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UNCONDITIONAL CONVERGENCE OF HIGH-ORDER EXTRAPOLATIONS OF THE CRANK-NICOLSON, FINITE ELEMENT METHOD FOR THE NAVIER-STOKES EQUATIONS

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Abstract. Error estimates for the Crank-Nicolson in time, Finite Element in space (CNFE) discretization of the Navier-Stokes equations require application of the discrete Gronwall inequality, which leads to a time-step (Δt) restriction. All known convergence analyses of the fully discrete CNFE with linear extrapolation rely on a similar Δt-restriction. We show that CNFE with arbitrary-order extrapolation (denoted CNLE) is *convergences optimally in the energy norm* without any Δt-restriction. We prove that CNLE velocity and corresponding discrete time-derivative converge optimally in $l^\infty(H^1)$ and $l^2(L^2)$ respectively under the mild condition $\Delta t \leq Mh^{1/4}$ *for any arbitrary* $M > 0$ (e.g. independent of problem data, h, and Δt) where $h > 0$ is the maximum mesh element diameter. Convergence in these higher order norms is needed to prove convergence estimates for pressure and the drag/lift force a fluid exerts on an obstacle. Our analysis exploits the extrapolated convective velocity to avoid any Δt-restriction for convergence in the energy norm. However, the coupling between the extrapolated convecting velocity of usual CNLE and the *a priori* control of *average* velocities (characteristic of CN methods) rather than pointwise velocities (e.g. backward-Euler methods) in $l^2(H^1)$ is precisely the source of Δt-restriction for convergence in higher-order norms.

Key words. Navier-Stokes, Crank-Nicolson, finite element, extrapolation, linearization, error, convergence, linearization

1. Introduction

The usual Crank-Nicolson (CN) in time Finite Element (FE) in space discretization of the Navier-Stokes (NS) Equations (NSE) denoted by CNFE is well-known to be unconditionally (energetically) stable. The error analysis of the CNFE method is based on a discrete Gronwall inequality which introduces a time-step ($\Delta t > 0$) restriction (for convergence, not for stability) of the form

$$\Delta t \leq C(Re, h), \qquad \text{e.g.} \quad \Delta t \leq \mathcal{O}(Re^{-3}) \tag{1}$$

(see Appendix A for a derivation, Theorem A.1 with e.g. (157)). Here $h > 0$ is the maximum mesh element diameter and $Re > 0$ is the Reynolds number. Condition (1)(a) implies *conditional convergence* whereas (1)(b) is a *robustness condition* and both are prohibitively restrictive in practice; for example, (1)(b) suggests

$$Re = 100 \text{ (low-to-moderate value)} \quad \Rightarrow \quad \Delta t \leq \mathcal{O}(10^{-6}).$$

Consequently, an important open question regards whether condition (1) is

- an artifact of imperfect mathematical technique, or
- a special feature of the CN time discretization.

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We consider the necessity of a Δt-restriction in a linear, fully implicit variant of CNFE obtained by extrapolation of the convecting velocity $\mathbf{u}$: for example, suppressing spatial discretization, given $\mathbf{u}^0$, $\mathbf{u}^1$, and p^1, for each $n = 1, 2, \ldots$ find velocity $\mathbf{u}^{n+1}$ and pressure p^{n+1} satisfying

$$\begin{aligned}\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} &+ (\frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1})\cdot\nabla\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2} \\ &-Re^{-1}\Delta\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2} + \nabla\frac{p^{n+1}+p^n}{2} = \frac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2} \\ \nabla\cdot\mathbf{u}^{n+1} &= 0.\end{aligned} \tag{2}$$

Here $\mathbf{f}$ is body-force term, and $z^i := z(x, t^i)$ and $t^i = i\Delta t$. This method is often called CNLE and was first studied by Baker [3]. CNLE is linearly implicit, unconditionally (energetically) stable, and second-order accurate. In this report, we show that *no Δt-restriction* is required for the convergence of CNLE (Proposition 3.1, Theorem 3.5). In particular,

$$||error(CNLE)||_{l^\infty(L^2)\cap l^2(H^1)} \le C(h^k + \Delta t^2), \qquad k = \text{ degree of FE space}$$

(Theorem 3.5). This result was proved for the semi-discrete case as $\Delta t \to 0$ in [10] and the fully discrete Backward Euler (BE) scheme with Constant Extrapolation (BECE) in [32]. The analysis depends on

- *Gronwall inequality* - exploit time-lagged convecting velocity (Section 1.1)
- Estimate (74) - bound convecting velocity in L^2 (Section 1.1.1)

Indeed, for extrapolated CN, we apply a discrete Gronwall Lemma without any Δt-restriction; for general extrapolations we derive and apply the estimate (74)(b) of the *explicitly* skew-symmetric convective term. We explain our strategy for proving the CNLE error estimate and corresponding difficulties in detail in Section 1.1.

We also prove convergence estimates in higher-order norms. We show that the CNLE velocity approximation converges optimally in the $l^\infty(H^1)$-norm and the corresponding discrete time-derivative of the velocity approximation converges optimally in the $l^2(L^2)$-norm (Theorems 3.8, 3.10) under a modest Δt-restriction

$$\Delta t \le Mh^{1/4}, \quad \text{for any } M > 0 \text{ (no } Re\text{-dependence)}. \tag{3}$$

Note that M is completely arbitrary so that (3) only governs the rate at which $\Delta t \to 0$ and not the size of Δt. In particular, restriction (3) is not a typical artifact of the discrete Gronwall inequality since it does not depend problem data. The error estimate is obtained by a bootstrap argument that utilizes the error in the energy norm. Although such estimates have been proved for BECE in [32], the analysis of CNLE is distinctly different because CN methods only give *a priori* control of *average* velocities $\mathbf{u}^{n+1/2}$ rather than pointwise velocities $\mathbf{u}^{n+1}$ (e.g. BE methods) in $l^2(H^1)$. Our analysis depends on

- Estimate (75) - bound test-function of convective term in L^2 (Section 1.1.1)
- CN *a priori* estimates - introduce Δt-restriction (3) (Section 1.1.2)
- Stokes projection - preserve optimal convergence rate (Section 1.1.3).

Indeed, we derive and apply estimate (75)(b) of the *explicitly* skew-symmetric convective term; we obtain intermediate estimates in the convergence analysis of CNLE with limited options corresponding to limited control of *average* velocities (characteristic of CN methods) in $l^2(H^1)$; and we exploit the Stokes projection to preserve the optimal convergence rate for the FE and CN discretization.

It is a source of current research to determine whether (3) is strictly necessary and to develop linear, implicit variants of usual CNLE (2)(a) that are guaranteed to avoid (3). Indeed, the extrapolated convecting velocity in (2)(a) is the source of Δt-restriction for convergence in $l^\infty(H^1)$. Moreover, higher-order extrapolations reduce the modeling error of extrapolation. Consequently, we investigate the arbitrary-order extrapolation herein: for some $n_0 \geq 0$ and $a_i \geq 0$ for $0 \leq i \leq n_0$, consider

$$\int \frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2} \cdot \nabla \frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2} \cdot \mathbf{v} \approx \int (a_0\mathbf{u}^n + \ldots + a_{n_0}\mathbf{u}^{n-n_0}) \cdot \nabla \frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2} \cdot \mathbf{v}.$$

We consider the important case when the nonlocal compatibility condition is not satisfied (addressed by Heywood and Rannacher in [19,20] and more recently, e.g., by He in [12,14] and He and Li in [15,16]). Suppose, for example, that p^0 be the solution of (well-posed) Neumann problem

$$\begin{cases} \Delta p^0 = \nabla \cdot (\mathbf{f}^0 - \mathbf{u}^0 \cdot \nabla \mathbf{u}^0), \text{ in } \Omega, \\ \nabla p^0 \cdot \hat{\mathbf{n}}|_{\partial\Omega} = (\Delta \mathbf{u}^0 + \mathbf{f}^0 - \mathbf{u}^0 \cdot \nabla \mathbf{u}^0) \cdot \hat{\mathbf{n}}|_{\partial\Omega}. \end{cases} \tag{4}$$

In order to avoid the accompanying factor min $\{t^{-1}, 1\}$ in the estimates presented in Section 3, the following compatibility condition is necessarily required (e.g. see [19], Corollary 2.1):

$$\nabla p^0|_{\partial\Omega} = (\Delta \mathbf{u}^0 + \mathbf{f}^0 - \mathbf{u}^0 \cdot \nabla \mathbf{u}^0)|_{\partial\Omega}. \tag{5}$$

Replacing (4) with (4)(a), (5) defines an overdetermined Neumann-type problem. Condition (5) is a nonlocal condition relating $\mathbf{u}^0$ and $\mathbf{f}^0$. Condition (5) is satisfied for several practical applications including startup from rest with zero force, $\mathbf{u}^0 = 0$, $\mathbf{f}^0 = 0$. In general, however, condition (5) cannot be verified. In this case, it is shown that, e.g., $|\mathbf{u}_t(\cdot, t)|_1$, $||\mathbf{u}(\cdot, t)||_3 \to \infty$ as $t \to 0^+$.

In Section 1.1, we provide an overview of CN time-stepping schemes for NSE approximation and explain our methodology for CNLE convergence analysis and corresponding improved estimates. We provide the mathematical setting for CNLE in Section 2 for both the continuous and discrete function spaces (Section 2.1). In Section 2.2, we compile the fundamental estimates and assumptions of the FE space and extrapolated convecting velocity. Section 3 contains the main results of our report. Section 4 contains the proofs of these results. In Section 3.2, we compile the fundamental approximations and identities required for our error analysis. Section 3.3 is devoted to analysis of the trilinear convective term and the explicitly skew-symmetric convective term used in CNLE. In Section 3.4, we present the elliptic and Stokes projections.

1.1. Remark on improved estimate. The key difference between our convergence proof for CNLE and that of CNFE is the resulting intermediate estimate: for approximations $\mathbf{U}_h^n$ and constants $\kappa^n > 0$,

$$\text{CNFE} \Rightarrow ||\mathbf{U}_h^N||^2 + \ldots \leq \sum_{n=0}^{N} \kappa^n ||\mathbf{U}_h^n||^2 + \ldots \tag{6}$$

$$\text{CNLE} \Rightarrow ||\mathbf{U}_h^N||^2 + \ldots \leq \sum_{n=0}^{N-1} \kappa^n ||\mathbf{U}_h^n||^2 + \ldots. \tag{7}$$

Notice that the term $||\mathbf{U}_h^N||^2$ is included in the right-hand-side of (6), but not of (7). Estimates of the form (6) require a discrete Gronwall inequality (Lemma 3.14) to proceed, which is the source of a Δt-restriction. Conversely, estimates of the form

(7) allow application of an alternate discrete Gronwall inequality (Lemma 3.15), which does not require a Δt-restriction.

1.1.1. Key estimate for CNLE error analysis. The key difficulty resolved in our CNLE proof (resulting in suboptimal results reported in previous error analyses) is associated with the extrapolated convecting velocity $\frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$. Once again consider the error equation for CNLE:

$$\begin{aligned}||\mathbf{U}_h^N||^2 + Re^{-1}\Delta t \sum_n ||\nabla \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2}||^2 + \dots \\ = \Delta t \sum_n \int (\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1}) \cdot \nabla \mathbf{u} \cdot \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2} + \dots . \end{aligned} \tag{8}$$

To derive an *a priori* estimate from (8), each $\mathbf{U}_h^n$ from the right-hand side must be absorbed into the left-hand side. However, $\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1}$ cannot be written as a sum of averages. Indeed, suppose that $\mathbf{U}_h^n = -\mathbf{U}_h^{n+1} \neq 0$ so that $||\nabla \frac{\mathbf{U}_h^{n+1}+\mathbf{U}_h^n}{2}|| = 0$ whereas $||\nabla \mathbf{U}_h^{n+1}|| > 0$. Then, in this case, it is impossible to absorb any factor of $\sum_n ||\nabla(\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1})||^2 > 0$ into $\sum_n ||\nabla \frac{\mathbf{U}_h^{n+1}+\mathbf{U}_h^n}{2}||^2 = 0$. To proceed from (8) requires care so that only L^2-norms of $\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1}$ are introduced when majorizing the right-hand side. Indeed $\sum_n \kappa_n ||\mathbf{U}_h^n||^2$ can be *absorbed* via the discrete Gronwall Lemma. Assuming that $\mathbf{u} \in H^2$, the key estimate to prove is

$$|c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C||\mathbf{u}||_{H^r}||\mathbf{v}||_{H^2}||\mathbf{w}||_{H^{1-r}} \quad \forall \mathbf{u} \in H^r,\ \mathbf{v} \in H^2,\ \mathbf{w} \in H^{1-r}$$

for some $C > 0$ and $r = 0$ and 1 where $c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})$ is the explicitly skew-symmetric convective term (see estimates (74)(b), (75)(b)).

1.1.2. Introduction of Δt-restriction (3). It is illuminating to introduce the BE scheme to highlight the difficulty in convergence estimates for CN schemes in higher order norms. Let $i = 1$ for BE and $i = 2$ for CN. Write $z^{n+1/2} = \frac{1}{2}(z^{n+1} + z^n)$ and

$$\int \mathbf{u}^{n+1/i} \cdot \nabla \mathbf{u}^{n+1/i} \cdot \mathbf{v} \approx \int \xi^n(\mathbf{u}) \cdot \nabla \mathbf{u}^{n+1/i} \cdot \mathbf{v}, \quad \xi^n(\mathbf{u}) := a_0 \mathbf{u}^n + \dots + a_{n_0} \mathbf{u}^{n-n_0}. \tag{9}$$

Note that $\xi^n(\mathbf{u}) = \mathbf{u}^n$ and $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$ in (9) for BECE and CNLE respectively. The *energy difference* due to the numerical extrapolation (9) is the source of the Δt-restriction (3) for CNLE. Indeed, BECE velocities are shown to converge unconditionally in $l^\infty(H^1)$ (see e.g. [32]). Let $\mathbf{e}$ represent the fully discrete velocity error for CNLE or BECE. We show herein for CNLE (and it is known for BECE) that

$$\max_n ||\mathbf{e}^n|| + (Re^{-1}\Delta t \sum_n ||\nabla \mathbf{e}^{n+1/i}||^2)^{1/2} \le C(h^k + \Delta t^i) \tag{10}$$

for some $C > 0$ without any Δt-restriction. The key difference between our convergence proof for CNLE and that for BECE is the resulting error equation: for approximations $\mathbf{U}_h^n$,

$$Re^{-1}||\nabla \mathbf{U}_h^N||^2 + \Delta t \sum_n ||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||^2 + \dots$$

$$= \begin{cases} \Delta t \sum_n \int \mathbf{e}^n \cdot \nabla \mathbf{e}^{n+1} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} + \dots & \text{BECE} \\ \Delta t \sum_n \int (\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1}) \cdot \nabla \frac{\mathbf{e}^{n+1} + \mathbf{e}^n}{2} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} + \dots & \text{CNLE} \end{cases} \quad (11)$$

Estimate (10) gives us *a priori* control in $l^2(H^1)$ of two terms $\mathbf{e}^n$, $\mathbf{e}^{n+1}$ in (11)(a). However, we only have *a priori* control of $\frac{1}{2}(\mathbf{e}^{n+1}+\mathbf{e}^n)$ in (11)(b) since $\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1}$ cannot be written as a sum of averages. Indeed, suppose that $\mathbf{e}^n = -\mathbf{e}^{n+1} \neq 0$. Then $||\nabla \frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}|| = 0$ whereas $||\nabla \mathbf{e}^{n+1}|| > 0$. In this case it is impossible to bound $\sum_n ||\nabla(\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1})||^2 > 0$ above by any factor of $\sum_n ||\nabla \frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}||^2 = 0$. The remaining term in (11)(a) and 2 terms in (11)(b) must be absorbed to the left-hand side of the estimate or with the Gronwall Lemma. The limited control of the CNLE term $\int (\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1}) \cdot \nabla \frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2} \cdot \frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t}$ leads to the restriction (3).

