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RESIDUAL-BASED A POSTERIORI ESTIMATORS FOR THE T/Ω MAGNETODYNAMIC HARMONIC FORMULATION OF THE MAXWELL SYSTEM

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Abstract. In this paper, we focus on an a posteriori residual-based error estimator for the $\mathbf{T}/\Omega$ magnetodynamic harmonic formulation of the Maxwell system. Similarly to the $\mathbf{A}/\varphi$ formulation [7], the weak continuous and discrete formulations are established, and the well-posedness of both of them is addressed. Some useful analytical tools are derived. Among them, an ad-hoc Helmholtz decomposition for the $\mathbf{T}/\Omega$ case is derived, which allows to pertinently split the error. Consequently, an a posteriori error estimator is obtained, which is proven to be reliable and locally efficient. Finally, numerical tests confirm the theoretical results.

Key words. Maxwell equations, potential formulations, a posteriori estimators, finite element method.

1. Introduction

Let us consider the electromagnetic fields, modeled by the well-known Maxwell system :

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{2}$$

where $\mathbf{E}$ is the electrical field, $\mathbf{H}$ the magnetic field, $\mathbf{B}$ the magnetic flux density, $\mathbf{J}$ the current flux density (or eddy current) and $\mathbf{D}$ the displacement flux density. Equation (1) is classically called Maxwell-Faraday equation and equation (2) Maxwell-Ampère one. In the low frequency regime, the quasistatic approximation can be applied, which consists in neglecting the temporal variation of the displacement flux density with respect to the current density [12], such that the propagation phenomena are not taken into account. Consequently, equation (2) becomes :

$$\operatorname{curl}\mathbf{H} = \mathbf{J}. \tag{3}$$

Here, the current density $\mathbf{J}$ can be decomposed in two terms such that $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_{ec}$. $\mathbf{J}_s$ is a known distribution current density generally generated by a coil, and $\mathbf{J}_{ec}$ represents the eddy current. Both equations (1) and (3) are linked by the material constitutive laws :

$$\mathbf{B} = \mu\mathbf{H}, \tag{4}$$

$$\mathbf{J}_{ec} = \sigma\mathbf{E}, \tag{5}$$

where μ stands for the magnetic permeability and σ for the electrical conductivity of the material. Figure 1 displays the domains configuration we are interested

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in. We consider an open connected domain $D \subset \mathbb{R}^3$, with a Lipschitz boundary $\Gamma = \partial D$. We define an open simply connected conductor domain $D_c \subset D$ and we note $\Gamma_c = \partial D_c$ its boundary which is supposed to be Lipschitz and such that $\Gamma_c \cap \Gamma = \emptyset$. In D_c, the electrical conductivity σ is not equal to zero so that eddy currents can be created. Finally we define $D_e = D \backslash \overline{D_c}$ as the part of D where the electrical conductivity σ is equal to zero. Boundary conditions associated with the system (1)-(3) are given by:

$$\mathbf{B}.\mathbf{n} = 0 \text{ on } \Gamma, \tag{6}$$

$$\mathbf{J}_{ec}.\mathbf{n} = 0 \text{ on } \Gamma_c. \tag{7}$$

We intend to solve this problem by using the potential formulations often used for

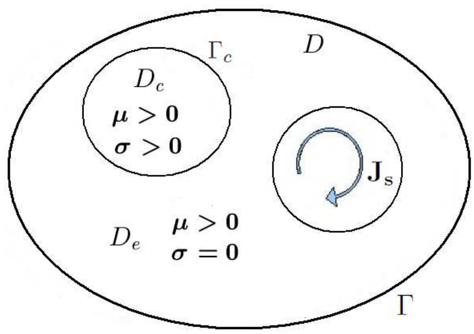


FIGURE 1. Domains configuration.

electromagnetic problems. A similar work was already done for the so-called $\mathbf{A}/\varphi$ formulation [7]. Another recent paper was concerned with the $\mathbf{A}/\varphi$ formulation [3] but in a different framework, having at last the potential vector $\mathbf{A}$ as an unique unknown, and considering the case where μ is constant. Here, we consider the $\mathbf{T}/\Omega$ formulation which is first described. Since $\operatorname{div} \mathbf{J}_s = 0$ in D, there exists a source magnetic field $\mathbf{H}_s$ such that [9]:

$$\operatorname{curl} \mathbf{H}_s = \mathbf{J}_s \text{ in } D,$$

and since the conductor domain D_c is simply connected, as $\operatorname{div} \mathbf{J}_{ec} = 0$, there exists a source magnetic field $\mathbf{T}$ such that:

$$\operatorname{curl} \mathbf{T} = \mathbf{J}_{ec} \text{ in } D_c.$$

From (3), a magnetic scalar potential Ω can be introduced so that the magnetic field $\mathbf{H}$ can be written by:

$$\mathbf{H} = \begin{cases} \mathbf{H}_s + \mathbf{T} - \nabla\Omega & \text{in} \quad D_c, \\ \\ \mathbf{H}_s - \nabla\Omega & \text{in} \quad D_e. \end{cases} \tag{8}$$

From (4), (5) and (8), equation (1) becomes:

$$\operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl}\mathbf{T}\right) + \frac{\partial}{\partial t}\left(\mu\left(\mathbf{T} - \nabla\Omega\right)\right) = -\frac{\partial}{\partial t}\left(\mu\mathbf{H}_s\right) \text{ in } D_c. \tag{9}$$

Consequently, we also have

$$\operatorname{div}\left(\mu\left(\mathbf{T} - \nabla\Omega\right)\right) = -\operatorname{div}\left(\mu\mathbf{H}_s\right) \text{ in } D_c. \tag{10}$$

Moreover since $\operatorname{div}\mathbf{B}=0$ in D, we get

$$\operatorname{div}\left(-\mu\nabla\Omega\right)=-\operatorname{div}\left(\mu\mathbf{H}_s\right) \text{ in } D_e. \tag{11}$$

The content of the paper is as follows. Section 2 establishes the weak formulation of the continuous and discrete problems, and the well-posedness of both of them is proven. In Section 3, we derive an ad-hoc Helmholtz decomposition of the error in the conductor domain D_c. Section 3 is devoted to several analytical tools needed in the following of the paper. In Section 4, the reliability and the efficiency of the derived estimator are established. Finally, some numerical tests are treated in Section 5 to evaluate the estimator capabilities.

2. Weak formulation and discretization of the problem

2.1. Weak Formulation. From (9), (10) and (11), the $\mathbf{T}/\Omega$ formulation of the magnetodynamic problem can be written:

$$\left\{\begin{array}{rcll} \operatorname{curl}\left(\dfrac{1}{\sigma}\operatorname{curl}\mathbf{T}\right)+\dfrac{\partial}{\partial t}\left(\mu\left(\mathbf{T}-\nabla\Omega\right)\right) & = & -\dfrac{\partial}{\partial t}\left(\mu\mathbf{H}_s\right) & \text{in} \quad D_c, \\ \operatorname{div}\left(\mu\left(\mathbf{T}-\nabla\Omega\right)\right) & = & -\operatorname{div}\left(\mu\mathbf{H}_s\right) & \text{in} \quad D_c, \\ \operatorname{div}\left(-\mu\nabla\Omega\right) & = & -\operatorname{div}\left(\mu\mathbf{H}_s\right) & \text{in} \quad D_e. \end{array}\right. \tag{12}$$

We suppose that $\mu\in L^\infty(D)$ and that there exists $\mu_{\min}\in\mathbb{R}_+^*$ such that $\mu>\mu_{\min}$ on D. We also assume that $\sigma\in L^\infty(D)$, that there exists $\sigma_{\min}\in\mathbb{R}_+^*$ such that $\sigma>\sigma_{\min}$ on D_c, and we recall that $\sigma_{|D_e}\equiv 0$. Taking the associated boundary conditions (6) and (7) into account, the harmonic formulation of the system (12) is given by: Find $(\mathbf{T},\Omega)\in V$ such that for all $(\mathbf{T}',\Omega')\in V$, we have

$$\left\{\begin{array}{l} \displaystyle\int_{D_c}\frac{1}{\sigma}\operatorname{curl}\mathbf{T}\cdot\overline{\operatorname{curl}\mathbf{T}'}+\int_{D_c}j\omega\mu\left(\mathbf{T}-\nabla\Omega\right)\cdot\overline{\mathbf{T}'}=\int_{D_c}-j\omega\mu\mathbf{H}_s\cdot\overline{\mathbf{T}'}, \\ \displaystyle -\int_{D_c}\mu\left(\mathbf{T}-\nabla\Omega\right)\cdot\overline{\nabla\Omega'}+\int_{D_e}\mu\nabla\Omega\cdot\overline{\nabla\Omega'}=\int_{D}\mu\mathbf{H}_s\cdot\overline{\nabla\Omega'}, \end{array}\right. \tag{13}$$

where $j^2=-1$ and $V=X^0(D_c)\times\widetilde{H^1}(D)$ with:

