1}{2}} \\
&\quad \leq (\sum_{i=1}^{n-1}(b_{i-1} - b_i)\|u_h^{n-i}\| + b_{n-1}\|u_h^0\|)^2.
\end{aligned} \tag{3.11}$$

We will prove the Theorem 3.1 by mathematical induction. When $n = 1$, by the expression (3.11), we can obtain the following inequalities immediately

$$\begin{aligned}
&\|u_h^1\|^2 + \beta\sum_{j=1}^{N}([p_h^1]^2 + [q_h^1]^2 + \lambda[r_h^1]^2)_{j-\frac{1}{2}} \\
&\quad + 2\beta\tau\sum_{j=1}^{N}[q_h^1 - \lambda s_h^1]^2 + [-r_h^1 + \lambda w_h^1]^2)_{j-\frac{1}{2}} \leq \|u_h^0\|^2,
\end{aligned} \tag{3.12}$$

and $\|u_h^1\| \leq \|u_h^0\|$.

Now, supposing the following inequality holds

$$\|u_h^m\| \leq \|u_h^0\|, m = 2, 3, \cdots, K, \tag{3.13}$$

and let $n = K + 1$ in the inequality (3.11), we can obtain

$$\begin{aligned}
&\|u_h^{K+1}\|^2 + \beta \sum_{j=1}^{N}([p_h^{K+1}]^2 + [q_h^{K+1}]^2 + \lambda[r_h^{K+1}]^2)_{j-\frac{1}{2}} \\
&\quad + 2\beta\tau \sum_{j=1}^{N}[q_h^{K+1} - \lambda s_h^{K+1}]^2 + [-r_h^{K+1} + \lambda w_h^{K+1}]^2)_{j-\frac{1}{2}} \\
&\leq (\sum_{i=1}^{K}((b_{i-1} - b_i)\|u_h^{K+1-i}\| + b_K\|u_h^0\|))^2.
\end{aligned}$$

Using the property (3.2) and assumption (3.13), we can obtain the following inequality easily

$$\begin{aligned}
&\|u_h^{K+1}\|^2 + \beta \sum_{j=1}^{N}([p_h^{K+1}]^2 + [q_h^{K+1}]^2 + \lambda[r_h^{K+1}]^2)_{j-\frac{1}{2}} \\
&\quad + 2\beta\tau \sum_{j=1}^{N}[q_h^{K+1} - \lambda s_h^{K+1}]^2 + [-r_h^{K+1} + \lambda w_h^{K+1}]^2)_{j-\frac{1}{2}} \\
&\leq \|u_h^0\|^2.
\end{aligned} \tag{3.14}$$

The desired result is obtained. $\square$

Next we state the error estimate of the scheme for the linear case.

Theorem 3.2. *Let $u(x, t_n)$ be the exact solution of the problem (1.1), u_h^n be the numerical solution of the fully discrete LDG scheme (3.5), then there exists a positive constant C, such that the following error estimate holds*

$$\|u(x, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}).$$

Proof. Without loss of generality, we consider the linear time-fractional sKdV equation

$$D_t^\alpha u(x,t) + u_x + u_{3x} - u_{5x} + u_{7x} = 0. \tag{3.15}$$

It is easy to verify that the exact solution of the above PDE (3.15) satisfies

$$\begin{aligned}
&\int_\Omega u(x,t_n)vdx - \beta\mathbf{C}(u(x,t_n), u(x,t_n); v) - \beta\mathbf{C}(q^*(x,t_n), q^*(x,t_n); v) \\
&\quad + \beta\mathbf{C}(s(x,t_n), s(x,t_n); v) - \beta\lambda\mathbf{C}(z(x,t_n), z(x,t_n); v) \\
&= \sum_{i=1}^{n-1}(b_{i-1} - b_i)\int_\Omega u_h^{n-i}vdx + b_{n-1}\int_\Omega u_h^0 vdx,
\end{aligned} \tag{3.16}$$