1.1.3. Preserving optimal convergence rate in $l^\infty(H^1)$. We utilize the Stokes projections (79) in the convergence analysis of Theorems 3.8, 3.10. The Stokes projection requires additional regularity of the pressure p, but is necessary to establish the optimal convergence rate for velocity in $l^\infty(H^1)$ reported in Theorem 3.10. The crucial estimate involves the error in the pressure: for each $n \geq 0$, fix $\tilde{q}_h^{n+1} \approx p^{n+1}$ and $\mathbf{U}_h^{n+1} \in H_0^1$ so that

$$(p^{n+1} - \tilde{q}_h^{n+1}, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \leq \begin{cases} ||p^{n+1} - \tilde{q}_h^{n+1}|| \, ||\nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}|| \\ ||p^{n+1} - \tilde{q}_h^{n+1}||_1 \, ||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}|| \end{cases} . \quad (12)$$

The first option in (12) must be avoided, because we have no *a priori* control of $||\nabla \cdot \frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t}||$. The second option (12) is applicable, but ultimately leads to a suboptimal error estimate. Indeed, approximation theory for FE functions suggests

$$||p^{n+1} - \tilde{q}_h^{n+1}||_m \leq Ch^{s+1-m}, \qquad s = \text{ degree of FE space} \quad (13)$$

for some $C > 0$ so that a factor of h is *lost* in the case $m = 1$. Alternatively, let $(\tilde{\mathbf{v}}_h^{n+1}, \tilde{q}_h^{n+1}) \approx (\mathbf{u}^{n+1}, p^{n+1})$ be the Stokes projection. Then

$$Re^{-1}(\nabla(\mathbf{u}^{n+1} - \tilde{\mathbf{v}}_h^{n+1}), \nabla \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - (p^{n+1} - \tilde{q}_h^{n+1}, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) = 0. \quad (14)$$

Identity (14) eliminates the need to bound (12). Instead, the error is shifted to the time derivative of the Stokes projection and requires

$$||\frac{(p^{n+1} - \tilde{q}_h^{n+1}) - (p^n - \tilde{q}_h^n)}{\Delta t}|| \leq Ch^{s+1} \quad (15)$$

for some $C > 0$.

1.2. Overview of CN schemes. There are many analyses of CN time-stepping methods for the NSE. Heywood and Rannacher [20] provide a well-cited and comprehensive analysis of CNFE. The 2nd and 3rd order CNLE methods are introduced and analyzed in [3,4]. Numerous error analyses of CNLE have been provided since: e.g. multilevel based CNLE [18,21], LES turbulence modeling [6], stochastic CNLE [7], and a stabilized CNLE method [24]. Each of these analyses requires, explicitly stated or implicitly, a Δt-restriction of the form (1) to guarantee convergence. For instance, in [6], the first Gronwall Lemma 3.14 must be applied with Δt-restriction (p. 223) since Equation (4.11) includes the pertinent term up to and including the current time-step (see 4th-term on RHS of Equation (4.11)). The convergence analysis in [12–17] requires $\Delta t \leq c_T$ or $\Delta t \leq c_T|\log h|^{-1}$. Convergence estimates for fully-discrete BECE is derived and family of semi-discrete multi-step CNLE-type methods are derived in [32] and [10] respectively. Concurrent to our work, the authors of [26] correctly perform the convergence analysis in energy norm, but not in higher-orders and do not consider general case when the nonlocal compatibility condition (4), (5) is not satisfied.

Applications of CNLE are also widespread: e.g. stability analysis of NSE and MHD equations with CNLE in [2], a 1st order CNLE applied in conjunction with a coupled multigrid and pressure Schur complement scheme for the NSE is proposed in [23], a velocity-vorticity formulation of CNLE analyzed in [29], and the NS-α/NS-ω regularization method with CNLE [25]. The CN method is also applied, for example, to a general class of non-stationary partial differential equations encompassing reaction-diffusion type equations including the nonlinear Sobolev equations [28] and the Ginzburg-Landau model [22]. Time-step restrictions of type (1)(b) (where Re has a different meaning) are implicitly required in the convergence analyses of these discrete models.

A CN/Adams-Bashforth (CN-AB) time-stepping, scheme is another linear variant of CNFE. Unlike CNLE, CN-AB is explicit in the nonlinearity and only *conditionally* stable [17] (i.e. a Δt-restriction of form (1)(a) is required for *stability*). CN-AB is a popular method for approximating NS flows because it is fast and easy to implement. Each time-step requires only one discrete Stokes system and linear solve. For example, CN-AB is used to model turbulent flows induced by wind turbine motion [31], turbulent flows transporting particles [27], and reacting flows in complex geometries (e.g. gas turbine combustors) [1].

Lastly, the compatibility condition (4), (5) is not satisfied in general, see [20] for an overview of this problem. When the (4), (5) is not satisfied, we have limited regularity that greatly restricts convergence analysis of high-order methods (both in time and *space*), e.g. $\partial_t^{(2)}\mathbf{u} \notin L^2(H^1)$. Corresponding regularity in the case that (4), (5) is satisfied is assumed the analyses of [3, 4, 6, 7, 10, 24, 26, 32]. See [12, 14–16, 19, 20, 30] for a rigorous treatment of the general case when the compatibility condition is not satisfied. We also provide details herein.

2. Problem formulation

Let $\mathbf{a} := (a_0, a_1, \ldots, a_{n_0}) \in \mathbb{R}^{n_0+1}$ for some $n_0 \in \{0\} \cup \mathbb{N}$ be equipped with the standard l^q norm denoted by $|\mathbf{a}|_q$ for $1 \leq q \leq \infty$. Fix $p \geq 1$. Let $L^p(\Omega)$ denote the linear space of all real Lebesgue-measurable functions $\mathbf{u}$ and bounded in the usual norm denoted by $||\mathbf{u}||_{L^p(\Omega)}$. Let $(\cdot,\cdot)_\Omega$ and $||\cdot||_\Omega$ be the standard $L^2(\Omega)$-inner product and norm. Fix $k \in \mathbb{R}$. The Sobolev space $W^{k,p}(\Omega)$ is equipped with the usual norm denoted by $||\mathbf{u}||_{W^{k,p}(\Omega)}$. Identify $||\cdot||_{k,p,\Omega} := ||\cdot||_{W^{k,p}(\Omega)}$, $H^k(\Omega) := W^{k,2}(\Omega)$, $||\cdot||_{k,\Omega} := ||\cdot||_{W^{k,2}(\Omega)}$ with $|\cdot|_{k,\Omega}$ the corresponding semi-norm. Let the context determine whether $W^{k,p}(\Omega)$ denotes a scalar, vector, or tensor function space. For example let $\mathbf{v}:\Omega\to\mathbb{R}^d$. Then, $\mathbf{v}\in H^1(\Omega)$ implies that $\mathbf{v}\in H^1(\Omega)^d$ and $\nabla\mathbf{v}\in H^1(\Omega)$ implies that $\nabla\mathbf{v}\in H^1(\Omega)^{d\times d}$. Define

$$H_0^1(\Omega) := \left\{\mathbf{v}\in H^1(\Omega)\ :\ \mathbf{v}|_{\partial\Omega}=0\right\},\quad V(\Omega) := \left\{\mathbf{v}\in H_0^1(\Omega)\ :\ \nabla\cdot\mathbf{v}=0\right\}.$$

The dual space of $H_0^1(\Omega)$ is denoted $W^{-1,2}(\Omega) := (H_0^1(\Omega))'$ and equipped with the norm

$$||\mathbf{f}||_{-1,\Omega} := \sup_{0\neq\mathbf{v}\in H_0^1(\Omega)} \frac{<\mathbf{f},\mathbf{v}>_{W^{-1,2}(\Omega)\times H_0^1(\Omega)}}{|\mathbf{v}|_{1,\Omega}}.$$

Define

$$L_0^2(\Omega) := \left\{q\in L^2(\Omega) : (q,1)_\Omega = 0\right\}.$$

For brevity, omit Ω in the definitions above. For example, $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $H^1 = H^1(\Omega)$, and $V = V(\Omega)$.

Fix time $T>0$ and $m\geq 1$. Let $W^{m,q}(0,T;W^{k,p}(\Omega))$ denote the linear space of all Lebesgue measurable functions from $(0,T)$ onto $W^{k,p}$ equipped with and bounded in the norm

$$||\mathbf{u}||_{W^{m,q}(0,T;W^{k,p})} := (\int_0^T \sum_{i=0}^m ||\partial_t^{(i)}\mathbf{u}(\cdot,t)||^q_{W^{k,p}}dt)^{1/q}.$$

Write $W^{m,q}(W^{k,p}) = W^{m,q}(0,T;W^{k,p}(\Omega))$ and $C^m(W^{k,p}) = C^m([0,T];W^{k,p}(\Omega))$.

2.1. Discrete function setting. Fix $h>0$. Let $\mathcal{T}_h$ be a family of subdivisions (e.g. triangulation) of $\overline{\Omega}\subset\mathbb{R}^d$ satisfying $\overline{\Omega} = \bigcup_{E\in\mathcal{T}_h} E$ so that *diameter*$(E)\leq h$ and any two closed elements E_1, $E_2\in\mathcal{T}_h$ are either disjoint or share exactly one face, side, or vertex. Suppose further that $\mathcal{T}_h$ is quasi-uniformly regular as $h\to 0$. See [5] (Definition 4.4.13) for a precise definition and treatment of the inherited properties of such a space (see Appendix II.A in [11] for more on this subject in context of Stokes problem). For example, $\mathcal{T}_h$ consists of triangles for $d=2$ or tetrahedra for $d=3$ that are nondegenerate as $h\to 0$.

Let $X_{h,\cdot}\subset(H^1)^d$ and $Q_{h,\cdot}\subset L^2$ be a FE space. For example, let $X_{h,\cdot}$ and $Q_{h,\cdot}$ be continuous, piecewise (on each $E\in\mathcal{T}_h$) polynomial spaces. Define $X_h := X_{h,\cdot}\cap H_0^1$, $Q_h := Q_{h,\cdot}\cap L_0^2$. The discretely divergence-free space is given by

$$V_h = \left\{\mathbf{v}_h\in X_h : (q_h,\nabla\cdot\mathbf{v}_h) = 0\quad \forall q_h\in Q_{h,\cdot}\right\}.$$

Note that in general $V_h\not\subset V$ (e.g. Taylor-Hood elements).

Set $0 = t^0 < t^1 < \ldots < t^n = T < \infty$ with $\Delta t = t^n - t^{n-1}$. Write $z^n := z(t^n)$ and $z^{n+1/2} := \frac{1}{2}(z(t^{n+1}) + z(t^n))$. Define

$$||\mathbf{u}||_{l^q(m_1,m_2;W^{k,p})} := \begin{cases} (\Delta t\sum_{n=m_1}^{m_2} ||\mathbf{u}^n||^q_{k,p})^{1/q}, & q\in[1,\infty) \\ \max_{m_1\leq n\leq m_2} ||\mathbf{u}^n||_{k,p}, & q=\infty \end{cases}$$

for any $0\leq n = m_1, m_1+1,\ldots,m_2\leq N$. Write $||\mathbf{u}||_{l^q(W^{k,p})} = ||\mathbf{u}||_{l^q(0,N;W^{k,p})}$. We say that $\mathbf{u}\in l^q(m_1,m_2;W^{k,p})$ if the associated norm defined above stays finite as $\Delta t\to 0$. Define the discrete time-derivative

$$\partial^{n+1}_{\Delta t}\mathbf{v} := \frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\Delta t}.$$

Explicit skew-symmetrization of the convective term in NS-type equations ensures stability of the corresponding numerical approximation:

$$c_h(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}). \tag{16}$$

Fix $a_i \in \mathbb{R}$ for $i = -1, 0, 1, \ldots, n_0 \geq -1$ and $n \in \{0\} \cup \mathbb{N}$. Define $\xi^n(\mathbf{u})$ so that

$$c_h(\mathbf{u},\mathbf{v},\mathbf{w}) \approx c_h(\xi^n(\mathbf{u}),\mathbf{v},\mathbf{w}), \qquad \xi^n(\mathbf{u}) := a_{-1}\mathbf{u}^{n+1} + a_0\mathbf{u}^n + \ldots + a_{n_0}\mathbf{u}^{n-n_0}.$$

To summarize,

$$\begin{array}{lll} \text{No linearization} & \Rightarrow & a_{-1} = 1, \quad a_i = 0 \text{ for all } i \geq 0 \\ \text{Linearization} & \Rightarrow & a_{-1} = 0, \quad a_i \neq 0 \text{ for some } i \geq 0 \end{array}$$

For example,

$$\begin{array}{lll} \xi^n(\mathbf{u}) = \mathbf{u}^n & \Rightarrow & \xi^n(\mathbf{u}) = \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t) \\ \xi^n(\mathbf{u}) = \frac{1}{2}(3\mathbf{u}^n - \mathbf{u}^{n-1}) & \Rightarrow & \xi^n(\mathbf{u}) = \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t^2) \\ \xi^n(\mathbf{u}) = \frac{1}{8}(15\mathbf{u}^n - 10\mathbf{u}^{n-1} + 3\mathbf{u}^{n-2}) & \Rightarrow & \xi^n(\mathbf{u}) = \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t^3) \end{array}$$

CNLE is a particularly attractive method because it is Δt^2-accurate, implicit in the nonlinearity (a source of stiffness), and linearized which avoids issues of nonlinear solvers converging and greatly reduces the computational cost. Fix the kinematical viscosity $\nu > 0$. Note that $\nu \propto Re^{-1}$.

Problem 2.1 (CNLE). *Suppose that $\mathbf{u}_h^i \in V_h$ for $i = 0, 1, \ldots, n_0$ and $p_h^{n_0} \in Q_h$. For each $n = n_0, n_0+1, \ldots, N-1$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_h \times Q_h$ satisfying*

$$\begin{aligned} &(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) \\ &\qquad + \nu(\nabla\mathbf{u}_h^{n+1/2}, \nabla\mathbf{v}_h) - (p_h^{n+1/2}, \nabla\cdot\mathbf{v}_h) = (\mathbf{f}^{n+1/2}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h \end{aligned} \tag{17}$$

$$(q_h, \nabla\cdot\mathbf{u}^{n+1}) = 0, \quad \forall q_h \in Q_h. \tag{18}$$

Remark 2.2. *Note that $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^{n+1/2}$ defines the CNFE method analyzed in e.g. [20], $\xi^n(\mathbf{u}_h) = \frac{1}{2}(3\mathbf{u}_h^n - \mathbf{u}_h^{n-1})$ defines the CNLE method of e.g. [3, 13, 24], and $\xi^n(\mathbf{u}_h) = \mathbf{u}^{n-1/2}$ the $\mathcal{O}(\Delta t)$ CNLE method of e.g. [6].*

2.2. Assumptions and approximations. Let $C > 0$ be a generic data-independent constant throughout; specifically, C is independent of h, Δt, $\nu \to 0$. Error estimates for the elliptic projection (78) and Stokes projection (80) in L^2 and $W^{-1,2}$ require regularity of solutions to the following auxiliary problem.

Assumption 2.3. *Given $\theta \in W^{-1,2}$, find $(\mathbf{w}_\theta, r_\theta) \in H_0^1 \times L_0^2$ satisfying*

$$(\nabla\mathbf{w}_\theta, \nabla\mathbf{v}) - (r_\theta, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\mathbf{w}) = (\theta, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in H_0^1 \times L^2.$$

This problem is well-known to be well-posed. Suppose further that $(\mathbf{w}_\theta, r_\theta) \in (H^{m+2} \cap V) \times (H^{m+1} \cap L_0^2)$ satisfy

$$||\mathbf{w}_\theta||_{m+2} + ||r_\theta||_{m+1} \leq C||\theta||_m \tag{19}$$

when $m = 0$, 1 and $\theta \in H_0^m$ (with $H_0^0 = L^2$).

Indeed, (19) is true if Ω is smooth enough.

Preserving an abstract framework for the FE spaces, we assume that $X_{h,\cdot} \times Q_{h,\cdot}$ inherit several fundamental approximation properties.