$$X(D_c)=H_0(\operatorname{curl},D_c)=\left\{\mathbf{T}\in\mathbf{L}^2(D_c);\operatorname{curl}\mathbf{T}\in\mathbf{L}^2(D_c)\text{ and }\mathbf{T}\times\mathbf{n}=\mathbf{0}\text{ on }\Gamma_c\right\},$$

$$X^0(D_c)=\left\{\mathbf{T}\in X(D_c);\int_{D_c}\mathbf{T}\,.\,\nabla\xi\,dx=0,\forall\xi\in H_0^1(D_c)\right\},$$

$$\widetilde{H^1}(D)=\left\{\Omega\in H^1(D);\int_D\Omega=0\right\}.$$

In the following, on a given domain D, the $L^2(D)$ norm will be denoted by $||\cdot||_D$, and the corresponding $L^2(D)$ inner product by $(\cdot,\cdot)_D$. The usual norm and semi-norm of $H^1(D)$ will be denoted by $||\cdot||_{1,D}$ and $|\cdot|_{1,D}$ respectively. Now, $X(D_c)$ is equipped with its usual norm :

$$\|\mathbf{T}\|^2_{X(D_c)}=\|\mathbf{T}\|^2_{L^2(D_c)}+\|\operatorname{curl}\mathbf{T}\|^2_{L^2(D_c)},$$

and the natural norm $||.||_V$ associated with the Hilbert space V is given by :

$$\|(\mathbf{T},\Omega)\|^2_V=||\mathbf{T}||^2_{X(D_c)}+|\Omega|^2_{1,D}.$$

It can be directly obtained that an equivalent variational formulation to (13) is given by : Find $(\mathbf{T}, \Omega) \in V$ solution of

$$a((\mathbf{T}, \Omega), (\mathbf{T}', \Omega')) = l((\mathbf{T}', \Omega')), \forall (\mathbf{T}', \Omega') \in V, \tag{14}$$

where a and l are respectively the following bilinear and linear forms defined by:

$$a((\mathbf{T}, \Omega), (\mathbf{T}', \Omega')) =$$

$$\int_{D_c} \frac{1}{\sigma} \operatorname{curl} \mathbf{T} \cdot \overline{\operatorname{curl} \mathbf{T}'} + \int_{D_c} j\omega\mu(\mathbf{T} - \nabla\Omega) \cdot \overline{(\mathbf{T}' - \nabla\Omega')} + \int_{D_e} j\omega\mu\nabla\Omega \cdot \overline{\nabla\Omega'},$$

$$l((\mathbf{T}', \Omega')) = -\int_{D_c} j\omega\mu\mathbf{H}_s \cdot \overline{(\mathbf{T}' - \nabla\Omega')} + \int_{D_e} j\omega\mu\mathbf{H}_s \cdot \overline{\nabla\Omega'}.$$

2.2. Well-posedness of the problem.

Lemma 2.1. *The bilinear form $\sqrt{2}e^{-j\frac{\pi}{4}}a$ is coercive on V, namely there exists $C > 0$ such that:*

$$\left|\sqrt{2}e^{-j\frac{\pi}{4}}a((\mathbf{T}, \Omega), (\mathbf{T}, \Omega))\right| \geq C\left\|(\mathbf{T}, \Omega)\right\|_V^2.$$

Proof. First, let us notice that:

$$\Re\left[\sqrt{2}e^{-j\frac{\pi}{4}}a\left((\mathbf{T}, \Omega), (\mathbf{T}, \Omega)\right)\right] = \int_{D_c} \frac{1}{\sigma}\left|\operatorname{curl} \mathbf{T}\right|^2 + \int_{D_c} \omega\mu\left|\mathbf{T} - \nabla\Omega\right|^2 + \int_{D_e} \omega\mu\left|\nabla\Omega\right|^2.$$

Our aim is to prove that there exists $C > 0$ such that for all $(\mathbf{T}, \Omega) \in V$,

$$\int_{D_c} \frac{1}{\sigma}\left|\operatorname{curl} \mathbf{T}\right|^2 + \int_{D_c} \omega\mu\left|\mathbf{T} - \nabla\Omega\right|^2 + \int_{D_e} \omega\mu\left|\nabla\Omega\right|^2 \geq C\left\|(\mathbf{T}, \Omega)\right\|_V^2.$$

This is done by an usual contradiction argument and using the fact that for $\mathbf{T} \in X^0(D_c)$, the Friedrichs-Poincaré inequality $\|\mathbf{T}\|_{L^2(D_c)} \leq C\|\operatorname{curl} \mathbf{T}\|_{L^2(D_c)}$ holds (see [11]). We refer to [7] for a similar proof in the $\mathbf{A}/\varphi$ formulation case. □

Theorem 2.2. *The weak formulation (14) admits a unique solution $(\mathbf{T}, \Omega) \in V$.*

Proof. The sesquilinear form $|\sqrt{2}e^{-j\frac{\pi}{4}}a|$ is obviously continuous on $V \times V$ and coercive on V by Lemma 2.1. So Lax-Milgram's lemma ensures existence and uniqueness of a solution $(\mathbf{T}', \Omega') \in V$ to (14). □

Lemma 2.3. *Let $(\mathbf{T}', \Omega') \in V$ be the unique solution of (14). Then for all $(\mathbf{T}', \Omega') \in X(D_c) \times H^1(D)$, we have :*

$$a\left((\mathbf{T}, \Omega), (\mathbf{T}', \Omega')\right) = l\left((\mathbf{T}', \Omega')\right).$$

Proof. Since $\mathbf{T}' \in X(D_c)$, we can decompose it using the following Helmholtz decomposition [11, p. 66]:

$$\mathbf{T}' = \Psi + \nabla\tau,$$

where $\Psi \in X^0(D_c)$ and $\tau \in H_0^1(D_c)$. It is known that for any function $\Omega' \in H^1(D)$, we can find $\widehat{\Omega'} \in \widetilde{H^1}(D)$ such that $\nabla\widehat{\Omega'} = \nabla\Omega'$ by the definition:

$$\widehat{\Omega'} = \Omega' - \frac{1}{|D|}\int_D \Omega'.$$

We write

$$a((\mathbf{T},\Omega),(\mathbf{T}',\Omega')) = a((\mathbf{T},\Omega),(\Psi+\nabla\tau,\Omega'))$$

$$= \int_{D_c} \frac{1}{\sigma}\operatorname{curl}\mathbf{T}\cdot\overline{\operatorname{curl}\Psi} + \int_{D_c} j\omega\mu(\mathbf{T}-\nabla\Omega)\cdot\overline{(\Psi-\nabla(\widehat{\Omega'}-\tau))} + \int_{D_e} j\omega\mu\nabla\Omega\cdot\overline{\nabla\widehat{\Omega'}}$$

$$= \int_{D_c} \frac{1}{\sigma}\operatorname{curl}\mathbf{T}\cdot\overline{\operatorname{curl}\Psi} + \int_{D_c} j\omega\mu(\mathbf{T}-\nabla\Omega)\cdot\overline{(\Psi-\nabla\widetilde{\Omega'})} + \int_{D_e} j\omega\mu\nabla\Omega\cdot\overline{\nabla\widetilde{\Omega'}},$$

where we define $\widetilde{\Omega'} \in \widetilde{H^1}(D)$ by

$$\widetilde{\Omega'} = \begin{cases} \widehat{\Omega'} - \tau + \dfrac{1}{|D|}\displaystyle\int_{D_c}\tau & \text{in } D_c, \\[2ex] \widehat{\Omega'} + \dfrac{1}{|D|}\displaystyle\int_{D_c}\tau & \text{in } D_e. \end{cases}$$

Therefore we conclude that:

$$a((\mathbf{T},\Omega),(\Psi+\nabla\tau,\Omega')) = a((\mathbf{T},\Omega),(\Psi,\widetilde{\Omega'})). \tag{15}$$

Similarly it is clear that

$$l((\mathbf{T}',\Omega')) = l((\Psi+\nabla\tau,\Omega'))$$

$$= -\int_{D_c} j\omega\mu\mathbf{H}_s\cdot\overline{(\Psi-\nabla\widetilde{\Omega'})} + \int_{D_e} j\omega\mu\mathbf{H}_s\cdot\overline{\nabla\widetilde{\Omega'}}$$

$$= l((\Psi,\widetilde{\Omega'})).$$

Taking into account that $\Psi \in X^0(D_c)$ and $\widetilde{\Omega'} \in \widetilde{H^1}(D)$, from (14), we get

$$a((\mathbf{T},\Omega),(\Psi,\widetilde{\Omega'})) = l((\Psi,\widetilde{\Omega'})). \tag{16}$$

From (15) and (16) we conclude that

$$a((\mathbf{T},\Omega),(\mathbf{T}',\Omega')) = l((\mathbf{T}',\Omega')).$$

□

2.2.1. Discrete formulation. Now, the boundaries Γ_c and Γ are supposed to be polyhedral such that the domain D can be discretized by a conforming mesh $\mathscr{T}_h$ made of tetrahedra, each element T of $\mathscr{T}_h$ belonging either to D_c or D_e. The faces of $\mathscr{T}_h$ are denoted by F and its edges by E. Let us denote by h_T the diameter of T and ρ_T the diameter of its largest inscribed ball. We suppose that for any element T, the ratio h_T/ρ_T is bounded by a constant $\alpha > 0$ independant of T and of the mesh size $h = \max\limits_{T\in\mathscr{T}_h} h_T$. The set of faces (resp. edges and nodes) of the triangulation is denoted $\mathcal{F}$ (resp. $\mathcal{E}$ and $\mathcal{N}$), and we denote h_F the diameter of the face F. The set of internal faces (resp. internal edges and internal nodes) to D is denoted $\mathcal{F}_{int}$ (resp. $\mathcal{E}_{int}$ and $\mathcal{N}_{int}$). The coefficients μ and σ arising in (12) are moreover supposed to be constant on each tetrahedron of the mesh, and we will note $\mu_T = \mu_{|T}$ and $\sigma_T = \sigma_{|T}$ for all $T \in \mathscr{T}_h$.