$$\int_\Omega z(x,t_n)\rho dx + \mathbf{C}(w(x,t_n),w(x,t_n);\rho) = 0,$$
$$\int_\Omega w(x,t_n)\psi dx + \mathbf{C}(s(x,t_n),s(x,t_n);\psi) = 0,$$
$$\int_\Omega s(x,t_n)\phi dx + \mathbf{C}(r(x,t_n),r(x,t_n);\phi) = 0,$$
$$\int_\Omega p(x,t_n)\eta dx + \mathbf{C}(u(x,t_n),u(x,t_n);\eta) = 0,$$
$$\int_\Omega r(x,t_n)\varphi dx + \mathbf{C}(q(x,t_n),q(x,t_n);\varphi) + \tau\sum_{j=1}^{N}[q(x,t_n)-\lambda s(x,t_n)][\varphi]_{j-\frac{1}{2}} = 0,$$
$$\int_\Omega q(x,t_n)\xi dx + \mathbf{C}(p(x,t_n),p(x,t_n);\xi) + \tau\sum_{j=1}^{N}[\lambda w(x,t_n)-r(x,t_n)][\xi]_{j-\frac{1}{2}} = 0,$$
$$\int_\Omega q^*(x,t_n)\theta dx + \mathbf{C}(p(x,t_n),p(x,t_n);\theta) = 0.$$

We introduce the following notations

$$\begin{aligned}
e_u^n &= u(x,t_n) - u_h^n = \mathcal{P}^- e_u^n - (\mathcal{P}^- u(x,t_n) - u(x,t_n)),\\
e_{q^*}^n &= q^*(x,t_n) - (q_h^n)^* = \mathcal{P} e_{q^*}^n - (\mathcal{P} q^*(x,t_n) - q^*(x,t_n)),\\
e_s^n &= s(x,t_n) - s_h^n = \mathcal{P} e_s^n - (\mathcal{P} s(x,t_n) - s(x,t_n))\\
e_z^n &= z(x,t_n) - z_h^n = \mathcal{P} e_z^n - (\mathcal{P} z(x,t_n) - z(x,t_n))\\
e_w^n &= w(x,t_n) - w_h^n = \mathcal{P} e_w^n - (\mathcal{P} w(x,t_n) - w(x,t_n))\\
e_p^n &= p(x,t_n) - p_h^n = \mathcal{P} e_p^n - (\mathcal{P} p(x,t_n) - p(x,t_n)),\\
e_r^n &= r(x,t_n) - r_h^n = \mathcal{P} e_r^n - (\mathcal{P} r(x,t_n) - r(x,t_n)),\\
e_q^n &= q(x,t_n) - q_h^n = \mathcal{P} e_q^n - (\mathcal{P} q(x,t_n) - q(x,t_n)).
\end{aligned} \tag{3.17}$$

Subtracting (3.5) from (3.16), and using the fluxes (3.6), we can obtain the error equation

$$\begin{aligned}
&\int_\Omega e_u^n v dx - \beta\mathbf{C}(e_u^n,(e_u^n)^-;v) - \beta\mathbf{C}(e_{q^*}^n,(e_{q^*}^n)^+;v) + \beta\mathbf{C}(e_s^n,(e_s^n)^+;v)\\
&\quad - \beta\lambda\mathbf{C}(e_z^n,(e_z^n)^+;v) + \int_\Omega e_z^n\rho dx + \mathbf{C}(e_w^n,(e_w^n)^+;\rho)\\
&\quad + \int_\Omega e_w^n\psi dx + \mathbf{C}(e_s^n,(e_s^n)^-;\psi) + \int_\Omega e_s^n\phi dx + \mathbf{C}(e_r^n,(e_r^n)^+;\phi)\\
&\quad + \int_\Omega e_r^n\varphi dx + \mathbf{C}(e_q^n,(e_q^n)^+;\varphi) + \int_\Omega e_q^n\xi dx + \mathbf{C}(e_p^n,(e_p^n)^-;\xi)\\
&\quad + \int_\Omega e_p^n\eta dx + \mathbf{C}(e_u^n,(e_u^n)^-;\eta) + \int_\Omega e_{q^*}^n\theta dx + \mathbf{C}(e_p^n,(e_p^n)^+;\theta)\\
&\quad - \sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega e_u^{n-i} v dx - b_{n-1}\int_\Omega e_u^0 v dx + \beta\int_\Omega \gamma^n(x) v dx\\
&\quad + \tau\sum_{j=1}^{N}([e_q^n-\lambda e_s^n][\varphi] + [\lambda e_w^n - e_r^n][\xi])_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{3.18}$$