Assumption 2.4. *The FE spaces $X_h \times Q_h$ satisfy:*

Uniform inf-sup (LBB) condition:

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in X_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{|\mathbf{v}_h|_1 \; ||q||} \geq C > 0 \tag{20}$$

FE-approximation:

$$\begin{aligned} \inf_{\mathbf{v}_h \in X_h} ||\mathbf{u} - \mathbf{v}_h||_1 &\leq Ch^k ||\mathbf{u}||_{k+1} \\ \inf_{q_h \in Q_h} ||p - q_h|| &\leq Ch^{s+1} ||p||_{s+1} \end{aligned} \tag{21}$$

for $k \geq 0$, and $s \geq -1$ when $\mathbf{u} \in H^{k+1} \cap H_0^1$, $p \in H^{s+1} \cap L_0^2$

Inverse-estimate:

$$|\mathbf{v}_h|_1 \leq Ch^{-1} ||\mathbf{v}_h||, \qquad \forall \mathbf{v}_h \in X_h \tag{22}$$

The well-known Taylor-Hood mixed FE is one such example satisfying Assumption 2.4. Estimates in (23), (24), (25), (28) stated below are used in proving error estimates for time-dependent problems. First define

$$\sigma(t) := \min\{1, t\}.$$

Then for any $n = n_0, n_0 + 1, \ldots, N-1$, $k \geq -1$,

$$||\partial_{\Delta t}^{n+1} \mathbf{u}||_k^2 \leq \Delta t^{-1} \int_{t^n}^{t^{n+1}} ||\partial_t \mathbf{u}(\cdot, t)||_k^2 dt \tag{23}$$

$$||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_k^2 \leq C\Delta t^3 \sigma(t^{n+1/2})^{-2} \int_{t^n}^{t^{n+1}} \sigma(t)^2 ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt \tag{24}$$

$$||\partial_{\Delta t}^{n+1} \mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_k^2 \leq C\Delta t^3 \sigma(t^{n+1/2})^{-2} \int_{t^n}^{t^{n+1}} \sigma(t)^2 ||\partial_t^{(3)} \mathbf{u}(\cdot, t)||_k^2 dt \tag{25}$$

when $\partial_t \mathbf{u} \in L^2(H^k)$, $t^2 \; \partial_t^{(2)} \mathbf{u}(\cdot, t) \in L^2(H^k)$, $t^2 \; \partial_t^{(3)} \mathbf{u}(\cdot, t) \in L^2(H^k)$. Derivation of these estimates follows from application of an appropriate Taylor expansion with integral remainder, see Appendix B. Moreover, (24) and (25) are replaced by

$$||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_k^2 \leq C\Delta t^3 \int_{t^n}^{t^{n+1}} ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt \tag{26}$$

$$||\partial_{\Delta t}^{n+1} \mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_k^2 \leq C\Delta t^3 \int_{t^n}^{t^{n+1}} ||\partial_t^{(3)} \mathbf{u}(\cdot, t)||_k^2 dt \tag{27}$$

if $\partial_t^{(2)} \mathbf{u} \in L^2(H^k)$, $\partial_t^{(3)} \mathbf{u} \in L^2(H^k)$ respectively. The operator $\xi^n(\mathbf{u})$ should be chosen to preserve the Δt^2-convergence rate of the fully-nonlinear CN scheme. This is made precise by *assuming* (28) holds.

Assumption 2.5. *Suppose that $t^2 \; \partial_t^{(2)} \mathbf{u} \in L^2(H^k)$ for some $k \geq 0$. Then for each $n = n_0$, $n_0 + 1$, ..., $N - 1$,*

$$||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_k^2 \leq C\Delta t^3 \sigma(t^{n+1/2})^{-2} \int_{t^{n-n_0}}^{t^{n+1}} \sigma(t)^2 ||\mathbf{u}(\cdot, t)||_k^2 dt \tag{28}$$

Note that if $\partial_t^{(2)} \mathbf{u} \in L^2(H^k)$, then (28) becomes

$$||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_k^2 \leq C\Delta t^3 \int_{t^{n-n_0}}^{t^{n+1}} ||\mathbf{u}(\cdot, t)||_k^2 dt \tag{29}$$

We summarize the continuous-in-space and discrete-in-time counterparts to the regularity results of Theorems 2.2, 2.3., 2.4 reported in [19] when the nonlocal compatibility condition (4), (5) is *not* satisfied. The same results (with extensions) are also reported and applied in [12, 14–16]). We summarize the results here as an assumption. The results follow if $\mathbf{u} \in L^\infty(H^1)$ and $\partial\Omega$ and problem data are smooth enough.

Assumption 2.6. *Suppose that for any $i \geq 0$, $k \geq -1$, and $2i + k > 0$ that*

$$\int_{t^{n_0}}^{T} \sigma(t)^{2i+k-3} ||\partial_t^{(i)} \mathbf{u}(\cdot, t)||_k^2 dt < \infty$$

and

$$\Delta t \sum_{n=1}^{N} \sigma(t^n)^{2i+k-2} ||\partial_t^{(i)} \mathbf{u}^n||_{k+1}^2 + \max_{1 \leq n \leq N} \sigma(t^n)^{2i+k-2} ||\mathbf{u}^n||_k^2 + \Delta t \sum_{k=1}^{N} \sigma(t^n)^{2i+k-2} ||\partial_t^{(i)} p^n||_k^2 < \infty.$$

3. Convergence estimate for CNLE

We state the main results in this section. A list of the constants found in these theorems (referenced throughout this section) are compiled in Section 3.1. The error equation required for the convergence proofs is derived in Section 4. Throughout, require that the continuous problem data minimally satisfies $\mathbf{u}^0 \in H^2 \cap V$ and $\mathbf{f} \in W^{1,\infty}(L^2)$. In this setting, we state the NSE: find $\mathbf{u} \in L^\infty(L^2) \cap L^2(H_0^1)$ and $p \in W^{-1,\infty}(L_0^2)$ satisfying

$$\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0^1 \tag{30}$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0 \quad \text{a.e. } (\mathbf{x}, t) \in \Omega \times (0, T] \tag{31}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega. \tag{32}$$

See e.g. [9] and references therein for NSE existence and regularity results. Define $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$. Proposition 3.1 provides sufficient conditions to conclude that $\mathbf{e} \in l^\infty(L^2) \cap l^2(H^1)$. Indeed, Proposition 3.1 holds for $\mathbf{u} \in C^0([t^{n_0}, T]; H^1)$ without a Gronwall exponential factor (use *a priori* estimate derivable for $\mathbf{u}_h^{n+1}$). Alternatively, the proof of Proposition 3.1 herein is obtained as an intermediate step in the proof of Theorem 3.5. We gain insight to the basic regularity requirements in our method of proof for the error estimate (34).

Proposition 3.1. *Suppose that the FE space satisfies Assumption 2.4. Suppose further that Assumption 2.3 is satisfied along with*

$$\mathbf{u} \in l^2(H^2) \cap l^\infty(V) \quad \partial_t \mathbf{u} \in l^2(n_0, N; W^{-1,2}), \quad p \in l^2(n_0, N; L_0^2).$$

and $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^1)$. Then

$$||\mathbf{e}||_{l^\infty(n_0+1,N;L^2)} + \nu^{1/2} (\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} \leq C(||\mathbf{u}^N|| + \nu^{1/2} ||\nabla \mathbf{u}||_{l^2(n_0,N;L^2)}) + \exp(C\nu^{-1} ||\mathbf{u}||_{l^2(n_0,N;H^2)}^2)(K_0 ||\mathbf{e}||_{l^\infty(0,n_0;L^2)} + K_1 + K_2) \tag{33}$$

The constants K_0, K_1, $K_2 > 0$ are given in (41), (42), (43) respectively so that the right hand side of (33) remains bounded as h, $\Delta t \to 0$.

Proof. See Section 4.1. □

Remark 3.2. *The regularity* $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^1)$ *assumption in Proposition 3.1 is more than necessary. Indeed, we can replace the elliptic projection with the* L^2*-projection to eliminate the time-derivative term* $(\partial_{\Delta t}^{n+1}\eta, \mathbf{U}_h^{n+1/2})$ *(the source of the resulting regularity restriction) in the proof of Proposition 3.1. However, incorporation of the* L^2*-projection from* V *into* V_h *requires additional (technical) development and does not benefit the main error estimate in Theorem 3.5 for* $k > 2$*.*

Remark 3.3. *The exponential Gronwall factor* $\exp(C\nu^{-1}||\mathbf{u}||^2_{l^2(n_0,N;H^2)})$ *in* K *above typically takes the form* $\exp(C\nu^{-3}||\nabla\mathbf{u}||^4_{l^4(n_0,N;L^2)})$*. However, a factor of* $\sum_n ||\nabla\mathbf{u}^n||^2$ *must be absorbed into* $\sum_n ||\nabla\mathbf{u}^{n+1/2}||^2$ *to proceed with the latter estimate. As discussed in Section 1.1.1, this is not possible in general for the usual CNLE (without a great restriction on time interval length* T*). In particular, since the dissipative term for CN schemes has the form* $\nu\sum_n ||\nabla\mathbf{u}^{n+1/2}||^2$ *and the usual extrapolation for CNLE is* $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$ *(which is not a sum of average velocities), errors manifested from the extrapolated velocity cannot be absorbed (as usual) into the dissipative term. Consequently all errors propagated from the extrapolated convecting velocity must be absorbed via the discrete Gronwall Lemma.*

The optimal convergence rate proved in Theorem 3.5 requires that initial iterates $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$ are *accurate enough*. We make this precise in the following assumption.

Assumption 3.4. *Fix* $k \geq 0$*,* $s \geq -1$ *and* $\alpha > 0$*. Suppose* $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$ *satisfy*

$$K_0||\mathbf{e}||_{l^\infty(0,n_0;L^2)} \leq \alpha(K_u h^k + K_p h^{s+1} + K_t\Delta t^2)$$

where K_0*,* K_u*,* K_p*,* $K_t > 0$ *are given in (41), (44), (45), (46) respectively and remain bounded as* h*,* $\Delta t \to 0$*.*

Note that Assumption 3.4 reduces to, when $s = k-1$,

$$||\mathbf{e}||_{l^\infty(0,n_0;L^2)} \leq \alpha(h^k + \Delta t^2).$$

The following theorem provides sufficient conditions to ensure CNLE converges optimally in the energy norm: e.g. under usual regularity conditions and $s = k-1$

$$||\mathbf{e}||_{l^\infty(n_0+1,N;L^2)} + \nu^{1/2}(\Delta t\sum_{n=n_0}^{N-1}|\mathbf{e}^{n+1/2}|_1^2)^{1/2} \leq C(h^k + \Delta t^2)$$

without any Δt-restriction!

Theorem 3.5. *Fix* $k > 0$*,* $s > -1$*. Under the assumptions of Proposition 3.1, suppose further that Assumptions 2.5, 3.4 are satisfied along with* $\mathbf{u} \in l^\infty(H^k \cap H^2) \cap l^2(H^{k+1})$*,* $\partial_t\mathbf{u} \in L^2(t^{n_0}, T; H^{k-1} \cap H^1) \cap l^\infty(n_0, N; H^1)$*,* $\partial_t^{(2)}\mathbf{u} \in L^2(L^2)$*,* $\partial_t^{(3)}\mathbf{u} \in L^2(t^{n_0}, T; W^{-1,2})$*, and* $p \in l^2(n_0, N; H^{s+1})$*. Then*

$$\begin{aligned} ||\mathbf{e}||_{l^\infty(n_0+1,N;L^2)} + \nu^{1/2}(\Delta t\sum_{n=n_0}^{N-1}|\mathbf{e}^{n+1/2}|_1^2)^{1/2} &\leq G^N K_p h^{s+1} \\ + (C||\mathbf{u}^N||_k + C\nu^{1/2}||\mathbf{u}||_{l^2(n_0,N;H^{k+1})} + G^N K_u)h^k &+ G^N K_t\Delta t^2 \qquad (34) \end{aligned}$$

where $G^N := \exp(C\nu^{-1}||\mathbf{u}||^2_{l^2(n_0,N;H^2)})$*. The constants* K_u*,* K_p*,* $K_t > 0$ *are given in (44), (45), (46) respectively and remain bounded as* h*,* $\Delta t \to 0$*.*

Proof. See Section 4.1. □

In the general case that the compatibility condition (4), (5) is not satisfied, the result of Theorem 3.5 holds under a reduced convergent rate: e.g. under usual regularity conditions and $s = k - 1$

$$||\mathbf{e}||_{l^\infty(n_0+1,N;L^2)} + \nu^{1/2}(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} \leq C\sigma(t^1)^{-1}(\sigma(t^1)^{-(k-3)/2}h^k + \Delta t^2)$$

Theorem 3.6. *Fix $k > 0$, $s > -1$. Under the assumptions of Proposition 3.1, suppose further that Assumptions 2.5, 3.4, and 2.6 are satisfied. Then (34) holds with K_u, K_p, K_t replaced by $\sigma(t^1)^{-(k-1)/2}\overline{K}_u$, $\sigma(t^1)^{-(k-2)/2}\overline{K}_p$, and $\sigma(t^1)^{-1}\overline{K}_t$ respectively where the constants $\overline{K}_u$, $\overline{K}_p$, $\overline{K}_t > 0$ are given in (47), (48), (49) respectively and remain bounded as h, $\Delta t \to 0$.*

Remark 3.7. *If we use the L^2-projection rather than elliptic projection in proving Theorem 3.6, the result is improvable so that K_u is replaced by $\sigma(t^1)^{-(k-2)/2}\overline{K}_u$.*

Proof. See Section 4.1. □

An estimate for $\Delta t \sum_n ||(\mathbf{e}^{n+1} - \mathbf{e}^n)/\Delta t||^2$ is needed in the error analysis for pressure and the drag/lift forces by the fluid on embedded obstacles. Theorem 3.8 provides sufficient regularity of $(\mathbf{u}, p)$ solving (81), (82), (83) to ensure $\mathbf{u}_h \in l^\infty(H^1)$ and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$. Note that the regularity $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^1)$, $\partial_t p \in L^2(t^{n_0}, T; L^2)$ in Theorem 3.8 is a result of using the Stokes projection in corresponding proof herein and can be relaxed. The proof of Theorem 3.8 is obtained as an intermediate step in the proof of Theorem 3.10. Relaxing the regularity assumption (by using the L^2-projection or elliptic instead) leads to a suboptimal estimate. The additional regularity required here gives insight to the *extra* regularity required for the optimal error estimate (38).

Theorem 3.8. *Under the assumptions of Proposition 3.1, suppose further that $\mathbf{u} \in l^\infty(H^2)$, $p \in l^\infty(H^1)$, $\partial_t \mathbf{u} \in l^2(n_0, N; L^2) \cap L^2(t^{n_0}, T; H^1)$, and $\partial_t p \in L^2(t^{n_0}, T; L^2)$ so that*

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 < \infty, \quad \textit{as } h,\ \Delta t \to 0. \tag{35}$$

Then

$$\begin{aligned} &||\partial_{\Delta t}\mathbf{e}||_{l^2(n_0+1,N;L^2)} + \nu^{1/2}||\nabla \mathbf{e}||_{l^\infty(n_0+1,N;L^2)} \\ &\leq G^N(F_0||\nabla \mathbf{e}||_{l^\infty(0,n_0;L^2)} + F_1 + F_2) + C\nu^{-1/2}||p^N|| \\ &\quad + C\nu^{1/2}|\mathbf{u}^N|_1 + Ch\nu^{-1}||\partial_t p||_{L^2(t^{n_0},T;L^2)} + Ch||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^1)} \end{aligned} \tag{36}$$

where $G^N := \exp(C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1}(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2))$. The constants F_0, F_1, F_2 are given in (50), (51), (52) respectively so that the right hand side of (36) remains bounded as h, $\Delta t \to 0$.

Proof. See Section 4.2. □

The optimal convergence result proved in Theorem 3.10 requires that the initial iterates $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$ must be accurate enough. We make this precise in the following assumption.

Assumption 3.9. *Fix $k \geq 0$, $s \geq -1$ and $\alpha > 0$. Suppose $\{\mathbf{u}_h^i\}_i^{n_0}$ satisfy*

$$F_0||\nabla \mathbf{e}||_{l^\infty(0,n_0;L^2)} \leq \alpha(F_u h^k + F_p h^{s+1} + F_t \Delta t^2)$$

where F_0, F_u, F_p, $F_t > 0$ are given in (50), (53), (54), (56) respectively.