The approximation space V_h is defined by $V_h = X_h^0 \times \widetilde{\Theta}_h$, where :

$$X_h(D_c) = X(D_c) \cap \mathcal{ND}_1(D_c,\mathscr{T}_h) = \Big\{\mathbf{T}_h \in X(D_c); \mathbf{T}_{h|T} \in \mathcal{ND}_1(T),\ \forall\, T \in \mathscr{T}_h\Big\},$$

$$\mathcal{ND}_1(T) = \left\{ \mathbf{T}_h : \begin{array}{ccc} T & \longrightarrow & \mathbb{C}^3 \\ \mathbf{x} & \longrightarrow & \mathbf{a} + \mathbf{b} \times \mathbf{x} \end{array}, \mathbf{a}, \mathbf{b} \in \mathbb{C}^3 \right\},$$

$$\Theta_h = \left\{ \Omega_h \in H^1(D); \Omega_{h|T} \in \mathbb{P}_1(T) \ \forall\, T \in \mathscr{T}_h \right\},$$

$$\Theta_h^0 = \left\{ \xi_h \in H_0^1(D_c); \xi_{h|T} \in \mathbb{P}_1(T) \ \forall\, T \in \mathscr{T}_h \right\},$$

$$X_h^0(D_c) = \left\{ \mathbf{T}_h \in X_h(D_c); (\mathbf{T}_h, \nabla \xi_h)_{D_c} = 0 \ \forall\, \xi_h \in \Theta_h^0 \right\},$$

$$\widetilde{\Theta}_h = \left\{ \Omega_h \in \widetilde{H^1}(D); \Omega_{h|T} \in \mathbb{P}_1(T) \ \forall\, T \in \mathscr{T}_h \right\}.$$

Now, the discretized weak formulation associated with (14) consists in finding $(\mathbf{T}_h, \Omega_h) \in V_h$ such that for all $(\mathbf{T}'_h, \Omega'_h) \in V_h$, we have :

$$a((\mathbf{T}_h, \Omega_h), (\mathbf{T}'_h, \Omega'_h)) = l((\mathbf{T}'_h, \Omega'_h)). \tag{17}$$

Theorem 2.4. *The weak formulation (17) admits a unique solution* $(\mathbf{T}_h, \Omega_h) \in V_h$.

Proof. The proof is in any point similar to the one of Theorem 2.2 for the continuous case. The main point relies in proving the coercivity of a on V_h, which can be done by using the discrete Poincaré-Friedrichs inequality $||\mathbf{T}_h|| \leq C||\operatorname{curl} \mathbf{T}_h||$ for all $\mathbf{T}_h \in X_h^0(D_c)$, with $C > 0$ independent of h, see [11, p. 185]. □

Remark 2.5. *Let us notice that, because of the discrete Gauge condition arising in the definition of* X_h^0, V_h *is not included in* V, *so that the approximation is not a conforming one.*

Lemma 2.6. *For all* $(\mathbf{T}'_h, \Omega'_h) \in X_h(D_c) \times \Theta_h$, *we have:*

$$a((\mathbf{T}_h, \Omega_h), (\mathbf{T}'_h, \Omega'_h)) = l((\mathbf{T}'_h, \Omega'_h)).$$

Proof. : Since $\mathbf{T}'_h \in X_h(D_c)$, this time we can use the discrete Helmholtz decomposition [8, p. 272]:

$$\mathbf{T}'_h = \Psi_h + \nabla \tau_h \text{ with } \Psi_h \in X_h^0(D_c) \text{ and } \tau_h \in \Theta_h^0.$$

The proof is then very similar to the one for the continuous case, see Lemma 2.3. □

A direct consequence of Lemmas 2.3 and 2.6 is the following orthogonality property, despite the fact that the approximation is not a conforming one :

Lemma 2.7. *For all* $(\mathbf{T}'_h, \Omega'_h) \in X_h(D_c) \times \Theta_h$

$$a((\mathbf{T} - \mathbf{T}_h, \Omega - \Omega_h), (\mathbf{T}'_h, \Omega'_h)) = 0.$$

3. Helmholtz decomposition

In the following, the notations $a \lesssim b$ and $a \sim b$ mean the existence of positive constants C_1 and C_2 which are independent of the quantities a and b under consideration as well as of the mesh size h, the coefficients μ, σ and of the frequency ω, such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively.

Helmholtz decomposition. Compared to the work on the $\mathbf{A}/\varphi$ formulation [7], one suitable Helmholtz decomposition has to be found for the $\mathbf{T}/\Omega$ formulation. Let us define the errors on $\mathbf{T}$ and Ω by:

$$\mathbf{e_T} = \mathbf{T} - \mathbf{T}_h \in H_0(\text{curl}, D_c), \tag{18}$$

$$e_\Omega = \Omega - \Omega_h \in \widetilde{H^1}(D). \tag{19}$$

Let $X_N(D_c)$ denote the space:

$$X_N(D_c) = \{\mathbf{u} \in H(\text{curl}, D_c) \cap H(\text{div}\,, D_c), \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_c\}$$

equipped with the norm:

$$\|\mathbf{u}\|^2_{X_N(D_c)} = \|\mathbf{u}\|^2_{L^2(D_c)} + \|\text{div}\,\mathbf{u}\|^2_{L^2(D_c)} + \|\,\text{curl}\,\mathbf{u}\|^2_{L^2(D_c)}.$$

Theorem 3.1. *The error $\boldsymbol{e}_\mathbf{T} - \nabla e_\Omega$ admits the following Helmholtz decomposition in the conductor domain D_c:*

$$\boldsymbol{e}_\mathbf{T} - \nabla e_\Omega = \Psi + \nabla \Phi + \nabla(\psi - e_\Omega),$$

where $\Psi \in H^1(D_c) \cap X_N(D_c)$, $\Phi \in H^1_0(D_c)$ and $\psi \in H^1_0(D_c)$, with

$$\begin{aligned} &\|\Psi\|^2_{H^1(D_c)} + \|\nabla\Phi\|^2_{L^2(D_c)} + \|\nabla(\psi - e_\Omega)\|^2_{L^2(D_c)} \\ \lesssim\ &\|\boldsymbol{e}_\mathbf{T} - \nabla e_\Omega\|^2_{L^2(D_c)} + \|\nabla e_\Omega\|^2_{L^2(D_e)} + \|\,\text{curl}\,\boldsymbol{e}_\mathbf{T}\|^2_{L^2(D_c)}. \end{aligned}$$

Proof. Let us define

$$\widetilde{e_\Omega} = e_\Omega - \frac{1}{|D_e|}\int_{D_e} e_\Omega. \tag{20}$$

Since $e_\Omega \in \widetilde{H^1}(D)$, we have $e_\Omega \in H^1(D_c)$ and $e_\Omega \in H^1(D_e)$, which leads to $\widetilde{e_\Omega} \in H^1(D_c)$ and $\widetilde{e_\Omega} \in \widetilde{H^1}(D_e)$.