Using the notation in (3.17), the error equation (3.18) can be written as

$$
\begin{aligned}
&\int_\Omega \mathcal{P}^- e_u^n v dx - \beta \mathbf{C}(\mathcal{P}^- e_u^n, (\mathcal{P}^- e_u^n)^-; v) - \beta \mathbf{C}(\mathcal{P} e_{q^*}^n, (\mathcal{P} e_{q^*}^n)^+; v) \\
&\quad + \beta \mathbf{C}(\mathcal{P} e_s^n, (\mathcal{P} e_s^n)^+; v) - \beta\lambda \mathbf{C}(\mathcal{P} e_z^n, (\mathcal{P} e_z^n)^+; v) \\
&\quad + \int_\Omega \mathcal{P} e_z^n \rho dx + \mathbf{C}(\mathcal{P} e_w^n, (\mathcal{P} e_w^n)^+; \rho) + \int_\Omega \mathcal{P} e_w^n \psi dx + \mathbf{C}(\mathcal{P} e_s^n, (\mathcal{P} e_s^n)^-; \psi) \\
&\quad + \int_\Omega \mathcal{P} e_s^n \phi dx + \mathbf{C}(\mathcal{P} e_r^n, (\mathcal{P} e_r^n)^+; \phi) + \int_\Omega \mathcal{P} e_r^n \varphi dx + \mathbf{C}(\mathcal{P} e_q^n, (\mathcal{P} e_q^n)^+; \varphi) \\
&\quad + \int_\Omega \mathcal{P} e_q^n \xi dx + \mathbf{C}(\mathcal{P} e_p^n, (\mathcal{P} e_p^n)^-; \xi) + \int_\Omega \mathcal{P} e_p^n \eta dx + \mathbf{C}(\mathcal{P}^- e_u^n, (\mathcal{P}^- e_u^n)^-; \eta) \\
&\quad + \int_\Omega \mathcal{P} e_{q^*}^n \theta dx + \mathbf{C}(\mathcal{P} e_p^n, (\mathcal{P} e_p^n)^+; \theta) + \tau \sum_{j=1}^N ([\mathcal{P} e_q^n - \lambda \mathcal{P} e_s^n][\varphi] + [\lambda \mathcal{P} e_w^n - \mathcal{P} e_r^n][\xi])_{j-\frac{1}{2}} \\
&= \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega \mathcal{P}^- e_u^{n-i} v dx + b_{n-1} \int_\Omega \mathcal{P}^- e_u^0 v dx - \beta \int_\Omega \gamma^n(x) v dx \\
&\quad - \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega (\mathcal{P}^- u(x, t_{n-i}) - u(x, t_{n-i})) v dx - b_{n-1} \int_\Omega (\mathcal{P}^- u(x, t_0) - u(x, t_0)) v dx \\
&\quad + \int_\Omega (\mathcal{P}^- u(x, t_n) - u(x, t_n)) v dx - \beta \mathbf{C}(\mathcal{P}^- u(x, t_n) - u(x, t_n), (\mathcal{P}^- u(x, t_n) - u(x, t_n))^-; v) \\
&\quad - \beta \mathbf{C}(\mathcal{P} q^*(x, t_n) - q^*(x, t_n), (\mathcal{P} q^*(x, t_n) - q^*(x, t_n))^+; v) + \beta \mathbf{C}(\mathcal{P} s(x, t_n) - s(x, t_n), \\
&\quad (\mathcal{P} s(x, t_n) - s(x, t_n))^+; v) - \beta\lambda \mathbf{C}(\mathcal{P} z(x, t_n) - z(x, t_n), (\mathcal{P} z(x, t_n) - z(x, t_n))^+; v) \\
&\quad + \int_\Omega (\mathcal{P} z(x, t_n) - z(x, t_n)) \rho dx + \mathbf{C}(\mathcal{P} w(x, t_n) - w(x, t_n), (\mathcal{P} w(x, t_n) - w(x, t_n))^+; \rho) \\
&\quad + \int_\Omega (\mathcal{P} w(x, t_n) - w(x, t_n)) \psi dx + \mathbf{C}(\mathcal{P} s(x, t_n) - s(x, t_n), (\mathcal{P} s(x, t_n) - s(x, t_n))^-; \psi) \\
&\quad + \int_\Omega (\mathcal{P} s(x, t_n) - s(x, t_n)) \phi dx + \mathbf{C}(\mathcal{P} r(x, t_n) - r(x, t_n), (\mathcal{P} r(x, t_n) - r(x, t_n))^+; \phi) \\
&\quad + \int_\Omega (\mathcal{P} r(x, t_n) - r(x, t_n)) \varphi dx + \mathbf{C}(\mathcal{P} q(x, t_n) - q(x, t_n), (\mathcal{P} q(x, t_n) - q(x, t_n))^+; \varphi) \\
&\quad + \int_\Omega (\mathcal{P} q(x, t_n) - q(x, t_n)) \xi dx + \mathbf{C}(\mathcal{P} p(x, t_n) - p(x, t_n), (\mathcal{P} p(x, t_n) - p(x, t_n))^-; \xi) \\
&\quad + \int_\Omega (\mathcal{P} p(x, t_n) - p(x, t_n)) \eta dx + \mathbf{C}(\mathcal{P}^- u(x, t_n) - u(x, t_n), (\mathcal{P}^- u(x, t_n) - u(x, t_n))^-; \eta) \\
&\quad + \int_\Omega (\mathcal{P} q^*(x, t_n) - q^*(x, t_n) \theta dx + \mathbf{C}(\mathcal{P} p(x, t_n) - p(x, t_n), (\mathcal{P} p(x, t_n) - p(x, t_n))^+; \theta) \\
&\quad + \tau \sum_{j=1}^N ([\mathcal{P} q(x, t_n) - q(x, t_n) - \lambda(\mathcal{P} s(x, t_n) - s(x, t_n))][\varphi] \\
&\quad + [\lambda(\mathcal{P} w(x, t_n) - w(x, t_n)) - (\mathcal{P} r(x, t_n) - r(x, t_n))][\xi])_{j-\frac{1}{2}} = 0.
\end{aligned}
\tag{3.19}
$$