Note that Assumption 3.9 reduces to, when $s = k-1$,

$$||\nabla \mathbf{e}||_{l^\infty(0,n_0;L^2)} \leq \alpha(h^k + \Delta t^2).$$

Therefore, under usual regularity conditions we show in Theorem 3.10 that

$$||\partial_{\Delta t}\mathbf{e}||_{l^2(n_0+1,N;L^2)} + ||\nabla \mathbf{e}||_{l^\infty(n_0+1,N;L^2)} \leq C(h^k + \Delta t^2)$$

as long as $\Delta t \leq M h^{1/4}$ for any arbitrary $M > 0$ (i.e. no ν-dependence).

Theorem 3.10. *Fix $k > 0$, $s > -1$. Under the regularity of Theorem 3.8, suppose further that Assumptions 2.5, 3.9 are satisfied along with $\mathbf{u} \in l^\infty(H^{k+1})$, $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^k \cap H^3) \cap l^\infty(n_0, N; L^2)$, $\partial_t^{(2)}\mathbf{u} \in L^2(t^{n_0}, T; H^1)$, $\partial_t^{(3)}\mathbf{u} \in L^2(t^{n_0}, T; L^2)$, $p \in l^\infty(H^{s+1})$, $\partial_t p \in L^2(t^{n_0}, T; H^s)$, and*

$$\Delta t \leq M h^{1/4}, \quad \text{for any constant } M > 0 \text{ (no } \nu\text{-dependence).} \tag{37}$$

Then

$$\begin{aligned} &||\partial_{\Delta t}\mathbf{e}||_{l^2(n_0+1,N;L^2)} + \nu^{1/2}||\nabla \mathbf{e}||_{l^\infty(n_0+1,N;L^2)} \\ &\qquad \leq G^N(F_u h^k + F_p h^{s+1} + F_t \Delta t^2 + F_3(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2}) \end{aligned} \tag{38}$$

where $G^N := \exp(C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1}(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2))$. The constants F_u, F_p, F_3, $F_t > 0$ are given in (53), (54), (55), (56) respectively and remain finite as h, $\Delta t \to 0$.

Proof. See Section 4.2. □

In the general case that the compatibility condition (4), (5) is not satisfied, the result of Theorem 3.10 holds under a reduced convergent rate: e.g. under usual regularity conditions and $s = k-1$

$$||\partial_{\Delta t}\mathbf{e}||_{l^2(n_0+1,N;L^2)} + \nu^{1/2}||\nabla \mathbf{e}||_{l^\infty(n_0+1,N;L^2)} \leq C\sigma(t^1)^{-3/2}(\sigma(t^1)^{-(k-4)/2}h^k + \Delta t^2)$$

Theorem 3.11. *Fix $k > 0$, $s > -1$. Under the assumptions of Theorem 3.8, suppose further that Assumptions 2.5, 3.4, and 2.6 are satisfied. Then (38) holds with F_u, F_p, F_t replaced by $\sigma(t^1)^{-(k-1)/2}\overline{F}_u$, $\sigma(t^1)^{-(k-1)/2}\overline{F}_p$, and $\sigma(t^1)^{-3/2}\overline{F}_t$ respectively where the constants $\overline{F}_u$, $\overline{F}_p$, $\overline{F}_t > 0$ are given in (57), (58), (59) respectively and remain bounded as h, $\Delta t \to 0$.*

Proof. See Section 4.2. □

Remark 3.12. *It is common to assume that $\mathbf{u} \in L^\infty(W^{1,\infty})$ in the convergence analysis of NSE approximations (see e.g. [3, 24]). The conclusions of Proposition 3.1 3.1, Theorem 3.5, in addition to those of Theorems 3.8, 3.10 are preserved with the regularity condition $\mathbf{u}(\cdot, t) \in H^2$ replaced by $\mathbf{u}(\cdot, t) \in W^{1,\infty}$. Regardless, the analysis of [24] suggests an h, Δt-restriction for optimal convergence in $l^\infty(L^2) \cap l^2(H^1)$ (Theorem 3.1 in [24]) and a sub-optimal convergence estimate $||\mathbf{u} - \mathbf{u}_h||_{l^\infty(H^1)} \leq \mathcal{O}(h^k + h^{s+1} + h^{-3/2}\Delta t^4 + \Delta t^{3/2})$ (Theorem 4.1 in [24]). The*

$l^\infty(H^1)$-estimate in [24] requires, for instance, $\Delta t \leq h^{(3+2k)/8}$ for optimal convergence rate as $h \to 0$, but still predicts suboptimal convergence rate with respect to $\Delta t \to 0$.

The error estimates of Proposition 3.1, Theorem 3.5, 3.8, 3.10 give conditions in which $p_h(avg) \in l^2(n_0, N; L^2)$, $\mathbf{u}_h \in l^\infty(n_0, N; H^1)$, and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$. In particular, as a direct consequence of (36) and the conditions of Theorem 3.8, we have

$$||\partial_{\Delta t}\mathbf{u}_h||_{l^2(n_0,N;L^2)} + \nu||\nabla\mathbf{u}_h||_{l^\infty(n_0,N;L^2)} \leq C < \infty.$$

Estimates for pressure follow as well and are summarized in the next Corollary.

Corollary 3.13. *Under the conditions and conclusions of Theorem 3.8,*

$$\Delta t \sum_{n=n_0}^{N-1} ||p_h^{n+1/2}|| \leq C < \infty, \quad \textit{as } h,\ \Delta t \to 0. \tag{39}$$

Under the conditions and conclusions of Theorem 3.10, for $s = k - 1$,

$$\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - p_h^{n+1/2}|| \leq C(h^k + \Delta t^2). \tag{40}$$

Proof. See Section 4.3. □

3.1. Constant factors in convergence estimates. For Proposition 3.1, define the spatial-modeling error in $X_h \times Q_h$ by K_1, time modeling-error by K_2, and initial condition modeling error by K_0 so that

$$K_0 := C + C\nu^{-1/2}\begin{cases} ||\mathbf{u}||_{l^2(0,2n_0;H^2)} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

$$\begin{aligned} K_1 := {} & C\nu^{-1/2}||\nabla\mathbf{u}||_{l^\infty(L^2)}||\nabla\mathbf{u}||_{l^2(L^2)} + K_0||\mathbf{u}||_{l^\infty(0,n_0;L^2)} + \\ & \ldots + C\nu^{-1/2}||p||_{l^2(n_0,N;L^2)} + C\nu^{-1/2}h^2||\partial_t\mathbf{u}||_{L^2(t^{n_0},T;H^1)} \end{aligned} \tag{42}$$

$$\begin{aligned} K_2 := {} & C\nu^{-1/2}(||\nabla\mathbf{u}||_{l^\infty(L^2)}||\nabla\mathbf{u}||_{l^2(n_0,N;L^2)} + \ldots \\ & \ldots + ||\partial_t\mathbf{u}||^2_{L^2(t^{n_0},T;W^{-1,2})} + ||\partial_t\mathbf{u}||^2_{l^2(n_0,N;W^{-1,2})}). \end{aligned} \tag{43}$$

For Theorem 3.5 fix $k > 0$, $s > -1$, $k* = k$ (with $k* = 2$ if $k = 1$) and define the weight of spatial modeling error in X_h by K_u, spatial modeling error in Q_h by K_p, and time modeling error by K_t so that

$$\begin{aligned} K_u := {} & C(\nu^{-1/2}||\nabla\mathbf{u}||_{l^\infty(L^2)}||\mathbf{u}||_{l^2(H^{k+1})} + \nu^{-1/2}||\partial_t\mathbf{u}||_{L^2(t^{n_0},T;H^{k*-1})} + \ldots \\ & \ldots + K_0||\mathbf{u}||_{l^\infty(0,n_0;H^k)}) \end{aligned} \tag{44}$$

$$K_p := C\nu^{-1/2}||p||_{l^2(n_0,N;H^{s+1})} \tag{45}$$

$$\begin{aligned} K_t := {} & C\nu^{-1/2}(||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(L^2)} + \ldots \\ & \ldots + ||\partial_t\mathbf{u}||_{l^\infty(n_0,N;L^2)}||\partial_t\mathbf{u}||_{L^2(t^{n_0},T;H^2)}) \end{aligned} \tag{46}$$

$$\begin{aligned}
\overline{K}_u := {} & C\Delta t^{1/2}\sigma(t^1)^{(k-1)/2}\nu^{-1/2}||\nabla \mathbf{u}||_{l^\infty(L^2)}||\mathbf{u}^0||_{k+1} + K_0\sigma(t^1)^{(k-1)/2}||\mathbf{u}^0||_k \\
& + C\nu^{-1/2}\sigma(t^1)^{1/2}||\nabla \mathbf{u}||_{l^\infty(L^2)}(\Delta t\sum_{n=1}^{N}\sigma(t^n)^{k-2}||\mathbf{u}^n||_{k+1}^2)^{1/2} \\
& + C\nu^{-1/2}(\int_{t^{n_0}}^{T}\sigma(t)^{k-1}||\partial_t \mathbf{u}(\cdot,t)||_{k-1}^2 dt)^{1/2} \\
& + C\sigma(t^1)^{1/2}(K_0\max_{1\le n\le n_0}\sigma(t^n)^{(k-2)/2}||\mathbf{u}^n||_k) \qquad (47)
\end{aligned}$$

$$\overline{K}_p := C\nu^{-1/2}(\Delta t\sum_{n=n_0}^{N}\sigma(t^n)^{k-2}||p^n||_k^2)^{1/2} \qquad (48)$$

$$\begin{aligned}
\overline{K}_t := {} & C\nu^{-1/2}(\int_{t^{n_0}}^{T}\sigma(t)^2\ ||\partial_t^{(3)}\mathbf{u}(\cdot,t)||_{-1}dt)^{1/2} \\
& + C\nu^{-1/2}\sigma(t^{n_0})^{1/2}||\mathbf{u}||_{l^\infty(H^2)}(\int_{t^{n_0}}^{T}\sigma(t)\ ||\partial_t^{(2)}\mathbf{u}(\cdot,t)||dt)^{1/2} \\
& + C\nu^{-1/2}\sigma(t^{n_0})^{1/2}\Delta t||\partial_t \mathbf{u}||_{l^\infty(n_0,N;L^2)}(\int_{t^{n_0}}^{T}\sigma(t)\ ||\partial_t \mathbf{u}(\cdot,t)||_2 dt)^{1/2}. \qquad (49)
\end{aligned}$$

Analogous to K_1, K_2, K_0, for Theorem 3.8 define the spatial-modeling error in $X_h \times Q_h$ by F_1, time modeling-error by F_2, and initial condition modeling error by F_0 so that

$$F_0 := C\nu^{1/2} + C\begin{cases} (\Delta t\sum_{n=0}^{2n_0-1}(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2))^{1/2} & \text{if } n_0 \ge 1, \\ 0 & \text{otherwise} \end{cases} \qquad (50)$$

$$\begin{aligned}
F_1 := {} & C\nu^{-1}||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||p||_{l^2(L^2)} + \nu^{-1}F_0||p||_{l^\infty(0,n_0;L^2)} \\
& + C||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||\nabla \mathbf{u}||_{l^2(L^2)} + F_0||\nabla \mathbf{u}||_{l^\infty(0,n_0;L^2)} \\
& + C\nu^{-1}h||\partial_t p||_{L^2(t^{n_0},T;L^2)} + Ch||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^1)} \\
& + C(\nu^{-1}h^{1/2}||p||_{l^\infty(H^1)} + ||\mathbf{u}||_{l^\infty(H^2)})(\Delta t\sum_{n=n_0}^{N-1}|\mathbf{e}^{n+1/2}|_1^2)^{1/2} \qquad (51)
\end{aligned}$$

$$\begin{aligned}
F_2 := {} & C(||\mathbf{u}||_{l^\infty(H^2)}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^1)} + ||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0},T;L^2)} + \dots \\
& \dots + \nu||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^2)} + ||\partial_t p||_{L^2(t^{n_0},T;H^1)} + ||\partial_t \mathbf{f}||_{L^2(t^{n_0},T;L^2)}). \qquad (52)
\end{aligned}$$

For Theorem 3.10, fix $k > 0$, $s > -1$ and define the weight of spatial modeling error in X_h by F_u, spatial modeling error in Q_h by F_p, the bootstrapped modeling error of velocity in the energy norm by F_3, and time modeling error by F_t so that

$$\begin{aligned}
F_u := {} & C(||\mathbf{u}||_{l^\infty(H^2)}||\mathbf{u}||_{l^2(H^{k+1})} + ||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^k)} + \dots \\
& \dots + F_0||\mathbf{u}||_{l^\infty(0,n_0;H^{k+1})}) \qquad (53)
\end{aligned}$$

$$\begin{aligned}
F_p := {} & C\nu^{-1}(||\mathbf{u}||_{l^\infty(H^2)}||p||_{l^2(H^{s+1})} + ||\partial_t p||_{L^2(t^{n_0},T;H^s)} + \dots \\
& \dots + F_0||p||_{l^\infty(0,n_0;H^{s+1})}) \qquad (54)
\end{aligned}$$

$$F_3 := C||\mathbf{u}||_{l^\infty(H^2)}(||\mathbf{u}||_{l^\infty(H^2)} + \nu^{-1}h^{1/2}||p||_{l^\infty(H^1)}) \qquad (55)$$

$$\begin{aligned}
F_t := {} & C(||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0},T;L^2)} + ||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0},T;H^1)} + \dots \\
& \dots + ||\partial_t \mathbf{u}||_{l^\infty(n_0,N;L^2)}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^3)}) \qquad (56)
\end{aligned}$$

and

$$
\begin{aligned}
\overline{F}_u := &\ \sigma(t^1)^{(k-1)/2}(F_0 + C\Delta t^{1/2}||\mathbf{u}||_{l^\infty(n_0,N;H^2)})||\mathbf{u}^0||_{k+1}h^k \\
&+ F_0 \max_{1\le n\le n_0} \sigma(t^n)^{(k-1)/2}||\mathbf{u}^n||_{k+1}h^k \\
&+ C\sigma(t^1)^{1/2}||\mathbf{u}||_{l^\infty(n_0,N;H^2)}(\Delta t \sum_{n=1}^{N} \sigma(t^n)^{k-2}||\mathbf{u}^n||^2_{k+1})^{1/2}h^k \\
&+ C(\int_{t^{n_0}}^{T} \sigma(t)^{k-1}||\partial_t \mathbf{u}(\cdot,t)||^2_k dt)^{1/2}h^k \qquad (57)\\
\overline{F}_p := &\ \sigma(t^1)^{(k-1)/2}(F_0 + C\Delta t^{1/2}\nu^{-1}||\mathbf{u}||_{l^\infty(n_0,N;H^2)})||p^0||_k h^k \\
&+ F_0\nu^{-1} \max_{1\le n\le n_0} \sigma(t^n)^{(k-1)/2}||p^n||_k \\
&+ C\nu^{-1}\sigma(t^1)^{1/2}||\mathbf{u}||_{l^\infty(n_0,N;H^2)}(\Delta t \sum_{n=1}^{N} \sigma(t^n)^{k-2}||p^n||^2_k)^{1/2}h^k \\
&+ C\nu^{-1}(\int_{t^{n_0}}^{T} \sigma(t)^{k-1}||\partial_t p(\cdot,t)||^2_{k-1} dt)^{1/2}h^k \qquad (58)\\
\overline{F}_t := &\ C(\int_{t^{n_0}}^{T} \sigma(t)^3\ ||\partial_t^{(3)}\mathbf{u}(\cdot,t)||dt)^{1/2} \\
&+ C\sigma(t^{n_0})^{1/2}||\mathbf{u}||_{l^\infty(n_0,N;H^2)}(\int_{t^{n_0}}^{T} \sigma(t)^2\ ||\partial_t^{(2)}\mathbf{u}(\cdot,t)||_1 dt)^{1/2} \\
&+ C\sigma(t^{n_0})^{1/2}\Delta t||\partial_t\mathbf{u}||_{l^\infty(n_0,N;L^2)}(\int_{t^{n_0}}^{T} \sigma(t)^2\ ||\partial_t\mathbf{u}(\cdot,t)||_3 dt)^{1/2}. \qquad (59)
\end{aligned}
$$

3.2. Fundamentals of estimation. Standard error analysis for CNLE relies on the discrete Gronwall Lemma 3.14 which leads to a Δt-restriction of the form $\Delta t\kappa^n < 1$ for convergence. On the other hand, we show that this Δt-restriction is avoidable for CNLE because the second Gronwall Lemma 3.15 can be applied instead.