First of all, let us define $\psi \in H^1_0(D_c)$ such that

$$\begin{cases} \text{div}\,\nabla\psi &= \text{div}\,\mathbf{e_T} & \text{in } D_c, \\ \psi &= 0 & \text{on } \Gamma_c. \end{cases}$$

The definition $\mathbf{w} = \mathbf{e_T} - \nabla\psi$ yields

$$\text{curl}\,\mathbf{w} = \text{curl}\,\mathbf{e_T} \text{ in } D_c. \tag{21}$$

Clearly, $\mathbf{w} \in X_N(D_c)$. From (20) we get

$$\begin{aligned} \mathbf{e_T} - \nabla e_\Omega &= \mathbf{e_T} - \nabla\widetilde{e_\Omega} \\ &= \mathbf{w} + \nabla(\psi - \widetilde{e_\Omega}). \end{aligned}$$

From [1] we know that the embedding of $X_N(D_c)$ into $L^2(D_c)^3$ is compact, so that

$$\|\mathbf{w}\|^2_{X_N(D_c)} \lesssim \|\text{div}\,\mathbf{w}\|^2_{L^2(D_c)} + \|\,\text{curl}\,\mathbf{w}\|^2_{L^2(D_c)}.$$

Since $\text{div}\,\mathbf{w} = 0$, we have

$$\|\mathbf{w}\|^2_{X_N(D_c)} \lesssim \|\,\text{curl}\,\mathbf{w}\|^2_{L^2(D_c)}. \tag{22}$$

By (21), we can conclude that:

$$\|\mathbf{w}\|^2_{L^2(D_c)} \leq \|\mathbf{w}\|^2_{X_N(D_c)} \lesssim \|\,\text{curl}\,\mathbf{e_T}\|^2_{L^2(D_c)}, \tag{23}$$

and
(24)
$$\begin{aligned}\left\|\mathbf{e_T} - \nabla\widetilde{e_\Omega}\right\|^2_{L^2(D_c)} &= \int_{D_c} (\mathbf{w} + \nabla\psi - \nabla\widetilde{e_\Omega}) \cdot (\mathbf{w} + \nabla\psi - \nabla\widetilde{e_\Omega}) \\ &= \int_{D_c} |\mathbf{w}|^2 + \int_{D_c} |\nabla(\psi - \widetilde{e_\Omega})|^2 + 2\int_{D_c} \mathbf{w}\cdot\nabla(\psi - \widetilde{e_\Omega}).\end{aligned}$$

Using Green's formula
$$\begin{aligned}\int_{D_c} \mathbf{w}\cdot\nabla(\psi - \widetilde{e_\Omega}) &= \int_{\Gamma_c} \mathbf{w}\cdot\mathbf{n}\cdot(\psi - \widetilde{e_\Omega}) - \int_{D_c} \operatorname{div}\mathbf{w}\cdot(\psi - \widetilde{e_\Omega}) \\ &= -\int_{\Gamma_c} \mathbf{w}\cdot\mathbf{n}\cdot\widetilde{e_\Omega} \quad \text{since } \operatorname{div}\mathbf{w} = 0 \text{ and } \psi|_{\Gamma_c} = 0 \\ &\lesssim \left\|\mathbf{w}\cdot\mathbf{n}\right\|_{H^{-1/2}(\Gamma_c)} \cdot \left\|\widetilde{e_\Omega}\right\|_{H^{1/2}(\Gamma_c)}.\end{aligned}$$

Applying the trace theorem, we have:
$$\left\|\widetilde{e_\Omega}\right\|_{H^{1/2}(\Gamma_c)} \lesssim \left\|\widetilde{e_\Omega}\right\|_{H^1(D_e)},$$
and Theorem 2.5 in [8, p. 27] leads to:
$$\left\|\mathbf{w}\cdot\mathbf{n}\right\|_{H^{-1/2}(\Gamma_c)} \lesssim \|\mathbf{w}\|_{H(\operatorname{div},D_c)} = \|\mathbf{w}\|_{L^2(D_c)} .$$

From (23), we get
$$\left\|\mathbf{w}\cdot\mathbf{n}\right\|_{H^{-1/2}(\Gamma_c)} \lesssim \left\|\operatorname{curl}\mathbf{e_T}\right\|_{L^2(D_c)},$$
so that
$$\int_{D_c} \mathbf{w}\cdot\nabla(\psi - \widetilde{e_\Omega}) \lesssim \left\|\operatorname{curl}\mathbf{e_T}\right\|^2_{L^2(D_c)} + \left\|\widetilde{e_\Omega}\right\|^2_{H^1(D_e)}.$$

Taking into account (24), we get:
$$\left\|\mathbf{w}\right\|^2_{L^2(D_c)} + \left\|\nabla(\psi - \widetilde{e_\Omega})\right\|^2_{L^2(D_c)} \lesssim \left\|\mathbf{e_T} - \nabla\widetilde{e_\Omega}\right\|^2_{L^2(D_c)} + \left\|\widetilde{e_\Omega}\right\|^2_{H^1(D_e)} + \left\|\operatorname{curl}\mathbf{e_T}\right\|^2_{L^2(D_c)}.$$

Since $\widetilde{e_\Omega} \in \widetilde{H^1}(D_e)$, Poincaré's inequality $\left\|\widetilde{e_\Omega}\right\|^2_{H^1(D_e)} \lesssim \left\|\nabla\widetilde{e_\Omega}\right\|^2_{L^2(D_e)}$ holds. Hence we have:
(25)
$$\left\|\mathbf{w}\right\|^2_{L^2(D_c)} + \left\|\nabla(\psi - e_\Omega)\right\|^2_{L^2(D_c)} \lesssim \left\|\mathbf{e_T} - \nabla e_\Omega\right\|^2_{L^2(D_c)} + \left\|\nabla e_\Omega\right\|^2_{L^2(D_e)} + \left\|\operatorname{curl}\mathbf{e_T}\right\|^2_{L^2(D_c)}.$$

Finally from [5], we get
$$X_N(D_c) = X_N(D_c) \cap H^1(D_c)^3 + \nabla H^1_0(D_c),$$
which implies that
$$\mathbf{w} = \Psi + \nabla\Phi,$$
where $\Psi \in H^1(D_c)^3 \cap X_N(D_c)$ and $\Phi \in H^1_0(D_c)$, with:
$$\left\|\Psi\right\|_{H^1(D_c)} + \left\|\nabla\Phi\right\|_{L^2(D_c)} \lesssim \left\|\mathbf{w}\right\|_{X_N(D_c)}. \tag{26}$$

(25), (26) associated with (21) and (22) conclude the proof. □

Applying a change with variables, we can have the following corollary:

Corollary 3.2. *The error* $\boldsymbol{e}_{\mathbf{T}} - \nabla e_\Omega$ *admits the following decomposition in* D_c*:*

$$\boldsymbol{e}_{\mathbf{T}} - \nabla e_\Omega = \Psi + \nabla\widetilde{\Phi} + \nabla\widehat{e_\Omega},$$

where $\Psi \in H^1(D_c) \cap X_N(D_c)$, $\widetilde{\Phi} \in H_0^1(D)$ *and* $\widehat{e_\Omega} \in H^1(D)$*. Moreover,*

$$\big\|\Psi\big\|^2_{H^1(D_c)} + \big\|\nabla\widetilde{\Phi}\big\|^2_{L^2(D)} + \big\|\nabla\widehat{e_\Omega}\big\|^2_{L^2(D)} \lesssim \big\|\boldsymbol{e}_{\mathbf{T}} - \nabla e_\Omega\big\|^2_{L^2(D_c)} + \big\|\nabla e_\Omega\big\|^2_{L^2(D_e)} + \big\|\operatorname{curl}\boldsymbol{e}_{\mathbf{T}}\big\|^2_{L^2(D_c)}.$$

Proof. Using the notation of Theorem 3.1, let us define $\widetilde{\Phi}$ by:

$$\widetilde{\Phi} = \begin{cases} \Phi & \text{in } D_c, \\ 0 & \text{in } D_e. \end{cases} \tag{27}$$

Since $\Phi \in H_0^1(D_c)$, we have that $\widetilde{\Phi} \in H_0^1(D)$ and $\|\nabla\widetilde{\Phi}\|_{L^2(D)} = \|\nabla\Phi\|_{L^2(D_c)}$.

Let us also define $\widehat{e_\Omega}$ by:

$$\widehat{e_\Omega} = \begin{cases} \psi - e_\Omega & \text{in } D_c, \\ -e_\Omega & \text{in } D_e. \end{cases} \tag{28}$$

Since $\psi \in H_0^1(D_c)$ and $e_\Omega \in \widetilde{H^1}(D)$, clearly $\widehat{e_\Omega} \in H^1(D)$ and

$$\big\|\nabla\widehat{e_\Omega}\big\|^2_{L^2(D)} = \big\|\nabla(\psi - e_\Omega)\big\|^2_{L^2(D_c)} + \big\|\nabla e_\Omega\big\|^2_{L^2(D_e)}.$$

□

4. Analytical tools

For completeness, this section defines and recalls some usual analytical tools used in the following of the paper to make it self-contained.

4.1. Standard Clément interpolation operator.

For our further analysis, we need an interpolation operator that maps a function from $H_0^1(D)$ to $\Theta_h^0(D) = \Big\{\xi_h \in H^1(D); \xi_{h|T} \in \mathbb{P}_1(T)\ \forall\, T \in \mathscr{T}_h \cap D\Big\}$, as well as an interpolation operator that maps a function from $H^1(D)$ to Θ_h. Hence Lagrange interpolation is unsuitable, but Clément like interpolant is more appropriate. Recall that the nodal basis functions $\varphi_{\boldsymbol{x}} \in \Theta_h^0$ associated with a node $\boldsymbol{x}$ is uniquely determined by the condition :

$$\varphi_{\boldsymbol{x}}(\boldsymbol{y}) = \delta_{\boldsymbol{x},\boldsymbol{y}}, \quad \forall \boldsymbol{y} \in \mathcal{N}.$$

Moreover, for any $\boldsymbol{x} \in \mathcal{N}$, we define $D_{\boldsymbol{x}}$ as the set of tetrahedra containing the node $\boldsymbol{x}$.