For convenience, we denote the left-hand and right-hand terms in (3.19) by LHT and RHT, respectively. Let the test functions $v = \mathcal{P}^- e_u^n, \eta = -\beta \mathcal{P} e_{q^*}^n + \beta \mathcal{P} e_s^n - \beta\lambda \mathcal{P} e_z^n, \varphi = \beta \mathcal{P} e_q^n - \beta\lambda \mathcal{P} e_s^n, \theta = \beta \mathcal{P} e_p^n, \xi = -\beta \mathcal{P} e_r^n + \beta\lambda \mathcal{P} e_w^n, \phi = -\beta \mathcal{P} e_p^n + \beta\lambda \mathcal{P} e_r^n, \rho = \beta\lambda \mathcal{P} e_p^n, \psi = -\beta\lambda \mathcal{P} e_q^n$ in (3.19) with the fluxes (3.6). For the left-hand term LHT of (3.19), with the analogue argument for the equality (3.11), we have

$$
\begin{aligned}
LHT \geq & \|\mathcal{P}^- e_u^n\|^2 + \frac{\beta}{2}\sum_{j=1}^{N}([\mathcal{P}^- e_u^n]^2 + [\mathcal{P} e_p^n]^2 + [\mathcal{P} e_q^n]^2 + \lambda[\mathcal{P} e_r^n]^2)_{j-\frac{1}{2}} \\
& + \beta\tau\sum_{j=1}^{N}[\mathcal{P} e_q^n - \lambda\mathcal{P} e_s^n]^2 + [-\mathcal{P} e_r^n + \lambda\mathcal{P} e_w^n]^2)_{j-\frac{1}{2}}.
\end{aligned} \tag{3.20}
$$