Lemma 3.14 (Gronwall, Δt-restriction)**.** *Let $D \ge 0$ and κ^n, A^n, B^n, $C^n \ge 0$ for any integer $n \ge 0$ and satisfy*

$$A^N + \Delta t \sum_{n=0}^{N} B^n \le \Delta t \sum_{n=0}^{N} \kappa^n A^n + \Delta t \sum_{n=0}^{N} C^n + D, \quad \forall N \ge 0.$$

Suppose that for all n

$$\Delta t\kappa^n < 1$$

and set $\lambda^n = (1 - \Delta t\kappa^n)^{-1}$. Then,

$$A^N + \Delta t \sum_{n=0}^{N} B^n \le \exp(\Delta t \sum_{n=0}^{N} \lambda^n\kappa^n)(\Delta t \sum_{n=0}^{N} C^n + D), \quad \forall N \ge 0.$$

Lemma 3.15 (Gronwall, no Δt-restriction)**.** *Let $D \ge 0$ and κ^n, A^n, B^n, $C^n \ge 0$ for any integer $n \ge 0$ and satisfy*

$$A^N + \Delta t \sum_{n=0}^{N} B^n \le \Delta t \sum_{n=0}^{N-1} \kappa^n A^n + \Delta t \sum_{n=0}^{N} C^n + D, \quad \forall N \ge 0.$$

Then

$$A^N + \Delta t \sum_{n=0}^{N} B^n \leq \exp(\Delta t \sum_{n=0}^{N-1} \kappa^n)(\Delta t \sum_{n=0}^{N} C^n + D), \quad \forall N \geq 0.$$

Lemma 3.14 is proved in Lemma 5.1 of [20] with the proof of Lemma 3.15 explained in a subsequent remark. The following change of indices formula is required to resolve double sums that arise in the error analysis of CNLE.

Lemma 3.16. *Let κ^n, $\lambda^n \in \mathbb{R}$ for all $n \in \mathbb{N}$, $\alpha^i \in \mathbb{R}$ for all $i = 0, 1, \ldots, n_0$. Then,*

$$\sum_{n=n_0}^{N-1} \kappa^n (\sum_{i=0}^{n_0} \alpha^i \lambda^{n-i}) = \sum_{n=0}^{N-1} (\sum_{i=i_0(n)}^{i_1(n)} \alpha^i \kappa^{n+i}) \lambda^n \tag{60}$$

where

$$i_0(n) := \begin{cases} 0, & n \geq n_0 \\ n_0 - n, & \textit{otherwise} \end{cases}, \quad i_1(n) := \begin{cases} n_0, & n < N - 1 - n_0 \\ N - n, & \textit{otherwise} \end{cases}.$$

We use the fact that $||\nabla \cdot \mathbf{v}|| \leq \sqrt{d}|\mathbf{v}|_1$ throughout without further reference. Fix q, $q' \geq 1$ so that $1/q + (1/q') = 1$. The following estimates are used frequently in the analysis herein (for proofs see e.g. [8], Chapter II, and references therein):

- (Young) For any $a > 0$, $b > 0$, and $\delta > 0$

$$ab \leq \frac{1}{q\delta^{q/q'}} a^q + \frac{\delta}{q'} b^{q'} \tag{61}$$

- (Hölder) For any $\mathbf{v} \in L^q$, $\mathbf{w} \in L^{q'}$

$$|(\mathbf{v}, \mathbf{w})| \leq ||\mathbf{v}||_{0,q} ||\mathbf{w}||_{0,q'} \tag{62}$$

- (Ladyzhenskaya) For any $\mathbf{v} \in H^1$,

$$\begin{cases} ||\mathbf{v}||_{0,3} & \leq & C||\mathbf{v}||^{1/2}||\mathbf{v}||_1^{1/2} \\ ||\mathbf{v}||_{0,4} & \leq & C||\mathbf{v}||^{d/4}||\mathbf{v}||_1^{(4-d)/4} \\ ||\mathbf{v}||_{0,6} & \leq & C||\mathbf{v}||_1. \end{cases} \tag{63}$$

- (Sobolev) $H^2 \hookrightarrow L^\infty$, $W^{1,3}$ so that for any $\mathbf{v} \in H^2$,

$$||\mathbf{v}||_{0,\infty} \leq C||\mathbf{v}||_2, \qquad ||\mathbf{v}||_{1,3} \leq C||\mathbf{v}||_2. \tag{64}$$

3.3. Estimating $\int \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}$. Formulation of a stable FE discretization of NS and NS-type problems is subtle. We introduced the explicitly skew-symmetric convective term in (16) so that $c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \approx (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})$ and

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0. \tag{65}$$

In fact, $c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})$ only if $\nabla \cdot \mathbf{u} = 0$ when $\mathbf{v}$, $\mathbf{w} \neq 0$; i.e. in general, $c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \neq (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})$ when $\nabla \cdot \mathbf{u} \neq 0$. Consequently, it is worthwhile to carefully derive identities and estimates associated with both the convective and explicitly skew-symmetric forms.

Lemma 3.17. *Fix $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1$. Then*

$$|(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})| \leq C \begin{cases} ||\mathbf{u}||_1 |\mathbf{v}|_1 ||\mathbf{w}||_1 & \\ ||\mathbf{u}||\ ||\mathbf{v}||_2 ||\mathbf{w}||_1 & \forall \mathbf{v} \in H^2 \\ ||\mathbf{u}||_2 |\mathbf{v}|_1 ||\mathbf{w}|| & \forall \mathbf{u} \in H^2 \\ ||\mathbf{u}||\ ||\mathbf{v}||_3 ||\mathbf{w}|| & \forall \mathbf{u} \in H^3 \end{cases} \tag{66}$$

Proof. First, application of Hölder's inequality (62) with $1/p + 1/q + 1/r = 1$ and p, q, $r \geq 1$ gives

$$|(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})| \leq ||\mathbf{u}||_{0,p} ||\nabla \mathbf{v}||_{0,q} ||\mathbf{w}||_{0,r}, \quad \forall \mathbf{u} \in L^p, \ \mathbf{v} \in W^{1,q}, \ \mathbf{w} \in L^r.$$

For (66)(a), pick $p = r = 4$, $q = 2$ and apply (63)(b). For (66)(b), pick $p = 2$, $q = 3$, $r = 6$ and apply (63)(c), (64)(b). For (66)(c), pick $p = \infty$, $q = 2$, $r = 2$ and apply (64)(a). For (66)(d), pick $p = 2$, $q = \infty$, $r = 2$ and apply (64)(a). $\square$

Application of integration by parts and the divergence theorem give:

$$(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) - ((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}) + \int_{\partial\Omega} (\mathbf{u} \cdot \hat{\mathbf{n}})\mathbf{v} \cdot \mathbf{w}. \tag{67}$$

We conclude the following from (67).

Lemma 3.18. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1$. *Suppose that* $(\mathbf{u} \cdot \hat{\mathbf{n}})\mathbf{v} \cdot \mathbf{w}|_{\partial\Omega} = 0$. *Then*

$$(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) - ((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}). \tag{68}$$

Additionally,

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}). \tag{69}$$

It follows immediately from (67) and (66) that

$$|(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w})| \leq C||\mathbf{u}||_2 \ ||\mathbf{v}|| \ ||\mathbf{w}||_1 \quad \forall \mathbf{u} \in H^2 \cap V., \ (\mathbf{u} \cdot \hat{\mathbf{n}})\mathbf{v} \cdot \mathbf{w}|_{\partial\Omega} = 0 \tag{70}$$

Next, substitute (67) into (16) to obtain an additional (equivalent) formulation of $c_h(\cdot, \cdot, \cdot)$:

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}) - \frac{1}{2}\int_{\partial\Omega} (\mathbf{u} \cdot \hat{\mathbf{n}})\mathbf{v} \cdot \mathbf{w} \tag{71}$$

The following is an immediate consequence of (71).

Lemma 3.19. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1$. *Suppose that* $(\mathbf{u} \cdot \hat{\mathbf{n}})\mathbf{v} \cdot \mathbf{w}|_{\partial\Omega} = 0$. *Then*

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}) \tag{72}$$

Additionally,

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}). \tag{73}$$

Similar to the continuous case, we derive several important majorizations of the discrete trilinear form $c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})$ required in the analysis CNLE.

Lemma 3.20. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1$. *Then*

$$|c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \left\{ \begin{array}{ll} (||\mathbf{u}|| \ ||\mathbf{u}||_1)^{1/2} ||\mathbf{v}||_1 ||\mathbf{w}||_1 & \\ ||\mathbf{u}|| \ ||\mathbf{v}||_2 ||\mathbf{w}||_1 & \forall \mathbf{v} \in H^2 \end{array} \right. \tag{74}$$

Additionally, when $(\mathbf{u} \cdot \hat{\mathbf{n}})\mathbf{v} \cdot \mathbf{w}|_{\partial\Omega} = 0$,

$$|c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \left\{ \begin{array}{ll} ||\mathbf{u}||_1 ||\mathbf{v}||_1 (||\mathbf{w}|| \ ||\mathbf{w}||_1)^{1/2} & \\ ||\mathbf{u}||_1 ||\mathbf{v}||_2 ||\mathbf{w}|| & \forall \mathbf{v} \in H^2 \end{array} \right. \tag{75}$$

Proof. First, application of Hölder's inequality (62) to (16) with $1/p+1/q+1/r = 1$ and p, q, $r \geq 1$ and $1/p' + 1/q' + 1/r' = 1$ and p', q', $r' \geq 1$ gives

$$|c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{1}{2}||\mathbf{u}||_{0,p} ||\nabla \mathbf{v}||_{0,q} ||\mathbf{w}||_{0,r} + \frac{1}{2}||\mathbf{u}||_{0,p'} ||\nabla \mathbf{w}||_{0,q'} ||\mathbf{v}||_{0,r'}$$

For (74)(a), pick $p = 3$, $q = 2$, $r = 6$ and $p' = 3$, $q' = 2$, $r' = 6$ and apply (63)(a)(c). For (74)(b), pick $p = 2$, $q = 3$, $r = 6$ and $p' = 2$, $q' = 2$, $r' = \infty$ and apply (63)(c), and (64)(a)(b). Next apply Hölder's inequality (62) to (72) to get

$$|c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq ||\mathbf{u}||_{0,p}||\nabla\mathbf{v}||_{0,q}||\mathbf{w}||_{0,r} + \frac{1}{2}||\nabla\cdot\mathbf{u}||_{0,p'}||\mathbf{v}||_{0,q'}||\mathbf{w}||_{0,r'}$$

For (75)(a), pick $p = 6$, $q = 2$, $r = 3$ and $p' = 2$, $q' = 6$, $r' = 3$ and apply (63)(a)(c). For (75)(b), pick $p = 6$, $q = 3$, $r = 2$ and $p' = 2$, $q' = \infty$, $r' = 2$ and apply (63)(c) and (64)(a)(b). □

3.4. Elliptic and Stokes projections. We define the elliptic and Stokes projections for approximating H^1-functions in $X_{h,\cdot}$. Estimate (76) is necessary since the discrete pressure is eliminated from the error analysis for velocity by testing with functions in the discretely divergence free space V_h (proved e.g. in [11], see intermediate estimate (1.16) in Theorem II.1.1).

Lemma 3.21. *Suppose that the FE space satisfies Assumption 2.4. Then, for any* $\mathbf{u} \in V$*, there exists a constant* $0 < C < \infty$ *depending on (20) so that*

$$\inf_{\mathbf{v}_h\in V_h} |\mathbf{u} - \mathbf{v}_h|_1 \leq C \inf_{\mathbf{w}_h\in X_h} |\mathbf{u} - \mathbf{w}_h|_1. \tag{76}$$

Define the elliptic projection P_e: fix $\mathbf{u} \in V$ so that

$$P_e : V \to V_h, \qquad (\nabla(\mathbf{u} - P_e(\mathbf{u})), \nabla\mathbf{v}) = 0, \qquad \forall\mathbf{v} \in V_h. \tag{77}$$

We present an error estimate for P_e in H^1_0 below as well as L^2, $W^{-1,2}$ for a sufficiently regular domain Ω. Let $|\cdot|_0 = ||\cdot||$ and $|\cdot|_{-1} = ||\cdot||_{-1}$ throughout.

Lemma 3.22. *Fix* $\mathbf{u} \in H^1_0$*. Suppose that FE space satisfies Assumption 2.4. Then* P_e *given by (77) is well-defined and satisfies*

$$||\mathbf{u} - P_e(\mathbf{u})||_{-m} \leq Ch^{m+1} \inf_{\mathbf{v}_h\in X_h} |\mathbf{u} - \mathbf{v}_h|_1 \tag{78}$$

for $m = -1$*. Suppose further that Assumption 2.3 is satisfied. Then (78) also holds for* $m = 0$, 1.

Proof. For $m = -1$, apply Céa's Lemma to get $|\mathbf{u} - \tilde{\mathbf{v}}_h|_1 \leq 2\inf_{\mathbf{v}_h\in V_h} |\mathbf{u} - \mathbf{v}_h|_1$. To recover infimum over all $\mathbf{v}_h \in X_h$, apply estimate (76). To recover estimate for $m = 0$ and 1, follow the procedure in [11] (e.g. Theorem II.1.9). □

Define the Stokes projection: let $P_s : (V, L^2_0) \to (V_h, Q_h)$ so that $(\tilde{\mathbf{v}}_h, \tilde{q}_h) := P_s(\mathbf{u}, p)$ satisfies

$$\begin{aligned} &\forall\mathbf{v} \in X_h, \quad \nu(\nabla(\mathbf{u} - \tilde{\mathbf{v}}_h), \nabla\mathbf{v}) - (p - \tilde{q}_h, \nabla\cdot\mathbf{v}) = 0 \\ &\forall q \in Q_h, \quad (q, \nabla\cdot\tilde{\mathbf{v}}_h) = 0. \end{aligned} \tag{79}$$

Write $\tilde{\mathbf{v}}_h := P^{(1)}_s(\mathbf{u}, p)$. We prove an error estimate for $P^{(1)}_s$ in H^1_0 below as well as L^2, $W^{-1,2}$ for a sufficiently regular domain Ω.

Lemma 3.23. *Fix* $\mathbf{u} \in H^1_0$, $p \in L^2_0$*. Suppose that FE space satisfies 2.4. Then* P_s *given by (79) is well-defined so that*

$$||\mathbf{u} - P^{(1)}_s(\mathbf{u})||_{-m} \leq Ch^{m+1}(\nu^{-1} \inf_{q_h\in Q_h} ||p - q_h|| + \inf_{\mathbf{v}_h\in X_h} |\mathbf{u} - \mathbf{v}_h|_1) \tag{80}$$

for $m = -1$*. Suppose further that Assumption 2.3 is satisfied. Then (80) also holds for* $m = 0$, 1.