Definition 4.1. *We define the Clément interpolation operator* $I_{Cl}^0 : H_0^1(D) \to \Theta_h^0(D)$ *by :*

$$I_{Cl}^0 v = \sum_{\boldsymbol{x}\in\mathcal{N}_{int}} \frac{1}{|D_{\boldsymbol{x}}|}\Big(\int_{D_{\boldsymbol{x}}} v\Big)\varphi_{\boldsymbol{x}},$$

where $D_{\boldsymbol{x}}$ *is the set of tetrahedra containing the node* $\boldsymbol{x}$*.*

Definition 4.2. *We define the Clément interpolation operator* $I_{Cl} : H^1(D) \to \Theta_h$ *by :*

$$I_{Cl} v = \sum_{\boldsymbol{x}\in\mathcal{N}\cap\overline{D}} \frac{1}{|D_{\boldsymbol{x}} \cap D|}\Big(\int_{D_{\boldsymbol{x}}\cap D} v\Big)\varphi_{\boldsymbol{x}},$$

Then, we can state the following usual interpolation estimates :

Lemma 4.3. *For any $v^0 \in H_0^1(D)$ and $v \in H^1(D)$ it holds :*

$$\sum_{T\in\mathscr{T}_h} h_T^{-2}||v^0 - I_{Cl}^0 v^0||_T^2 \lesssim ||\nabla v^0||_{L^2(D)}^2, \tag{29}$$

$$\sum_{F\in\mathcal{F}_{int}} h_F^{-1}||v^0 - I_{Cl}^0 v^0||_F^2 \lesssim ||\nabla v^0||_{L^2(D)}^2, \tag{30}$$

$$\sum_{T\in\mathscr{T}_h, T\subset D} h_T^{-2}||v - I_{Cl} v||_T^2 \lesssim ||\nabla v||_{L^2(D)}^2, \tag{31}$$

$$\sum_{F\in\mathcal{F}, F\subset\overline{D}} h_F^{-1}||v - I_{Cl} v||_F^2 \lesssim ||\nabla v||_{L^2(D)}^2. \tag{32}$$

Proof. See [4]. □

4.2. Vectorial Clément-type interpolation operator. Since our problem also involves functions in $X(D_c)$, we further need a Clément-type interpolant mapping a (vector) function in $X(D_c)$ to $X_h(D_c)$. This operator was introduced in [6] in an anisotropic context (for an isotropic version, see [2]), we recall it here. It is defined with the help of the basis functions $\boldsymbol{w}_E \in X_h$, $E \in \mathcal{E}$, defined by the condition :

$$\int_{E'} \boldsymbol{w}_E \,.\, \mathbf{T}_{E'} = \delta_{\mathbf{E},\mathbf{E'}}, \quad \forall E' \in \mathcal{E},$$

where $\mathbf{T}_E$ means the unit vector directed along E. Let us define $PH^1(D_c)$ as the set of functions which are piecewise H^1 on the domain D_c, as well as ∇_P the so called "broken gradient" associated with this decomposition.

Definition 4.4. *For any edge $E \in \mathcal{E}$ fix one of its adjacent faces that we call $F_E \in \mathcal{F}$. Then define the Clément type interpolation operator $\mathcal{P}_{Cl} : [PH^1(D_c)] \cap X(D_c) \to X_h(D_c)$ by :*

$$\mathcal{P}_{Cl}\boldsymbol{v} = \sum_{E\in\mathcal{E}} \Big(\int_{F_E} (\boldsymbol{v} \times \boldsymbol{n}_{F_E}) \cdot \boldsymbol{f}_E^{F_E} \Big) \boldsymbol{w}_E,$$

where the (vector) functions $\boldsymbol{f}_{E'}^{F_E}$ are determined by the condition :

$$\int_{F_E} (\boldsymbol{w}_{E'} \times \boldsymbol{n}_{F_E}) \cdot \boldsymbol{f}_{E''}^{F_E} = \delta_{E',E''}, \quad \forall E', E'' \in \mathcal{E} \cup \partial F_E.$$

Then, we can state the following usual interpolation estimates :

Lemma 4.5. *For all $\boldsymbol{v} \in [PH^1(D_c)] \cap X(D_c)$, we have :*

$$\sum_{T\in\mathscr{T}_h} h_T^{-2}||\boldsymbol{v} - \mathcal{P}_{Cl}\boldsymbol{v}||_T^2 \lesssim ||\nabla_P \boldsymbol{v}||^2, \tag{33}$$

$$\sum_{F\in\mathcal{F}} h_F^{-1}||\boldsymbol{v} - \mathcal{P}_{Cl}\boldsymbol{v}||_F^2 \lesssim ||\nabla_P \boldsymbol{v}||^2. \tag{34}$$

Proof. See [2]. □

5. A posteriori error estimation

5.1. Definition of the residual. For all $(\mathbf{T}', \Omega') \in X(D_c) \times H^1(D)$, the residual is defined by :

$$r((\mathbf{T}', \Omega')) = l((\mathbf{T}', \Omega')) - a((\mathbf{T}_h, \Omega_h), (\mathbf{T}', \Omega')) \tag{35}$$

Lemma 5.1. *Let us recall that* $\boldsymbol{e}_{\mathbf{T}}$ *and* e_Ω *are respectively defined by (18) and (19). Then we have:*

(36)

$$\Re\left[\sqrt{2}e^{-j\frac{\pi}{4}}r\left((\boldsymbol{e}_{\mathbf{T}},e_\Omega)\right)\right]=\int_{D_c}\frac{1}{\sigma}\left|\operatorname{curl}\boldsymbol{e}_{\mathbf{T}}\right|^2+\int_{D_c}\omega\mu\left|\boldsymbol{e}_{\mathbf{T}}-\nabla e_\Omega\right|^2+\int_{D_e}\omega\mu\left|\nabla e_\Omega\right|^2$$

Now, we are interested in deriving an a posteriori error estimator in order to control the error defined as the right-hand-side of (36). Consequently, we are reduced to bound from above the quantity $\Big|r\left((\mathbf{e}_{\mathbf{T}},e_\Omega)\right)\Big|$.

5.2. Definition of the estimators. Let us consider T a given tetrahadron of the triangulation. $\mathbf{H}_h$ is defined on T by:

$$\mathbf{H}_h=\begin{cases}\mathbf{H}_s+\mathbf{T}_h-\nabla\Omega_h & \text{if } T\subset D_c,\\[2ex] \mathbf{H}_s-\nabla\Omega_h & \text{if } T\subset D_e.\end{cases}$$

Let F be a common face of the tetrahedra T_1 and T_2, we define the normal jump of the quantity $\mathbf{v}$ through the face F by:

$$\Big[\mathbf{v}\cdot\mathbf{n}\Big]_F=\mathbf{v}_{T_1}\cdot\mathbf{n}_{T_1}+\mathbf{v}_{T_2}\cdot\mathbf{n}_{T_2}$$

and the tangential jump of the quantity $\mathbf{v}$ through the face F by:

$$\Big[\mathbf{v}\times\mathbf{n}\Big]_F=\mathbf{v}_{T_1}\times\mathbf{n}_{T_1}+\mathbf{v}_{T_2}\times\mathbf{n}_{T_2},$$

where, $\mathbf{n}_{T_i}$ stands for the outward unit normal of T_i on F (i=1,2). In the case where F is on the boundary Γ, we set: $\mathbf{v}_{T_2}=\mathbf{0}$.