Now, we consider the right-hand term RHT of (3.19). Using the properties (2.2) and (2.3), further noticing the fact that $\phi\varphi \leq \varepsilon\phi^2 + \frac{1}{4\varepsilon}\varphi^2$, we get,

$$
\begin{aligned}
RHT \leq & (\sum_{i=1}^{n-1}(b_{i-1} - b_i)\|\mathcal{P}^- e_u^{n-i}\| + b_{n-1}\|\mathcal{P}^- e_u^0\|)^2 + \frac{1}{2}\|\mathcal{P}^- e_u^n\|^2 \\
& + \beta\varepsilon(2+\lambda)\sum_{j=1}^{N}[\mathcal{P}^- e_u^n]_{j-\frac{1}{2}}^2 + (\sum_{i=1}^{n-1}(b_{i-1} - b_i)\|\mathcal{P}^- u(x,t_{n-i}) - u(x,t_{n-i})\| \\
& + b_{n-1}\|\mathcal{P}^- u(x,t_0) - u(x,t_0)\| + \|\mathcal{P}^- u(x,t_n) - u(x,t_n)\| + \beta\|\gamma^n(x)\|)^2 \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} q^*(x,t_n) - q^*(x,t_n))^+)_{j-\frac{1}{2}}^2 + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} p(x,t_n) - p(x,t_n))^+)_{j-\frac{1}{2}}^2 \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} s(x,t_n) - s(x,t_n))^+)_{j-\frac{1}{2}}^2 + \frac{\beta\lambda}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} z(x,t_n) - z(x,t_n))^+)_{j-\frac{1}{2}}^2 \\
& + \frac{\beta\lambda}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} w(x,t_n) - w(x,t_n))^+)_{j-\frac{1}{2}}^2 + \beta\varepsilon(3+\lambda)\sum_{j=1}^{N}[\mathcal{P} e_p^n]_{j-\frac{1}{2}}^2 \\
& + \frac{\beta\lambda}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} s(x,t_n) - s(x,t_n))^-)_{j-\frac{1}{2}}^2 + \beta\lambda\varepsilon\sum_{j=1}^{N}[\mathcal{P} e_q^n]_{j-\frac{1}{2}}^2 \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} r(x,t_n) - r(x,t_n))^+)_{j-\frac{1}{2}}^2 + 2\beta\lambda\varepsilon\sum_{j=1}^{N}[\mathcal{P} e_r^n]_{j-\frac{1}{2}}^2 \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} q(x,t_n) - q(x,t_n))^+)_{j-\frac{1}{2}}^2 + 3\beta\varepsilon\sum_{j=1}^{N}[\mathcal{P} e_q^n - \lambda\mathcal{P} e_s^n]_{j-\frac{1}{2}}^2 \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P} p(x,t_n) - p(x,t_n))^-)_{j-\frac{1}{2}}^2 + 3\beta\varepsilon\sum_{j=1}^{N}[-\mathcal{P} e_r^n + \lambda\mathcal{P} e_w^n]_{j-\frac{1}{2}}^2 \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}(([\mathcal{P} q(x,t_n) - q(x,t_n)]^2 + \lambda[\mathcal{P} s(x,t_n) - s(x,t_n)]^2)_{j-\frac{1}{2}} \\
& + \frac{\beta}{4\varepsilon}\sum_{j=1}^{N}((\lambda[\mathcal{P} w(x,t_n) - w(x,t_n)]^2 + [\mathcal{P} r(x,t_n) - r(x,t_n)]^2)_{j-\frac{1}{2}}.
\end{aligned} \tag{3.21}
$$

Choosing a small enough ε such that the terms in (3.21) can be controlled by the corresponding ones in (3.20), and using (3.20), (3.21) and the property (2.5) , there exists a positive constant C such that

$$\|\mathcal{P}^-e_u^n\|^2 \leq 2(\sum_{i=1}^{n-1}(b_{i-1}-b_i)\|\mathcal{P}^-e_u^{n-i}\| + b_{n-1}\|\mathcal{P}^-e_u^0\|)^2 + C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}})^2. \tag{3.22}$$