Proof. For $m = -1$, a similar (but simpler) proof for the error estimate of the (nonlinear) NSE in Section 4.1 proves $|\mathbf{u}-\tilde{\mathbf{v}}_h|_1 \le 2\inf_{\mathbf{v}_h\in V_h}|\mathbf{u}-\mathbf{v}_h|_1+\nu^{-1}\inf_{q_h\in Q_h}||p-q_h||$. Take the infimum over all $\mathbf{v}_h \in V_h$ and all $\tilde{q}_h \in Q_h$ and apply (76) to prove (80) for $m = -1$. To recover estimate for $m = 0$, 1, follow the procedure in [11] (e.g. Theorem II.1.9). $\square$

4. Proof of CNLE convergence estimates

For the numerical analysis that follows, we require, as is assuredly a fact for many pertinent flows, strong solutions of (30), (31), (32) that satisfy $\mathbf{u} \in C^0(V)$, $\partial_t\mathbf{u} \in C^0([t^{n_0}, T]; W^{-1,2})$, and $p \in C^0([t^{n_0}, T]; L^2_0)$ and hence, for each $n = n_0$, $n_0 + 1, \ldots, N - 1$,

$$
\begin{aligned}
&((\partial_t\mathbf{u})^{n+1/2},\mathbf{v}) + \frac{1}{2}(\mathbf{u}^{n+1}\cdot\nabla\mathbf{u}^{n+1},\mathbf{v}) + \frac{1}{2}(\mathbf{u}^{n}\cdot\nabla\mathbf{u}^{n},\mathbf{v}) \\
&\quad + \nu(\nabla\mathbf{u}^{n+1/2},\nabla\mathbf{v}) - (p^{n+1/2},\nabla\cdot\mathbf{v}) = (\mathbf{f}^{n+1/2},\mathbf{v}), \quad \forall\mathbf{v}\in H^1_0 \qquad (81)\\
&\nabla\cdot\mathbf{u} = 0 \qquad (82)\\
&\mathbf{u}(\cdot,0) = \mathbf{u}^0. \qquad (83)
\end{aligned}
$$

The consistency error for the time-discretization is given by, for any $\mathbf{v} \in H^1_0$,

$$R^{n+1}(\mathbf{v}) : = (\partial^{n+1}_{\Delta t}\mathbf{u} - (\partial_t\mathbf{u})^{n+1/2},\mathbf{v}) + \hat{R}^{n+1}(\mathbf{v}) \qquad (84)$$

$$\hat{R}^{n+1}(\mathbf{v}) : = c_h(\xi^n(\mathbf{u}),\mathbf{u}^{n+1/2},\mathbf{v}) - \frac{1}{2}(\mathbf{u}^{n+1}\cdot\nabla\mathbf{u}^{n+1},\mathbf{v}) - \frac{1}{2}(\mathbf{u}^{n}\cdot\nabla\mathbf{u}^{n},\mathbf{v}). \qquad (85)$$

Recall (73). Then (81), (84) give

$$
\begin{aligned}
&(\partial^{n+1}_{\Delta t}\mathbf{u},\mathbf{v}) + c_h(\xi^n(\mathbf{u}),\mathbf{u}^{n+1/2},\mathbf{v}) - (p^{n+1/2},\nabla\cdot\mathbf{v})\\
&\quad + \nu(\nabla\mathbf{u}^{n+1/2},\nabla\mathbf{v}) = (\mathbf{f}^{n+1/2},\mathbf{v}) + R^{n+1}(\mathbf{v}), \quad \forall\mathbf{v}\in H^1_0. \qquad (86)
\end{aligned}
$$

Decompose the velocity error, for some $\tilde{\mathbf{v}}^n_h \in V_h$,

$$\left\{\begin{array}{rcl} \mathbf{e}^n &=& \mathbf{u}^n_h - \mathbf{u}^n = \mathbf{U}^n_h - \eta^n\\ \mathbf{U}^n_h &=& \mathbf{u}^n_h - \tilde{\mathbf{v}}^n_h \in V_h\\ \eta^n &=& \mathbf{u}^n - \tilde{\mathbf{v}}^n_h. \end{array}\right. \qquad (87)$$

Write

$$R^{n+1}_h(\mathbf{v}) := c_h(\xi^n(\mathbf{u}_h),\mathbf{u}^{n+1/2}_h,\mathbf{v}) - c_h(\xi^n(\mathbf{u}),\mathbf{u}^{n+1/2},\mathbf{v}). \qquad (88)$$

Fix $\tilde{q}^n_h \in Q_h$. Note that $(p_h, \nabla\cdot\mathbf{v}) = 0$ for any $\mathbf{v} \in V_h$. Subtract (86) from (17) to get the error equation, for any $\mathbf{v} \in V_h$,

$$
\begin{aligned}
&(\partial^{n+1}_{\Delta t}\mathbf{U}_h,\mathbf{v}) + \nu(\nabla\mathbf{U}^{n+1/2}_h,\nabla\mathbf{v}) = -R^{n+1}(\mathbf{v}) - R^{n+1}_h(\mathbf{v})\\
&\quad + (\partial^{n+1}_{\Delta t}\eta,\mathbf{v}) + \nu(\nabla\eta^{n+1/2},\nabla\mathbf{v}) - (p^{n+1/2} - \tilde{q}^{n+1/2}_h,\nabla\cdot\mathbf{v}). \qquad (89)
\end{aligned}
$$

Specifying different $\mathbf{v}$ in (89) results in error estimates in different norms:

$$\begin{array}{lll} \mathbf{v} = \mathbf{U}^{n+1/2}_h & \in V_h & \Rightarrow \text{ Theorems 3.1, 3.5}\\ \mathbf{v} = \partial^{n+1}_{\Delta t}\mathbf{U}_h & \in V_h & \Rightarrow \text{ Theorems 3.8, 3.10} \end{array}.$$

4.1. Proof of $\mathbf{u}_h \to \mathbf{u}$ in $l^2(H^1) \cap l^\infty(L^2)$.

Proof. (Theorems 3.1, 3.5)

Fix $n = n_0,\ n_0+1, \dots, N-1$. Set $\tilde{\mathbf{v}}_h^n = P_e(\mathbf{u}^n)$ defined by (77) in (87). Set $\mathbf{v} = \mathbf{U}_h^{n+1/2} \in V_h$ in (89). Fix $\tilde{q}_h^{n+1} \in Q_h$ so that $(\tilde{q}_h^{n+1}, \nabla \cdot \mathbf{U}_h^{n+1}) = 0$. Then

$$\frac{1}{2\Delta t}(||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2) + \nu |\mathbf{U}_h^{n+1/2}|_1^2 = -R^{n+1}(\mathbf{U}_h^{n+1/2})$$
$$- R_h^{n+1}(\mathbf{U}_h^{n+1/2}) + (\partial_{\Delta t}^{n+1}\eta, \mathbf{U}_h^{n+1/2}) - (p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla \cdot \mathbf{U}_h^{n+1/2}). \quad (90)$$

Apply the duality estimate on $W^{-1,2} \times H_0^1$ and Cauchy-Schwarz (62) to get

$$(\partial_{\Delta t}^{n+1}\eta, \mathbf{U}_h^{n+1/2}) \le ||\partial_{\Delta t}^{n+1}\eta||_{-1} |\mathbf{U}_h^{n+1/2}|_1 \quad (91)$$
$$(p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla \cdot \mathbf{U}_h^{n+1/2}) \le \sqrt{d}||p^{n+1/2} - \tilde{q}_h^{n+1/2}||\ |\mathbf{U}_h^{n+1/2}|_1. \quad (92)$$

The remaining terms in (90) are bounded in the next 2 lemmas.

Lemma 4.1. *Suppose that the FE space satisfies Assumption 2.4. Suppose further that $\mathbf{u}(\cdot, t) \in V$ for any $t \in [0, T]$ and $\mathbf{u}(\cdot, t) \in H^2$ for any $t \in [t^{n_0}, T]$. Then, for each $n = n_0, n_0+1, \dots, N-1$,*

$$|R_h^{n+1}(\mathbf{U}_h^{n+1/2})| \le C(||\mathbf{u}^{n+1/2}||_2 \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i}|| + \dots$$
$$\dots + \sum_{i=0}^{n_0} |\mathbf{u}^{n-i}|_1 |\eta^{n+1/2}|_1 + |\mathbf{u}^{n+1/2}|_1 \sum_{i=0}^{n_0} |\eta^{n-i}|_1)|\mathbf{U}^{n+1/2}|_1 \quad (93)$$

Proof. Add/subtract $c_h(\xi^n(\mathbf{u}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})$ and apply (87), (65) to (88) to get

$$R_h^{n+1}(\mathbf{U}_h^{n+1/2}) = c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})$$
$$- c_h(\xi^n(\mathbf{u}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) - c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})$$

Apply $||\sum_{i=0}^{n_0} a_i \mathbf{v}_i|| \le \sum_{i=0}^{n_0} |a_i|\ ||\mathbf{v}_i||$ throughout. Absorb $|a_i|$ into C. Apply (74)(b) to get

$$|c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C||\mathbf{u}^{n+1/2}||_2 \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i}||\ |\mathbf{U}_h^{n+1/2}|_1. \quad (94)$$

Apply (74)(a) along with $\mathbf{u} \in l^\infty(H^1)$ and $\mathbf{U}_h^{n+1/2} \in H_0^1$ to get

$$|c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C|\mathbf{u}^{n+1/2}|_1 \sum_{i=0}^{n_0} |\eta^{n-i}|_1 |\mathbf{U}_h^{n+1/2}|_1. \quad (95)$$

Apply (73) along with $\xi^n(\mathbf{u}) \in V$ and $\mathbf{U}_h^{n+1/2} \in H_0^1$ to rewrite the remaining trilinear term:

$$c_h(\xi^n(\mathbf{u}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) = (\xi^n(\mathbf{u}) \cdot \nabla\eta^{n+1/2}, \mathbf{U}_h^{n+1/2})$$
$$- c_h(\xi^n(\eta), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) + c_h(\xi^n(\mathbf{U}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}).$$

Estimate (66)(a) gives

$$|(\xi^n(\mathbf{u}) \cdot \nabla\eta^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C \sum_{i=0}^{n_0} |\mathbf{u}^{n-i}|_1 |\eta^{n+1/2}|_1\ |\mathbf{U}_h^{n+1/2}|_1. \quad (96)$$

Recall that (78), (21)(a) give $|\eta|_1 \leq Ch^k||\mathbf{u}||_{k+1}$. Then (74)(a) and (78), (21)(a) with $k = 0$ gives

$$|c_h(\xi^n(\eta), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2})| \leq C \sum_{i=0}^{n_0} |\mathbf{u}^{n-i}|_1 |\eta^{n+1/2}|_1 \ |\mathbf{U}_h^{n+1/2}|_1. \tag{97}$$

Similarly (74)(a) and (78), (21)(a) with $k = 1$ and inverse estimate (22) give

$$|c_h(\xi^n(\mathbf{U}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2})| \leq Ch^{1/2}||\mathbf{u}^{n+1/2}||_2 \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i}|| \ |\mathbf{U}_h^{n+1/2}|_1. \tag{98}$$

Estimates (94), (95), (96), (97), (98) imply (93). □

Lemma 4.2. *Suppose that $\mathbf{u}(\cdot, t) \in H^1 \cap V$ for any $t \in [0, T]$ and $\partial_t \mathbf{u}(\cdot, t) \in W^{-1,2}$ for any $t \in [t^{n_0}, T]$. Then, for each $n = n_0, n_0 + 1, \ldots, N - 1$,*

$$|R^{n+1}(\mathbf{U}_h^{n+1/2})| \leq (C_1^{n+1})^{1/2} |\mathbf{U}_h^{n+1/2}|_1 \tag{99}$$

where

$$C_1^{n+1} := C||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_{-1}^2 + C \begin{cases} ||\nabla \mathbf{u}||_{l^\infty(L^2)}^2 (|\mathbf{u}^{n+1}|_1^2 + |\mathbf{u}^n|_1^2), \quad \textit{or} \\ ||\mathbf{u}||_{l^\infty(n_0,N;H^2)}^2 (||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||^2 + ||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||^2) \\ \quad \ldots + \dfrac{\Delta t^3}{t^{n+1/2}} ||\mathbf{u}||_{l^\infty(n_0,N;H^2)}^2 \displaystyle\int_{t^n}^{t^{n+1}} t \ ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||^2 dt \\ \quad \ldots + \dfrac{\Delta t^3}{t^{n+1/2}} ||\partial_t \mathbf{u}||_{l^\infty(n_0,N;L^2)}^2 \displaystyle\int_{t^n}^{t^{n+1}} t \ ||\partial_t \mathbf{u}(\cdot, t)||_2^2 dt. \end{cases} \tag{100}$$

Remark 4.3. *If $\partial_t^{(2)} \mathbf{u} \in L^2(t^{n_0}, t^N; L^2)$ and $\partial_t \mathbf{u} \in L^2(t^{n_0}, t^N; H^2)$ then (100) is replaced by*

$$\begin{aligned} C_1^{n+1} &= C||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_{-1}^2 \\ &+ C||\mathbf{u}||_{l^\infty(n,n+1;H^2)}^2 (||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||^2 + ||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||^2) \\ &+ C\Delta t^3 ||\mathbf{u}||_{l^\infty(n,n+1;H^2)}^2 \int_{t^n}^{t^{n+1}} ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||^2 dt \\ &+ C\Delta t^3 ||\partial_t \mathbf{u}||_{l^\infty(n,n+1;L^2)}^2 \int_{t^n}^{t^{n+1}} ||\partial_t \mathbf{u}(\cdot, t)||_2^2 dt. \end{aligned} \tag{101}$$

Proof. Duality estimate on $W^{-1,2} \times H_0^1$ gives

$$|(\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}, \mathbf{U}_h^{n+1/2})| \leq ||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_{-1} |\mathbf{U}_h^{n+1/2}|_1. \tag{102}$$

Taylor-expansion about $t^{n+1/2}$ with integral remainder gives

$$\begin{aligned} \frac{1}{2}(\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}) + \frac{1}{2}(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{v}) &= (\mathbf{u}(\cdot, t^{n+1/2}) \cdot \nabla \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v}) \\ &+ \frac{1}{2} \int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t) \frac{d^2}{dt^2} (\mathbf{u}(\cdot, t) \cdot \nabla \mathbf{u}(\cdot, t), \mathbf{v}) dt \\ &+ \frac{1}{2} \int_{t^n}^{t^{n+1/2}} (t - t^n) \frac{d^2}{dt^2} (\mathbf{u}(\cdot, t) \cdot \nabla \mathbf{u}(\cdot, t), \mathbf{v}) dt. \end{aligned} \tag{103}$$

Add/subtract $(\xi^n(\mathbf{u})\cdot\nabla\mathbf{u}(\cdot,t^{n+1/2}),\mathbf{v})$ and apply (103) to (85) to get

$$\begin{aligned}\hat{R}^{n+1}(\mathbf{U}_h^{n+1}) &= (\xi^n(\mathbf{u})\cdot\nabla(\mathbf{u}^{n+1/2}-\mathbf{u}(\cdot,t^{n+1/2})),\mathbf{U}_h^{n+1})\\ &+((\xi^n(\mathbf{u})-\mathbf{u}(\cdot,t^{n+1/2}))\cdot\nabla\mathbf{u}(\cdot,t^{n+1/2}),\mathbf{U}_h^{n+1})\\ &-\frac{1}{2}\int_{t^{n+1/2}}^{t^{n+1}}(t^{n+1}-t)\int(\partial_t^{(2)}\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\partial_t^{(2)}\mathbf{u}+2\partial_t\mathbf{u}\cdot\nabla\partial_t\mathbf{u})\cdot\mathbf{U}_h^{n+1}dt\\ &-\frac{1}{2}\int_{t^n}^{t^{n+1/2}}(t-t^n)\int(\partial_t^{(2)}\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\partial_t^{(2)}\mathbf{u}+2\partial_t\mathbf{u}\cdot\nabla\partial_t\mathbf{u})\cdot\mathbf{U}_h^{n+1}dt \quad (104)\end{aligned}$$

Then we majorize (85) either directly with (66)(a) to get

$$|\hat{R}^{n+1}(\mathbf{U}_h^{n+1})|\leq C||\nabla\mathbf{u}||_{l^\infty(L^2)}(|\mathbf{u}^{n+1}|_1+|\mathbf{u}^n|_1)|\mathbf{U}_h^{n+1}|_1 \quad (105)$$

or with (66)(b), (70) and Hölder's inequality (in time) applied to (104) to get

$$\begin{aligned}&|\hat{R}^{n+1}(\mathbf{U}_h^{n+1})|\leq C||\mathbf{u}||_{l^\infty(H^2)}\times\dots\\ &\quad\dots\times(||\mathbf{u}^{n+1/2}-\mathbf{u}(\cdot,t^{n+1/2})||+||\xi^n(\mathbf{u})-\mathbf{u}(\cdot,t^{n+1/2})||)|\mathbf{U}_h^{n+1}|_1\\ &\quad+\frac{C\Delta t^{3/2}}{\sqrt{t^{n+1/2}}}||\mathbf{u}||_{l^\infty([n,n+1];H^2)}(\int_{t^n}^{t^{n+1}}t\,||\partial_t^{(2)}\mathbf{u}(\cdot,t)||^2dt)^{1/2}|\mathbf{U}_h^{n+1}|_1\\ &\quad+\frac{C\Delta t^{3/2}}{\sqrt{t^{n+1/2}}}||\partial_t\mathbf{u}||_{l^\infty([n,n+1];L^2)}(\int_{t^n}^{t^{n+1}}t\,||\partial_t\mathbf{u}(\cdot,t)||_2^2dt)^{1/2}|\mathbf{U}_h^{n+1}|_1 \quad (106)\end{aligned}$$