In the case where F is on the internal boundary Γ_c, $T_1\subset D_c$ and $T_2\subset D_e$, as $\mathbf{T}_h$ does not exist in D_e, we define:

$$\Big[\mathbf{n}\times\frac{1}{\sigma}\operatorname{curl}\mathbf{T}_h\Big]_F=\mathbf{n}_{T_1}\times\frac{1}{\sigma_{T_1}}\operatorname{curl}\mathbf{T}_{h\,T_1}.$$

Definition 5.2. *The local error estimator on the tetrahedron* T *is defined by:*

- *1st case:* $T\subset D_c$

$$\eta_T^2=\eta_{T;1}^2+\eta_{T;2}^2+\sum_{F\subset\partial T}(\eta_{F;1}^2+\eta_{F;2}^2),$$

- *2nd case:* $T\subset D_e$

$$\eta_T^2=\eta_{T;2}^2+\sum_{F\subset\partial T}\eta_{F;2}^2,$$

with

$$\begin{aligned}\eta_{T;1}&=h_T\left\|-\operatorname{curl}(\frac{1}{\sigma}\operatorname{curl}\mathbf{T}_h)-j\omega\mu\mathbf{H}_h\right\|_T,\\ \eta_{T;2}&=h_T\left\|\operatorname{div}(j\omega\mu\mathbf{H}_h)\right\|_T=0,\\ \eta_{F;1}&=h_F^{1/2}\left\|\Big[\boldsymbol{n}\times\frac{1}{\sigma}\operatorname{curl}\mathbf{T}_h\Big]_F\right\|_F,\\ \eta_{F;2}&=h_F^{1/2}\omega\left\|\Big[\boldsymbol{n}\cdot\mu\mathbf{H}_h\Big]_F\right\|_F.\end{aligned}$$

Moreover, the global error estimator is defined by:

$$\eta^2 = \sum_{T\in\mathcal{T}} \eta_T^2.$$

Remark 5.3. *Despite the fact that $\eta_{T;2} = 0$ because of the low order finite element spaces used here, we let them for the sake of completeness, having in mind that their contributions have to be taken into account for higher degree discretizations.*

5.3. Reliability.

Theorem 5.4. *We have :*

$$\left(\int_{D_c} \frac{1}{\sigma}\big|\operatorname{curl} \boldsymbol{e}_{\mathbf{T}}\big|^2 + \int_{D_c} \omega\mu\big|\boldsymbol{e}_{\mathbf{T}} - \nabla e_\Omega\big|^2 + \int_{D_e} \omega\mu\big|\nabla e_\Omega\big|^2\right)^{1/2} \quad \lesssim \quad C_{up}\,\eta,$$

with

$$C_{up} = \max\left\{\frac{1}{\omega^{1/2}\min\limits_{T\in D}\mu_T^{1/2}}, \max_{T\in D_c}\sigma_T^{1/2}\right\}.$$

Proof. Proof is similar to the ones given in the $\mathbf{A}/\varphi$ formulation [7], but with the new Helmholtz decomposition given in section 3.

By (35), we have

$$\begin{aligned}
r((\mathbf{e_T}, e_\Omega)) &= l((\mathbf{e_T}, e_\Omega)) - a((\mathbf{T}_h, \Omega_h), (\mathbf{e_T}, e_\Omega))\\
&= -\int_{D_c} j\omega\mu\mathbf{H}_s \cdot \overline{(\mathbf{e_T} - \nabla e_\Omega)} + \int_{D_e} j\omega\mu\mathbf{H}_s \cdot \overline{\nabla e_\Omega}\\
&\quad -\int_{D_c} \frac{1}{\sigma}\operatorname{curl}\mathbf{T}_h \cdot \overline{\operatorname{curl}\mathbf{e_T}} - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h)\cdot\overline{(\mathbf{e_T} - \nabla e_\Omega)}\\
&\quad -\int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla e_\Omega}.
\end{aligned}$$

Taking into account (20) and the fact that $\operatorname{curl}\nabla\widetilde{e_\Omega} = 0$, we have

$$\begin{aligned}
r((\mathbf{e_T}, e_\Omega)) &= -\int_{D_c} j\omega\mu\mathbf{H}_s \cdot \overline{(\mathbf{e_T} - \nabla\widetilde{e_\Omega})} + \int_{D_e} j\omega\mu\mathbf{H}_s \cdot \overline{\nabla e_\Omega}\\
&\quad -\int_{D_c} \frac{1}{\sigma}\operatorname{curl}\mathbf{T}_h \cdot \overline{\operatorname{curl}(\mathbf{e_T} - \nabla\widetilde{e_\Omega})}\\
&\quad -\int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h)\cdot\overline{(\mathbf{e_T} - \nabla\widetilde{e_\Omega})}\\
&\quad -\int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla e_\Omega}
\end{aligned}$$

Applying the Helmholtz decomposition in Corollary 3.2,

$$\mathbf{e_T} - \nabla\widetilde{e_\Omega} = \Psi + \nabla\Phi + \nabla(\psi - e_\Omega),$$

we get

$$\begin{aligned}
r((\mathbf{e_T}, e_\Omega)) &= -\int_{D_c} j\omega\mu\mathbf{H}_s \cdot \overline{(\Psi + \nabla\Phi + \nabla(\psi - e_\Omega))}\\
&\quad +\int_{D_e} j\omega\mu\mathbf{H}_s \cdot \overline{\nabla e_\Omega} - \int_{D_c} \frac{1}{\sigma}\operatorname{curl}\mathbf{T}_h \cdot \overline{\operatorname{curl}\Psi}\\
&\quad -\int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h)\cdot\overline{(\Psi + \nabla\Phi + \nabla(\psi - e_\Omega))}\\
&\quad -\int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla e_\Omega}
\end{aligned}$$

From (27) and (28), we get:

$$\begin{aligned} |r((\mathbf{e_T}, e_\Omega))| &= \Big| -\int_{D_c} j\omega\mu\mathbf{H}_s \cdot \overline{\Psi} - \int_D j\omega\mu\mathbf{H}_s \cdot \overline{\nabla\widetilde{\Phi}} - \int_D j\omega\mu\mathbf{H}_s \cdot \overline{\nabla\widehat{e_\Omega}} \\ &\quad - \int_{D_c} \frac{1}{\sigma} \operatorname{curl} \mathbf{T}_h \cdot \overline{\operatorname{curl}\Psi} - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h) \cdot \overline{\Psi} \\ &\quad - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h) \cdot \overline{\nabla\widetilde{\Phi}} - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h) \cdot \overline{\nabla\widehat{e_\Omega}} \\ &\quad + \int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla\widehat{e_\Omega}} + \int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla\widetilde{\Phi}} \Big| \end{aligned}$$

Introducing the three Clément interpolation operators P_{Cl}, I_{Cl} and $\widetilde{I^0_{Cl}}$, the orthogonality property from lemma 2.7 leads to:

$$\begin{aligned} |r((\mathbf{e_T}, e_\Omega))| &= \Big| -\int_{D_c} j\omega\mu\mathbf{H}_s \cdot \overline{\Psi - P_{Cl}\,\Psi} - \int_D j\omega\mu\mathbf{H}_s \cdot \overline{\nabla(\widetilde{\Phi} - \widetilde{I^0_{Cl}}\,\widetilde{\Phi})} \\ &\quad - \int_D j\omega\mu\mathbf{H}_s \cdot \overline{\nabla(\widehat{e_\Omega} - I_{Cl}\,\widehat{e_\Omega})} - \int_{D_c} \frac{1}{\sigma} \operatorname{curl}\mathbf{T}_h \cdot \overline{\operatorname{curl}(\Psi - P_{Cl}\,\Psi)} \\ &\quad - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h) \cdot \overline{\Psi - P_{Cl}\,\Psi} \\ &\quad - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h) \cdot \overline{\nabla(\widetilde{\Phi} - \widetilde{I^0_{Cl}}\,\widetilde{\Phi})} \\ &\quad - \int_{D_c} j\omega\mu(\mathbf{T}_h - \nabla\Omega_h) \cdot \overline{\nabla(\widehat{e_\Omega} - I_{Cl}\,\widehat{e_\Omega})} \\ &\quad + \int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla(\widehat{e_\Omega} - I_{Cl}\,\widehat{e_\Omega})} + \int_{D_e} j\omega\mu\nabla\Omega_h \cdot \overline{\nabla(\widetilde{\Phi} - \widetilde{I^0_{Cl}}\,\widetilde{\Phi})} \Big| \end{aligned}$$

Using Green's formula on each tetrahedron and the continuous as well as discrete Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} & |r((\mathbf{e_T}, e_\Omega))| \\ \lesssim\ & \Big(\sum_{T\in\mathscr{T}_h\cap D_c} \eta^2_{T;1}\Big)^{1/2} \Big(\sum_{T\in\mathscr{T}_h\cap D_c} h_T^{-2}\,\|\Psi - P_{Cl}\,\Psi\|^2_T\Big)^{1/2} \\ & + \Big(\sum_{T\in\mathscr{T}_h} \eta^2_{T;2}\Big)^{1/2} \Big(\sum_{T\in\mathscr{T}_h} h_T^{-2}\,\Big\|\widetilde{\Phi} - \widetilde{I^0_{Cl}}\,\widetilde{\Phi}\Big\|^2_T\Big)^{1/2} \\ & + \Big(\sum_{T\in\mathscr{T}_h} \eta^2_{T;2}\Big)^{1/2} \Big(\sum_{T\in\mathscr{T}_h} h_T^{-2}\,\|\widehat{e_\Omega} - I_{Cl}\,\widehat{e_\Omega}\|^2_F\Big)^{1/2} \\ & + \Big(\sum_{F\in\mathcal{F}\cap D_c} \eta^2_{F;1}\Big)^{1/2} \Big(\sum_{F\in\mathcal{F}\cap D_c} h_F^{-1}\,\|\Psi - P_{Cl}\,\Psi\|^2_F\Big)^{1/2} \\ & + \Big(\sum_{F\in\mathcal{F}} \eta^2_{F;2}\Big)^{1/2} \Big(\sum_{F\in\mathcal{F}} h_F^{-1}\,\Big\|\widetilde{\Phi} - \widetilde{I^0_{Cl}}\,\widetilde{\Phi}\Big\|^2_F\Big)^{1/2} \\ & + \Big(\sum_{F\in\mathcal{F}} \eta^2_{F;2}\Big)^{1/2} \Big(\sum_{F\in\mathcal{F}} h_F^{-1}\,\|\widehat{e_\Omega} - I_{Cl}\,\widehat{e_\Omega}\|^2_F\Big)^{1/2} \end{aligned}$$