Now we prove the error estimate by mathematical induction. When $n=1$, the equation (3.22) becomes

$$\|\mathcal{P}^-e_u^1\|^2 \leq 2\|\mathcal{P}^-e_u^0\|^2 + C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}})^2.$$

It is easy to see that $\|\mathcal{P}^-e_u^0\| \leq Ch^{k+1}$, then we deduce

$$\|\mathcal{P}^-e_u^1\| \leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}).$$

Next we suppose the following inequality holds

$$\|\mathcal{P}^-e_u^m\| \leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}), m=2,3,\cdots K. \tag{3.23}$$

When $n=K+1$, from the equation (3.22), we deduce

$$\|\mathcal{P}^-e_u^{K+1}\|^2 \leq 2(\sum_{i=1}^{K}(b_{i-1}-b_i)\|\mathcal{P}^-e_u^{K+1-i}\| + b_K\|\mathcal{P}^-e_u^0\|)^2 + C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}})^2.$$

Noticing the property (3.2), and using the assumption (3.23), we can get the following result immediately

$$\|\mathcal{P}^-e_u^{K+1}\| \leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}).$$

Then Theorem 3.2 follows from the triangle inequality and the standard approximation result (2.5). □

4. Numerical examples

In this section, we offer some numerical examples to illustrate the accuracy and capability of the method. For this purpose, we calculate the numerical results of the exact solutions (for the cases where exact solutions are available). We mainly focus on the spatial accuracy, so a small time step is used such that errors from the temporal approximation is negligible.

Example 4.1. In this example we show an accuracy test for the nonhomogeneous linear time-fractional sKdV equation in $[0,2\pi]\times[0,1]$

$$D_t^\alpha u(x,t) + u_x + u_{3x} - u_{5x} - 2u_{7x} = f(x,t), \tag{4.1}$$

and the forcing term

$$f(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\sin x + t^2\cos x,$$

then the exact solution is $u=t^2\sin x$. Choosing $\tau=0.5$ in numerical experiment, and fixing the time step $\Delta t = 1/1000$. In Table 1 we list the L^2 and L^∞ errors and the numerical orders of accuracy at time $T=1$ for different α, and from that we can see the order of convergence using piecewise P^2 elements gives a uniform third order of accuracy in both norms.

Example 4.2. We show an accuracy test for the nonlinear time-fractional sKdV equation

$$D_t^\alpha u(x,t) + uu_x + u_{3x} - u_{5x} + u_{7x} = f(x,t). \tag{4.2}$$

TABLE 1. The error and order of convergence of the scheme (3.5) for the linear time-fractional sKdV equation (4.1) using piecewise P^2 elements.

	N	L^2-error	order	L^∞-error	order
$\alpha = 0.1$	5	1.627915353246706E-002	-	2.958432094950025E-002	-
	10	2.101079671943452E-003	2.95	3.871267797717737E-003	2.93
	15	6.291098602417318E-004	2.97	1.202036665154643E-003	2.88
	20	2.668549716851078E-004	2.98	5.148238366143310E-004	2.95
$\alpha = 0.3$	5	1.627324721589849E-002	-	2.963000193552157E-002	-
	10	2.103578295177149E-003	2.95	3.828415221396320E-003	2.95
	15	6.336021836479708E-004	2.96	1.165652130645539E-003	2.93
	20	2.760545793091325E-004	2.89	4.750817692477094E-004	3.12
$\alpha = 0.5$	5	1.625798660761131E-002	-	2.977653317982076E-002	-
	10	2.100869882521485E-003	2.95	3.883060959286588E-003	2.94
	15	6.293239638002948E-004	2.97	1.207122483855794E-003	2.88
	20	2.674131017988163E-004	2.97	5.194149367315215E-004	2.93

We take the exact solution $u(x,t) = t^2 \cos x$, then the forcing term

$$f(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \cos x - \frac{1}{2} t^4 \sin(2x) + 3t^2 \sin x,$$

The solution is computed with a periodic boundary condition in $[0, 2\pi]$ using P^2 elements. Let $\tau = 0.1$. In Figure 1, we show the errors in L^2-norm and L^1-norm attains third order of accuracy for piecewise P^2 polynomials for three values of α: 0.2, 0.4 and 0.6. The numerical results are consistent with our theoretical results in Theorem 3.2. In Figure 2, we show the the variation of the error with alpha when $N = 30, T = 1$.