See Appendix B for more details on the derivation. Estimates (102), (103), (105)/(106) imply (99). □

Bound each term on the RHS of (90) with (91), (92), (93), (99). Successive application of Young (61) gives

$$\begin{aligned}&||\mathbf{U}_h^{n+1}||^2-||\mathbf{U}_h^n||^2+\nu\Delta t|\mathbf{U}_h^{n+1/2}|_1^2\\ &\qquad\leq\nu^{-1}\Delta t(C_2^{n+1}+C_1^{n+1}+C||\mathbf{u}^{n+1/2}||_2^2\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||^2) \quad (107)\end{aligned}$$

where

$$\begin{aligned}C_2^{n+1}&:=C(||\nabla\mathbf{u}||^2_{l^\infty(n_0,N;L^2)}\sum_{i=0}^{n_0}|\eta^{n-i}|_1^2+\dots\\ &\dots+\sum_{i=0}^{n_0}|\mathbf{u}^{n-i}|_1^2|\eta^{n+1/2}|_1^2+||\partial_{\Delta t}^{n+1}\eta||_{-1}^2+||p^{n+1/2}-\tilde{q}_h^{n+1/2}||^2). \quad (108)\end{aligned}$$

Lemma 4.4. *Suppose that the FE space satisfies Assumption 2.4. Fix* $k\geq0$, $k*\geq0$, $s\geq-1$. *Suppose further that Assumption 2.3 is satisfied and* $\mathbf{u}\in l^\infty(H^k\cap V)\cap l^2(H^{k+1}\cap H^2)$, $\partial_t\mathbf{u}\in L^2(t^{n_0},T;H^{k*+1})\cap l^2(n_0,N;W^{-1,2})$, $p\in l^2(n_0,N;H^{s+1})$. *Then,*

$$\begin{aligned}||\mathbf{U}_h^N||^2+\nu\Delta t\sum_{n=n_0}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2&\leq K_0^2||\mathbf{e}||^2_{l^\infty(0,n_0;L^2)}+K_1^2\\ &+\Delta t\sum_{n=n_0}^{N-1}(\nu^{-1}C_1^{n+1}+C\nu^{-1}||\mathbf{U}_h^n||^2\sum_{i=0}^{i_1(n)}||\mathbf{u}^{n+i+1/2}||_2^2) \quad (109)\end{aligned}$$

where

$$K_0 := C + C\nu^{-1/2}\begin{cases} ||\mathbf{u}||_{l^2(0,2n_0;H^2)} & \textit{if } n_0 \ge 1, \\ 0 & \textit{otherwise} \end{cases}$$

$$\begin{aligned} K_1 := &(C\nu^{-1/2}||\nabla \mathbf{u}||_{l^\infty(L^2)}||\mathbf{u}||_{l^2(H^{k+1})} + K_0||\mathbf{u}||_{l^\infty(0,n_0;H^k)})h^k \\ &+ C\nu^{-1/2}||p||_{l^2(n_0,N;H^{s+1})}h^{s+1} + C\nu^{-1/2}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^{k*+1})}h^{k*+2}. \end{aligned} \tag{110}$$

Proof. Recall that (78), (21)(a) give $|\eta|_1 \le Ch^k||\mathbf{u}||_{k+1}$. Fix $k* \ge 0$. Then (78), (21)(a) along with (23) gives

$$||\partial_{\Delta t}^{n+1}\eta||_{-1}^2 \le Ch^{2k*+4}\Delta t^{-1}\int_{t^n}^{t^{n+1}} ||\partial_t \mathbf{u}(\cdot,t)||_{k*+1}^2 dt. \tag{111}$$

Estimate (21)(b) gives

$$\inf_{\tilde{q}_h \in Q_h} ||p^{n+1/2} - \tilde{q}_h|| \le Ch^{s+1}||p^{n+1/2}||_{s+1}. \tag{112}$$

Write

$$\begin{aligned} \kappa_1^{n+1} := C(&||\nabla \mathbf{u}||_{l^\infty(L^2)}\sum_{i=-1}^{n_0} ||\mathbf{u}^{n-i}||_{k+1}^2 h^{2k} + \dots \\ &\dots + \Delta t^{-1}||\partial_t \mathbf{u}||_{L^2(t^n,t^{n+1};H^{k*+1})}^2 h^{2k*+4} + ||p^{n+1/2}||_{s+1}^2 h^{2s+2}). \end{aligned} \tag{113}$$

Application of (78), (21)(a), (111), (112) to (107), (108) proves $C_2^{n+1} \le \kappa_1^{n+1}$ so that

$$\begin{aligned} &||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2 + \nu\Delta t|\mathbf{U}_h^{n+1/2}|_1^2 \\ &\quad \le \nu^{-1}\Delta t(\kappa_1^{n+1} + C_1^{n+1} + C||\mathbf{u}^{n+1/2}||_2^2 \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i}||^2). \end{aligned} \tag{114}$$

Sum from $n = n_0$ to $n = N-1$ to get

$$\begin{aligned} &||\mathbf{U}_h^N||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} |\mathbf{U}_h^{n+1/2}|_1^2 \\ &\quad \le ||\mathbf{U}_h^{n_0}||^2 + \nu^{-1}\Delta t \sum_{n=n_0}^{N-1} (\kappa_1^{n+1} + C_1^{n+1} + C||\mathbf{u}^{n+1/2}||_2^2 \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i}||^2). \end{aligned}$$

Apply the change of indices (60) to get

$$\begin{aligned} &||\mathbf{U}_h^N||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} |\mathbf{U}_h^{n+1/2}|_1^2 \le ||\mathbf{U}_h^{n_0}||^2 + \gamma_1 \\ &\quad + \nu^{-1}\Delta t \sum_{n=n_0}^{N-1} (\kappa_1^{n+1} + C_1^{n+1} + C||\mathbf{U}_h^n||^2 \sum_{i=0}^{i_1(n)} ||\mathbf{u}^{n+i+1/2}||_2^2) \end{aligned}$$

where

$$\gamma_1 := \begin{cases} C\nu^{-1}\Delta t \displaystyle\sum_{n=0}^{2n_0-1} ||\mathbf{u}^{n+1/2}||_2^2 ||\mathbf{U}_h||_{l^\infty(0,n_0-1;L^2)}^2 & \text{if } n_0 \ge 1, \\ 0 & \text{otherwise} \end{cases} \tag{115}$$

Suppose that $n_0 \geq 1$. Then the triangle inequality $||\mathbf{U}_h|| \leq ||\mathbf{e}|| + ||\eta||$ and (78), (21)(a) give

$$||\mathbf{U}_h^{n_0}||^2 + \gamma_1 \leq ||\mathbf{e}^{n_0}||^2 + C\nu^{-1}\Delta t \sum_{n=0}^{2n_0-1} ||\mathbf{u}^{n+1/2}||_2^2 ||\mathbf{e}||^2_{l^\infty(0,n_0-1;L^2)}$$
$$+ C||\mathbf{u}^{n_0}||_k^2 h^{2k} + C\nu^{-1}\Delta t \sum_{n=0}^{2n_0-1} ||\mathbf{u}^{n+1/2}||_2^2 ||\mathbf{u}||^2_{l^\infty(0,n_0-1;H^k)} h^{2k}.$$

Combine this estimate with κ_1 defined in (113) to prove (109). $\square$

Write

$$G^N := \exp(C\nu^{-1}||\mathbf{u}||^2_{l^2(n_0,N;H^2)}). \tag{116}$$

Gronwall Lemma 3.15 applied to (109) gives

$$||\mathbf{U}_h^N||^2 + \nu\Delta t \sum_{n=n_0}^{N-1} |\mathbf{U}_h^{n+1/2}|_1^2$$
$$\leq G^N(K_0^2||\mathbf{e}||^2_{l^\infty(0,n_0;L^2)} + K_1^2 + \nu^{-1}\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1}) \tag{117}$$

Application of the triangle inequality $||\mathbf{e}|| \leq ||\mathbf{U}_h|| + ||\eta||$ along with (78), (21)(a), and (117) gives

$$||\mathbf{e}^N|| + \nu^{1/2}(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2}$$
$$\leq C(||\mathbf{u}^N||_k + \nu^{1/2}||\mathbf{u}||_{l^2(n_0,N;H^{k+1})} h^k$$
$$+ G^N(K_0||\mathbf{e}||_{l^\infty(0,n_0;L^2)} + K_1 + \nu^{-1/2}(\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1})^{1/2}) \tag{118}$$

Consider 3 cases: the first with minimal regularity (boundedness - Proposition 3.1), the second for optimal convergence rate (regularity matching the FE and CN approximation degree - Theorem 3.5), and the third with then compatibility condition is not satisfied.

(Proposition 3.1): Suppose that the regularity of Lemma 4.4 is satisfied for $k = 0$, $k* = 0$, and $s = -1$. Then

$$C||\mathbf{u}^N|| + C\nu^{1/2}||\mathbf{u}||_{l^2(n_0,N;H^1)}$$
$$+ G^N(K_0||\mathbf{e}||_{l^\infty(0,n_0;L^2)} + K_1) < \infty, \quad \text{as } h,\ \Delta t \to 0 \tag{119}$$

where $K_1 = K_1(k = 0, k* = 0, s = -1)$ is defined in (110). Suppose further that $\partial_t\mathbf{u} \in L^2(t^{n_0}, T; W^{-1,2}) \cap l^2(n_0, N; W^{-1,2})$. The triangle inequality and (23) gives

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t\mathbf{u})^{n+1/2}||_{-1}^2 \leq C(||\partial_t\mathbf{u}||^2_{L^2(t^{n_0},T;W^{-1,2})} + ||\partial_t\mathbf{u}||^2_{l^2(n_0,N;W^{-1,2})}).$$

Write

$$K_2 := C\nu^{-1/2}(||\nabla\mathbf{u}||_{l^\infty(L^2)}||\nabla\mathbf{u}||_{l^2(n_0,N;L^2)} + \dots$$
$$\dots + ||\partial_t\mathbf{u}||^2_{L^2(t^{n_0},T;W^{-1,2})} + ||\partial_t\mathbf{u}||^2_{l^2(n_0,N;W^{-1,2})}).$$

Then, recalling $C_1^{n+1} > 0$ given in (100),

$$\nu^{-1/2}(\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1})^{1/2} \leq K_2 < \infty, \quad \text{as } h,\ \Delta t \to 0. \tag{120}$$

Estimate (33) follows from (118) with boundedness as h, $\Delta t \to 0$.

(Theorem 3.5): Suppose that the regularity of Lemma 4.4 is satisfied for $k > 0$, $s > -1$, $k* = k-2$ (with $k* = k-1$ when $k = 1$). Suppose that $\mathbf{u} \in l^\infty(H^2)$, $\partial_t \mathbf{u} \in l^\infty(n_0, N; H^1) \cap L^2(t^{n_0}, T; H^1)$, $\partial_t^{(2)}\mathbf{u} \in L^2(L^2)$, and $\partial_t^{(3)}\mathbf{u} \in L^2(t^{n_0}, T; W^{-1,2})$. Write

$$K_t := C\nu^{-1/2}(||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(L^2)} + \dots$$
$$\dots + ||\partial_t\mathbf{u}||_{l^\infty(n_0,N;L^2)}||\partial_t\mathbf{u}||_{L^2(t^{n_0},T;H^2)}).$$

Then, (26), (27), and (28) (by Assumption 2.5) gives

$$\nu^{-1/2}(\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1})^{1/2} \leq K_t \Delta t^2$$

Apply above to (118) to get

$$||\mathbf{e}^N|| + \nu^{1/2}(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} \leq C(||\mathbf{u}^N||_k + \nu^{1/2}||\mathbf{u}||_{l^2(n_0,N;H^{k+1})}h^k$$
$$+ G^N(K_0||\mathbf{e}||_{l^\infty(0,n_0;L^2)} + K_1 + K_t\Delta t^2).$$

Estimate (34) follows with optimal convergence rate as h, $\Delta t \to 0$ under the assumed regularity. Note that K_u and K_p in the estimate are derived from K_1 in (110).

(Theorem 3.6): Suppose now that the compatibility condition (5) is not satisfied. Suppose further that $n_0 > 0$. Starting with (100), bound C_1 via (24), (25) (28) instead of (26), (27) (29). Then we can replace K_t with $\sigma(t^{n_0})^{-1}\overline{K}_t$ in Theorem 3.5 where $\overline{K}_t$ is given in (49). Apply estimates in Assumption 2.6. Then we can replace K_u, K_p in Theorem 3.5 with $\sigma(t^1)^{-(k-1)/2}\overline{K}_u$, $\sigma(t^1)^{-(k-2)/2}\overline{K}_p$ respectively where $\overline{K}_u$, $\overline{K}_p$ are given in (47), (48).

□

4.2. Proof of $\mathbf{u}_h \to \mathbf{u}$ in $l^\infty(H^1)$, $\partial_{\Delta t}(\mathbf{u}_h - \mathbf{u}) \to 0$ in $l^2(L^2)$.

Proof. Theorems 3.8, 3.10

Fix $n = n_0,\ n_0+1, \dots, N-1$. Set $(\tilde{\mathbf{v}}_h^n, \tilde{q}_h^n) = P_s(\mathbf{u}^n, p^n)$ in (87). Set $\mathbf{v} = \Delta t^{-1}(\mathbf{U}_h^{n+1} - \mathbf{U}_h^n) \in V_h$ in (89). Then

$$||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \frac{\nu}{2\Delta t}(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2)$$
$$= -R^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) - R_h^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) + (\partial_{\Delta t}^{n+1}\eta, \partial_{\Delta t}^{n+1}\mathbf{U}_h). \tag{121}$$

Cauchy-Schwarz (62) gives

$$(\partial_{\Delta t}^{n+1}\eta, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \leq ||\partial_{\Delta t}^{n+1}\eta||\ ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. \tag{122}$$

The remaining terms in (121) are bounded in the next 2 lemmas.