Now, we deduce from inequalities (29) to (34) that :

$$
\begin{aligned}
&|r((\mathbf{e_T}, e_\Omega))| \\
\lesssim\ & \left(\sum_{T\in\mathscr{T}_h\cap D_c} \eta_{T;1}^2\right)^{1/2} \|\Psi\|_{H^1(D_c)} + \left(\sum_{T\in\mathscr{T}_h} \eta_{T;2}^2\right)^{1/2} \|\nabla\widetilde{\Phi}\|_{L^2(D)} \\
&+ \left(\sum_{T\in\mathscr{T}_h} \eta_{T;2}^2\right)^{1/2} \|\nabla\widehat{e_\Omega}\|_{L^2(D)} + \left(\sum_{F\in\mathcal{F}\cap D_c} \eta_{F;1}^2\right)^{1/2} \|\Psi\|_{H^1(D_c)} \\
&+ \left(\sum_{F\in\mathcal{F}} \eta_{F;2}^2\right)^{1/2} \|\nabla\widetilde{\Phi}\|_{L^2(D)} + \left(\sum_{F\in\mathcal{F}} \eta_{F;2}^2\right)^{1/2} \|\nabla\widehat{e_\Omega}\|_{L^2(D)}
\end{aligned}
$$

Theorem 3.2 yields:

$$
\begin{aligned}
&\|\Psi\|^2_{H^1(D_c)} + \|\nabla\widetilde{\Phi}\|^2_{L^2(D)} + \|\nabla\widehat{e_\Omega}\|^2_{L^2(D)} \\
\lesssim\ & \|\mathbf{e_T} - \nabla e_\Omega\|^2_{L^2(D_c)} + \|\nabla e_\Omega\|^2_{L^2(D_e)} + \|\operatorname{curl}\mathbf{e_T}\|^2_{L^2(D_c)} \\
\lesssim\ & \Big\|\frac{\omega^{1/2}\mu^{1/2}}{\omega^{1/2}\mu^{1/2}}(\mathbf{e_T} - \nabla e_\Omega)\Big\|^2_{L^2(D_c)} + \Big\|\frac{\omega^{1/2}\mu^{1/2}}{\omega^{1/2}\mu^{1/2}}\nabla e_\Omega\Big\|^2_{L^2(D_e)} + \Big\|\frac{\sigma^{1/2}}{\sigma^{1/2}}\operatorname{curl}\mathbf{e_T}\Big\|^2_{L^2(D_c)} \\
\lesssim\ & \frac{1}{\omega \min\limits_{T\in D_c} \mu_T}\|\omega^{1/2}\mu^{1/2}(\mathbf{e_T} - \nabla e_\Omega)\|^2_{L^2(D_c)} + \frac{1}{\omega \min\limits_{T\in D_e} \mu_T}\|\omega^{1/2}\mu^{1/2}\nabla e_\Omega\|^2_{L^2(D_e)} \\
&+ \max_{T\in D_c}\sigma_T\Big\|\frac{1}{\sigma^{1/2}}\operatorname{curl}\mathbf{e_T}\Big\|^2_{L^2(D_c)} \\
\lesssim\ & C_{up}\left(\int_{D_c}\frac{1}{\sigma}|\operatorname{curl}\mathbf{e_T}|^2 + \int_{D_c}\omega\mu|\mathbf{e_T} - \nabla e_\Omega|^2 + \int_{D_e}\omega\mu|\nabla e_\Omega|^2\right)^{1/2}
\end{aligned}
$$

where $C_{up} = \max\left\{\frac{1}{\omega^{1/2}\min\limits_{T\in D}\mu_T^{1/2}}, \max\limits_{T\in D_c}\sigma_T^{1/2}\right\}$. □

5.4. Efficiency. Now, in order to derive the efficiency of our estimator, we have to bound each part of the estimator by the local error. Reiterating, proofs are based on the same kind of arguments than the ones given in $\mathbf{A}/\varphi$ formulation [7]. Due to this reason, these proofs can not be recalled here.

Theorem 5.5. *Let us define* $\mathscr{P}_T = \bigcup\limits_{T'\cap T\neq\emptyset} T'$*, we have the efficiency of our estimator:*

$$
\begin{aligned}
\eta_T \lesssim\ & C_{T,down}\Big(\Big\|\sigma^{-1/2}\operatorname{curl}\boldsymbol{e}_\mathbf{T}\Big\|^2_{\mathscr{P}_T} + \Big\|\omega^{1/2}\mu^{1/2}(\boldsymbol{e}_\mathbf{T} - \nabla\boldsymbol{e}_\Omega)\Big\|^2_{\mathscr{P}_T} \\
&+ \Big\|\omega^{1/2}\mu^{1/2}\nabla\boldsymbol{e}_\Omega\Big\|^2_{\mathscr{P}_T}\Big)^{1/2} + h.o.t,
\end{aligned}
$$

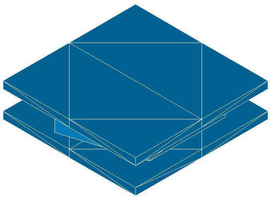
with

$$
C_{T,down} = \max_{T'\in\mathscr{P}_T}\Big\{\omega^{1/2}\mu_{T'}^{1/2}, \frac{1}{\sigma_{T'}^{1/2}}\Big\}.
$$

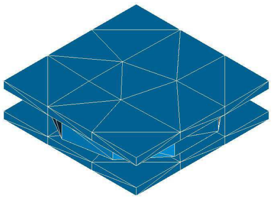
Proof. We have to bound each part of the estimator by a local error which is done with the so-called bubble functions and inverse inequalities. We refer to [7] for the details about the $\mathbf{A}/\varphi$ formulation. □

6. Numerical validation

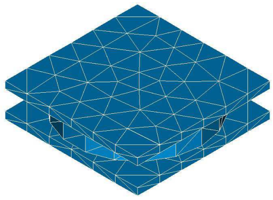
In this section, a physical test is considered [10]. The computation model consists of a coil between two symmetrical conductors shown in Figure 2 for five different refined meshes. Here, we set $\mu = 4\pi\ 10^{-7}$ H/m and $\sigma = 3.28\ 10^7\Omega$/m in the conductors. When a current is imposed in the coil, the eddy current $\mathbf{J}_{ec} = \operatorname{curl} \mathbf{T}$ is generated in the two conductor plates.



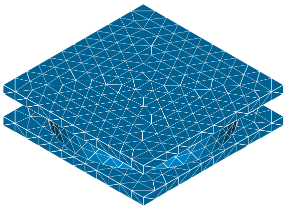
(a) Mesh 1



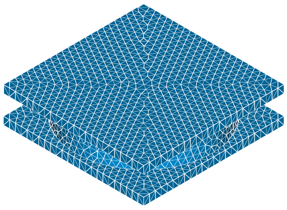
(b) Mesh 2



(c) Mesh 3



(d) Mesh 4



(e) Mesh 5

FIGURE 2. Domain and Mesh of the Problem.

To begin with, the value of the estimator η is displayed as a function of the maximum diameter of the tetrahedra h corresponding to each mesh (see Figure 3). It shows a good behavior of the estimator compared with the theoretically expected results since it converges well towards zero.

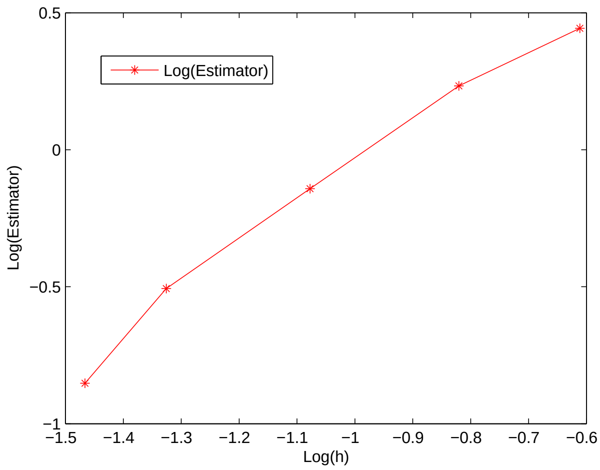


FIGURE 3. Estimator versus h.

Now, in Figure 4 a unique given mesh in chosen to perform the following tests. Let us note that the part of the mesh in the upper conductor is finer that the part of the mesh in the lower one.