5. Conclusion

In this paper, an implicit fully discrete local discontinuous Galerkin (LDG) finite element method is presented for solving the time-fractional sKdV equation. The stability and error analysis for the linear case is performed. The numerical experiments confirm the validity of the method and indicate that the scheme is a good tool to solve such equations. The method and analytical technique can also be extended to other kinds of time-fractional equations and higher-dimensional problems easily.

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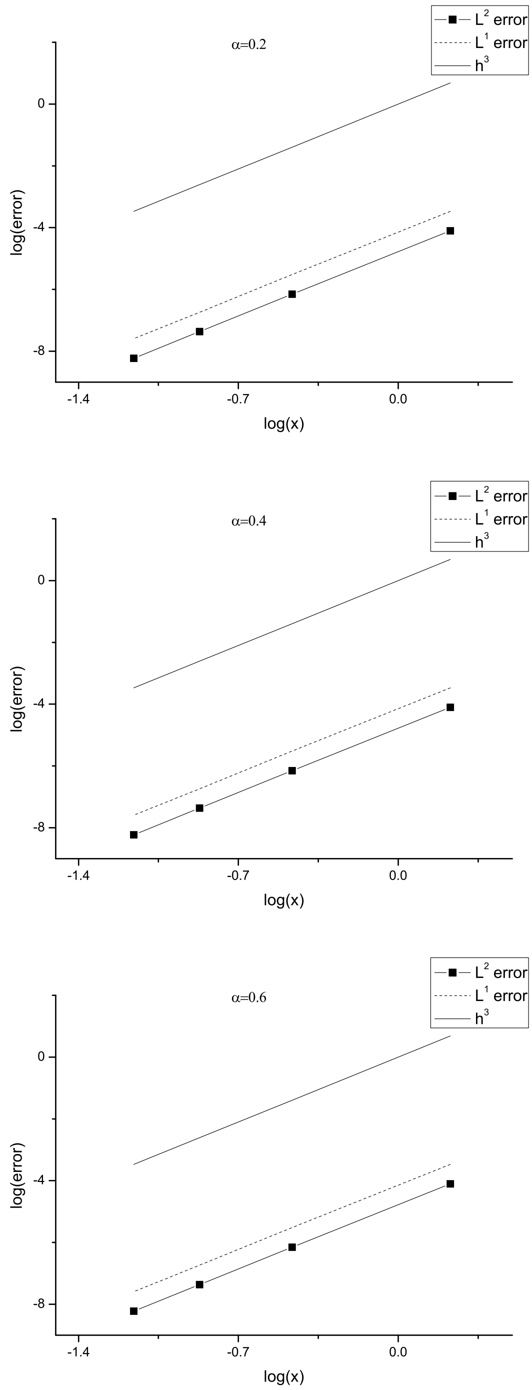

FIGURE 1. Convergence rate for different α when using piecewise P^2 polynomials.

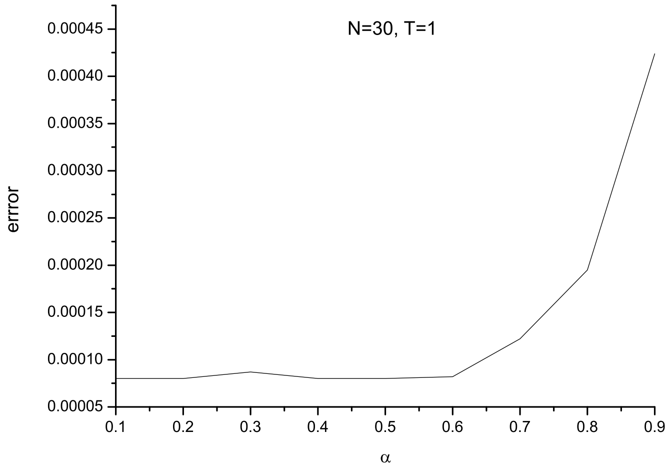

FIGURE 2. The variation of the L^2 error with alpha when $N = 30, T = 1$.

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