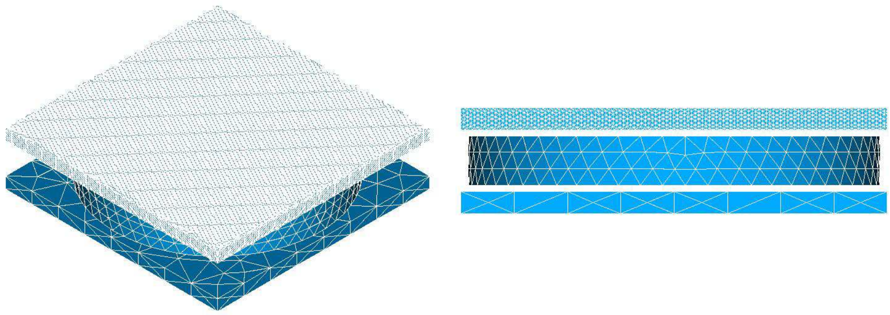
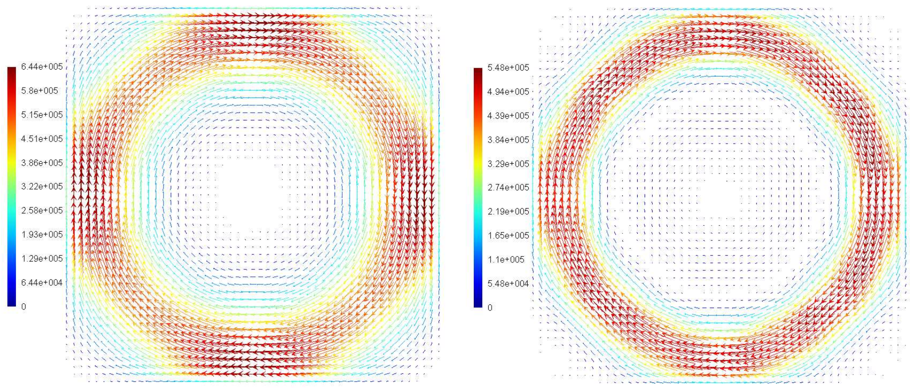


FIGURE 4. Domain and Mesh of the Problem.

As shown in Figure 5(a) and 5(b), the real and the imaginary parts of the eddy current vector field at the frequency of 50 Hz lies in the down surface of the upper conductor plate. The eddy current resembles the physically expected one qualitatively.



(a) Real part. (b) Imaginary part.

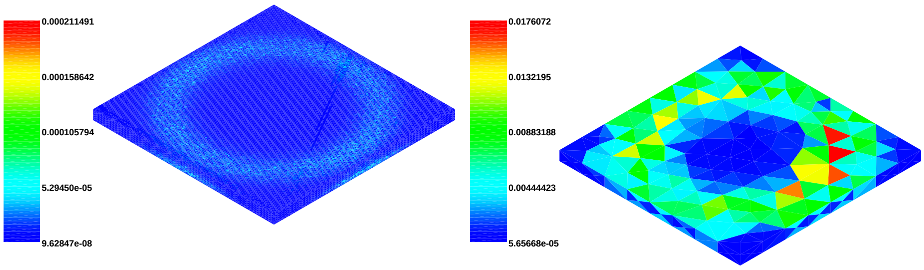
FIGURE 5. Top view of the eddy current at 50 Hz in the down surface of the upper conductor plate.

As numerically expected, the error in the lower plate is larger than the one above, as shown by the local estimators η_T maps in Figure 6(a) and 6(b).

At the end of this section, we want to underline the skin-effect phenomenon on the above conductor plate. In normal cases, the skin depth δ is well approximated as [15, p.504]:

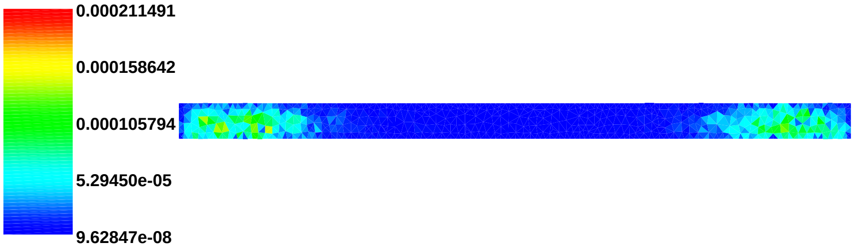
$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}.$$

As shown in Figure 7(a), 7(b) and 7(c), the estimator allows to detect the areas of the domain where the mesh needs to be refined in order to track this skin-effect phenomenon.

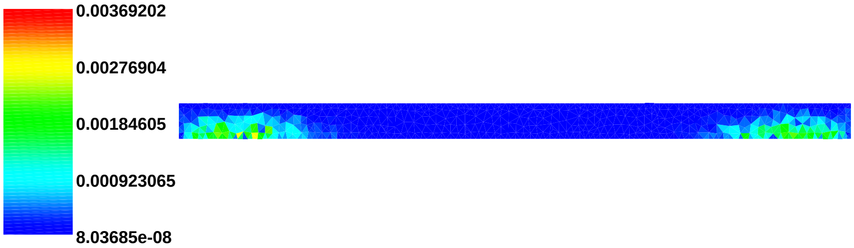


(a) Upper plate (b) Lower plate

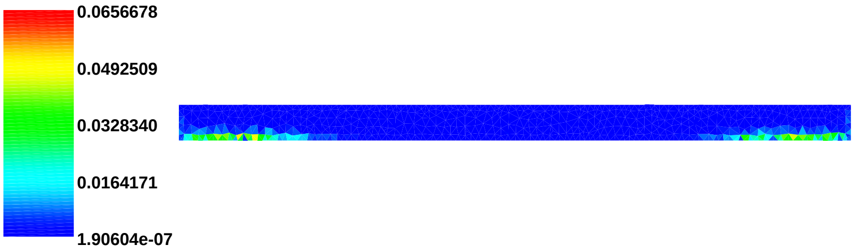
FIGURE 6. Local estimator η_T maps in the two conductor plates at frequency 50 Hz.



(a) 50 Hz.



(b) 500 Hz.



(c) 5000 Hz.

FIGURE 7. Local estimator η_T maps in the upper conductor plate.

Possible extension

The referee of this paper suggested to treat the case where D_c is no more simply connected, for which the variational formulation (14) is no more valid. For example, in the case where D_c is a torus, the weak formulation of the problem is stated in [13]. Unfortunately, we did not succeed in deriving the estimator in that case, because Theorem 3.1 is no more applicable.

References

[1] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector Potentials in Three-dimensional Non-smooth Domains, Math. Meth. in the Applied Sciences 21 (1998), 823-864.
[2] R. Beck, R. Hiptmair, R. H. W. Hoppe, B. Wohlmuth, Residual based a posteriori error estimators for eddy current computation, M2AN Math. Model. Numer. Anal., 34 (2000), 159-182.
[3] J. Chen, Z. Chen, T. Cui, L. Zhang, An adaptive finite element method for the eddy current model with circuit/field couplings, SIAM J. Sci. Comput. 32 (2010), no. 2, 1020-1042.
[4] P. Clément, Approximation by finite element functions using local regularization. Rev. Française. Automat. Informat. Recherche Opérationnelle Sér. R.A.I.R.O. Anal. Numŕ. 9(R2):77-84, 1975.
[5] M. Costabel, M. Dauge and S. Nicaise, Singularities of Maxwell Interface Problems, Modél Math. Anal. Numér. 33 (1999) 627-649.
[6] E. Creusé and S. Nicaise, Anisotropic a posteriori error estimation for the mixed discontinuous Galerkin approximation of the Stokes problem, Numerical Methods in Partial Differential Equations, vol. 22, no 2, 449-483, 2006.
[7] E. Creusé, S. Nicaise, Z. Tang, Y. Le Menach, N. Nemitz, F. Piriou, Residual-based a posteriori estimators for the $\mathbf{A}/\varphi$ magnetodynamic harmonic formulation of the Maxwell system, Mathematical Models and Methods in Applied Sciences, to appear.
[8] V. Girault and P.A. Raviart, Finite element methods for Navier-Stokes equations, volume 5 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 1986. Theory and algorithms.
[9] Y. Le Menach, S. Clénet, and F. Piriou, Determination and utilization of source field in 3-D magnetostatic problem, IEEE Trans. Magn., vol. 34, pp.2509 - 2511 , 1998.
[10] P. J. Leonard and D. Rodger, Voltage forced coils for 3-D finite element method electromagnetic model, IEEE Trans. Magn., vol. 24, 2579 - 2581, 1988.
[11] P. Monk, Finite element methods for Maxwell's equations. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.
[12] A.A. Rodriguez, R. Hiptmair, and A. Valli, Hybrid formulations of eddy current problems. Numer. Method. Part. Differ. Eq., 21(4):742-763, 2005.
[13] A.A. Rodríguez and A. Valli, Eddy current approximation of Maxwell equations, MS&A. Modeling, Simulation and Applications, vol. 4, Theory, algorithms and applications, Springer-Verlag Italia, Milan, 2010.
[14] R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement techniques. Chichester and Stuttgart: Wiley and Teubner, Amsterdam, 1996.
[15] J. A. Stratton, Electromagnetic Theory, McGraw - Hill (1941).

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