Lemma 4.5. *Suppose that the FE space satisfies Assumption 2.4 and* $\mathbf{u} \in l^\infty(H^2 \cap V)$*,* $p \in l^\infty(H^1)$*. Then for each* $n = n_0, n_0+1, \ldots, N-1$*,*

$$
\begin{aligned}
|R_h^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq C(||\mathbf{u}^{n+1/2}||_2 \sum_{i=0}^{n_0} |\eta^{n-i}|_1 + \ldots \\
\ldots + (\nu^{-1}h^{1/2}||p||_{l^\infty(H^1)} + ||\mathbf{u}||_{l^\infty(H^2)})|\mathbf{e}^{n+1/2}|_1 + \ldots \\
\ldots + (h^{-1/2}|\mathbf{e}^{n+1/2}|_1 + ||\mathbf{u}^{n+1/2}||_2) \sum_{i=0}^{n_0} |\mathbf{U}_h^{n-i}|_1)||\partial_{\Delta t}^{n+1}\mathbf{U}_h||
\end{aligned}
\tag{123}
$$

Proof. Add/subtract $(\xi^n(\mathbf{u}) \cdot \nabla \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v})$ and apply (87), (65) to (88) to get

$$
\begin{aligned}
R_h^{n+1}(\mathbf{U}_h^{n+1/2}) &= c_h(\xi^n(\mathbf{e}), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\
&\quad - c_h(\xi^n(\mathbf{u}), \mathbf{e}^{n+1/2}, \mathbf{U}_h^{n+1/2}) - c_h(\xi^n(\mathbf{e}), \mathbf{e}^{n+1/2}, \mathbf{U}_h^{n+1/2})
\end{aligned}
$$

Apply $||\sum_{i=0}^{n_0} a_i \mathbf{v}_i|| \leq \sum_{i=0}^{n_0} |a_i| \, ||\mathbf{v}_i||$ throughout. Absorb $|a_i|$ into C. Decompose $\mathbf{e} = \mathbf{U}_h - \eta$ to get

$$
\begin{aligned}
&c_h(\xi^n(\mathbf{e}), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\
&\quad = c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) - c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h).
\end{aligned}
$$

Estimates (66)(c), (75)(b) with $\mathbf{u} \in l^\infty(H^2)$ give

$$
\begin{aligned}
&|(\xi^n(\mathbf{u}), \mathbf{e}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) + c_h(\xi^n(\mathbf{e}), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)| \\
&\leq C(||\mathbf{u}||_{l^\infty(H^2)}|\mathbf{e}^{n+1/2}|_1 + \sum_{i=0}^{n_0}(|\eta^{n-i}|_1 + |\mathbf{U}_h^{n-i}|_1)||\mathbf{u}^{n+1/2}||_2)||\partial_{\Delta t}^{n+1}\mathbf{U}_h||.
\end{aligned}
\tag{124}
$$

Decompose $\mathbf{e} = \mathbf{U}_h - \eta$ again to get

$$
\begin{aligned}
&c_h(\xi^n(\mathbf{e}), \mathbf{e}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\
&\quad = c_h(\xi^n(\mathbf{U}_h), \mathbf{e}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) - c_h(\xi^n(\eta), \mathbf{e}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h).
\end{aligned}
$$

Recall that (80), (21) gives $|\eta|_1 \leq Ch(\nu^{-1}||p||_1 + ||\mathbf{u}||_2)$. Then (75)(a) with inverse estimate (22) and (80), (21) gives

$$
\begin{aligned}
|c_h(\xi^n(\mathbf{e}), \mathbf{e}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq Ch^{-1/2} \sum_{i=0}^{n_0} |\mathbf{U}_h^{n-i}|_1 |\mathbf{e}^{n+1/2}|_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \\
+ Ch^{1/2} \sum_{i=0}^{n_0} (\nu^{-1}||p^{n-i}||_1 + ||\mathbf{u}^{n-i}||_2)|\mathbf{e}^{n+1/2}|_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||
\end{aligned}
\tag{125}
$$

Estimates (124), (125) prove (123). □

Lemma 4.6. *Suppose that* $\mathbf{u}(\cdot, t) \in l^\infty(H^2 \cap V)$ *and* $\partial_t \mathbf{u}(\cdot, t) \in L^2$ *for any* $t \in [t^{n_0}, T]$*. Then, for each* $n = n_0, n_0+1, \ldots, N-1$*,*

$$
|R^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq (C_1^{n+1})^{1/2} ||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \tag{126}
$$

where

$$
C_1^{n+1} := C||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||^2
$$
$$
+ C \begin{cases} ||\mathbf{u}||^2_{l^\infty(H^2)}(|\mathbf{u}^{n+1}|_1^2 + |\mathbf{u}^n|_1^2), \quad \textit{or} \\ ||\mathbf{u}||^2_{l^\infty(n,n+1;H^2)}(|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})|_1^2 + |\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})|_1^2) \\ \quad \ldots + \dfrac{\Delta t^3}{(t^{n+1/2})^2}||\mathbf{u}||^2_{l^\infty(n,n+1;H^2)} \displaystyle\int_{t^n}^{t^{n+1}} t^2 \ ||\partial_t^{(2)}\mathbf{u}(\cdot,t)||_1^2 dt \\ \quad \ldots + \dfrac{\Delta t^3}{(t^{n+1/2})^2}||\partial_t \mathbf{u}||^2_{l^\infty(n,n+1;L^2)} \displaystyle\int_{t^n}^{t^{n+1}} t^2 \ ||\partial_t \mathbf{u}(\cdot,t)||_3^2 dt. \end{cases}
\tag{127}
$$

Remark 4.7. *If $\partial_t^{(2)}\mathbf{u} \in L^2(t^{n_0}, t^N; H^1)$ and $\partial_t \mathbf{u} \in L^2(t^{n_0}, t^N; H^3)$ then (127) gives*

$$
\begin{aligned}
C_1^{n+1} &= C||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||^2 \\
&+ C||\mathbf{u}||^2_{l^\infty(n,n+1;H^2)}(|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})|_1^2 + |\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})|_1^2) \\
&+ C\Delta t^3 ||\mathbf{u}||^2_{l^\infty(n,n+1;H^2)} \int_{t^n}^{t^{n+1}} ||\partial_t^{(2)}\mathbf{u}(\cdot,t)||_1^2 dt \\
&+ C\Delta t^3 ||\partial_t \mathbf{u}||^2_{l^\infty(n,n+1;L^2)} \int_{t^n}^{t^{n+1}} ||\partial_t \mathbf{u}(\cdot,t)||_3^2 dt.
\end{aligned}
\tag{128}
$$

Proof. Application of Cauchy-Schwarz (62) gives

$$
|(\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq ||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}|| \ ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. \tag{129}
$$

Then we majorize (85) either directly with (66)(a) to get

$$
|\hat{R}^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq C||\mathbf{u}||_{l^\infty(H^2)}(|\mathbf{u}^{n+1}|_1 + |\mathbf{u}^n|_1)||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \tag{130}
$$

or with (70), (66)(a)(b)(d), and Hölder's inequality (in time) applied to (104) (with $\mathbf{U}_h^{n+1}$ replaced by $\partial_{\Delta t}^{n+1}\mathbf{U}_h$) to get

$$
\begin{aligned}
&|\hat{R}^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq C||\mathbf{u}||_{l^\infty(H^2)} \times \ldots \\
&\quad \ldots \times (||\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2}))|| + ||\nabla(\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2}))||)||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \\
&\quad + \frac{C\Delta t^{3/2}}{t^{n+1/2}}||\mathbf{u}||_{l^\infty([n,n+1];H^2)}(\int_{t^n}^{t^{n+1}} t^2 \ ||\partial_t^{(2)}\mathbf{u}(\cdot,t)||_1^2 dt)^{1/2}||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \\
&\quad + \frac{C\Delta t^{3/2}}{t^{n+1/2}}||\partial_t \mathbf{u}||_{l^\infty([n,n+1];L^2)}(\int_{t^n}^{t^{n+1}} t^2 \ ||\partial_t \mathbf{u}(\cdot,t)||_3^2 dt)^{1/2}||\partial_{\Delta t}^{n+1}\mathbf{U}_h||
\end{aligned}
\tag{131}
$$

See Appendix B for more details on the derivation. Estimates (129), (130)/(131) give (126). □

Bound each term on the RHS of (121) with (122), (123), (126). Apply Young (61) to get

$$
\begin{aligned}
&\Delta t||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2) \\
&\leq \Delta t(C_2^{n+1} + C_1^{n+1} + C(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2)\sum_{i=0}^{n_0}|\mathbf{U}_h^{n-i}|_1^2)
\end{aligned}
\tag{132}
$$

where

$$C_2^{n+1} := C(||\mathbf{u}^{n+1/2}||_2^2 \sum_{i=0}^{n_0} |\eta^{n-i}|_1^2 + ||\partial_{\Delta t}^{n+1}\eta||^2 + \dots$$
$$\dots + (\nu^{-2}h||p||^2_{l^\infty(H^1)} + ||\mathbf{u}||^2_{l^\infty(H^2)})|\mathbf{e}^{n+1/2}|_1^2). \tag{133}$$

Lemma 4.8. *Suppose that the FE space together satisfies Assumptions 2.4. Fix* $k \geq 0$, $k* \geq 0$, $s \geq -1$, $s* \geq -1$. *Suppose further that Assumption 2.3 is satisfied and* $\mathbf{u} \in (l^2 \cap l^\infty)(H^{k+1} \cap H^2)$, $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^{k*+1})$, $p \in l^\infty(H^{s+1} \cap H^1) \cap l^2(H^{s+1})$, $\partial_t p \in L^2(t^{n_0}, T; H^{s*+1})$. *Then*

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu|\mathbf{U}_h^N|_1^2 \leq F_0^2||\nabla \mathbf{e}||^2_{l^\infty(0,n_0;L^2)} + F_1^2$$
$$+ \Delta t \sum_{n=n_0}^{N-1} (C_1^{n+1} + C|\mathbf{U}_h^n|_1^2 \sum_{i=0}^{i_1(n)} (||\mathbf{u}^{n+i+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+i+1/2}|_1^2)) \tag{134}$$

where

$$F_0 := C\nu^{1/2} + C \begin{cases} (\Delta t \displaystyle\sum_{n=0}^{2n_0-1} (||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2))^{1/2} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases}$$
$$F_1 := C\nu^{-1}(||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||p||_{l^2(H^{s+1})} + F_0||p||_{l^\infty(0,n_0;H^{s+1})})h^{s+1}$$
$$+ (C||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||\mathbf{u}||_{l^2(H^{k+1})} + F_0||\mathbf{u}||_{l^\infty(0,n_0;H^{k+1})})h^k$$
$$+ C\nu^{-1}||\partial_t p||_{L^2(t^{n_0},T;H^{s*+1})}h^{s*+2} + C||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^{k*+1})}h^{k*+1}$$
$$+ C(\nu^{-1}h^{1/2}||p||_{l^\infty(H^1)} + ||\mathbf{u}||_{l^\infty(H^2)})(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2}. \tag{135}$$

Proof. Recall that (80), (21) gives $||\eta|| \leq C(\nu^{-1}h^{s+2}||p||_{s+1} + h^{k+1}||\mathbf{u}||_{k+1})$. Fix $k* \geq 0$, $s* \geq -1$. Then (80), (21) with (23) gives

$$||\partial_{\Delta t}^{n+1}\eta||^2 \leq C\Delta t^{-1} \int_{t^n}^{t^{n+1}} (\nu^{-2}||\partial_t p(\cdot,t)||^2_{s*+1}h^{2s*+4} + \dots$$
$$\dots + ||\partial_t \mathbf{u}(\cdot,t)||^2_{k*+1}h^{2k*+2})dt. \tag{136}$$

Write

$$\kappa_1^{n+1} := C(||\mathbf{u}^{n+1/2}||_2^2 \sum_{i=0}^{n_0}(\nu^{-2}||p^{n-i}||^2_{s+1}h^{2s+2} + ||\mathbf{u}^{n-i}||^2_{k+1}h^{2k}) + \dots$$
$$\dots + \Delta t^{-1}\int_{t^n}^{t^{n+1}} (\nu^{-2}||\partial_t p(\cdot,t)||^2_{s*+1}h^{2s*+4} + ||\partial_t \mathbf{u}(\cdot,t)||^2_{k*+1}h^{2k*+2})dt + \dots$$
$$\dots + (\nu^{-2}h||p||^2_{l^\infty(H^1)} + ||\mathbf{u}||^2_{l^\infty(H^2)})|\mathbf{e}^{n+1/2}|_1^2). \tag{137}$$

Application of (80), (21), (136), to (132), (133) proves $C_2^{n+1} \leq \kappa_1^{n+1}$ so that

$$\Delta t||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2)$$
$$\leq \Delta t(\kappa_1^{n+1} + C_1^{n+1} + C(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2)\sum_{i=0}^{n_0}|\mathbf{U}_h^{n-i}|_1^2). \tag{138}$$

Sum from $n = n_0$ to $n = N - 1$ in (138). Apply the change of indices (60), group terms, and simplify to get

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu|\mathbf{U}_h^N|_1^2 \leq \nu|\mathbf{U}_h^{n_0}|_1^2 + \gamma_1$$
$$+ \Delta t \sum_{n=n_0}^{N-1} (\kappa_1^{n+1} + C_1^{n+1} + C|\mathbf{U}_h^n|_1^2 \sum_{i=0}^{i_1(n)} (||\mathbf{u}^{n+i+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+i+1/2}|_1^2)) \quad (139)$$

where

$$\gamma_1 := C \begin{cases} \Delta t \sum_{n=0}^{2n_0-1} (||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2) \times \dots & \\ \quad \dots \times ||\nabla \mathbf{U}_h||_{l^\infty(0,n_0-1;L^2)}^2 & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} . \quad (140)$$

Suppose that $n_0 \geq 1$. The triangle inequality $|\mathbf{U}_h|_1 \leq |\mathbf{e}|_1 + |\eta|_1$ and (80), (21) give

$$\nu|\mathbf{U}_h^{n_0}|_1^2 + \gamma_1 \leq \nu|\mathbf{e}^{n_0}|_1^2 + C\Delta t \sum_{n=0}^{2n_0-1} (||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2)||\nabla \mathbf{e}||_{l^\infty(0,n_0-1;L^2)}^2$$
$$+ C\nu(\nu^{-2}||p^{n_0}||_{s+1}^2 h^{2s+2} + ||\mathbf{u}^{n_0}||_{k+1}^2 h^{2k})$$
$$+ C\Delta t \sum_{n=0}^{2n_0-1} (||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2) \times \dots$$
$$\dots \times (\nu^{-2}||p||_{l^\infty(0,n_0-1;H^{s+1})}^2 h^{2s+2} + ||\mathbf{u}||_{l^\infty(0,n_0-1;H^{k+1})}^2 h^{2k}). \quad (141)$$

Combine this result with κ_1 defined in (137) to prove (134). □

Write

$$G^N := \exp(C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} (||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2)). \quad (142)$$

The Gronwall Lemma 3.15 applied to (134) gives

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu|\mathbf{U}_h^N|_1^2$$
$$\leq G^N(F_0^2||\nabla \mathbf{e}_h||_{l^\infty(0,n_0;L^2)}^2 + F_1^2 + \Delta t \sum_{n=n_0}^{N-1} C_1^{n+1}). \quad (143)$$

Application of the triangle inequality $||\mathbf{e}|| \leq ||\mathbf{U}_h|| + ||\eta||$ and (143), (80), (21), (136) give

$$(\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{e}||^2)^{1/2} + \nu^{1/2}|\mathbf{e}^N|_1$$
$$\leq C(\nu^{-1}||\partial_t p||_{L^2(t^{n_0},T;H^{s*+1})} h^{s*+2} + ||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^{k*+1})} h^{k*+1} + \dots$$
$$\dots + \nu^{-1/2}||p^N||_{s+1} h^{s+1} + \nu^{1/2}||\mathbf{u}^N||_{k+1} h^k)$$
$$+ G^N(F_0||\nabla \mathbf{e}_h||_{l^\infty(0,n_0;L^2)} + F_1 + (\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1})^{1/2}). \quad (144)$$

Consider 3 cases: first with minimal regularity (boundedness - Theorem 3.8), second for optimal convergence rate (regularity to match FE and CN approximation degree - Theorem 3.10), and the third with then compatibility condition is not satisfied.

(Theorem 3.8): Suppose that the regularity of Lemma 4.8 is satisfied for $k = 0$, $k* = 0$, and $s = -1$, $s* = -1$. Then

$$C(\nu^{-1}||\partial_t p||_{L^2(t^{n_0},T;H^{s*+1})}h^{s*+2} + ||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^{k*+1})}h^{k*+1} + \dots$$
$$\dots + \nu^{-1/2}||p^N||_{s+1}h^{s+1} + \nu^{1/2}||\mathbf{u}^N||_{k+1}h^k)$$
$$+ G^N(F_0||\nabla \mathbf{e}_h||_{l^\infty(0,n_0;L^2)} + F_1) < \infty, \quad \text{as } h,\ \Delta t \to 0 \tag{145}$$

as long as

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0$$

where $F_1 = F_1(k = 0, k* = 0, s = -1, s* = -1)$ in (135). Suppose that $\partial_t \mathbf{u} \in l^2(n_0, N; L^2) \cap L^2(t^{n_0}, T; L^2)$. The triangle inequality and (23) gives

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||^2 \leq C(||\partial_t \mathbf{u}||^2_{L^2(t^{n_0},T;L^2)} + ||\partial_t \mathbf{u}||^2_{l^2(n_0,N;L^2)}).$$

Write

$$F_2 := C(||\partial_t \mathbf{u}||^2_{L^2(t^{n_0},T;L^2)} + ||\partial_t \mathbf{u}||^2_{l^2(n_0,N;L^2)} + ||\mathbf{u}||^2_{l^\infty(H^2)}(|\mathbf{u}^{n+1}|_1^2 + |\mathbf{u}^n|_1^2)).$$

Then

$$\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1} \leq F_2 < \infty, \quad \text{as } h,\ \Delta t \to 0. \tag{146}$$

Estimate (36) follows from (144) with boundedness as h, $\Delta t \to 0$ under the assumed regularity.

(Theorem 3.10): Suppose that the regularity of Lemma 4.8 is satisfied for $k > 0$, $k* \geq 0$, $s > -1$, and $s* \geq -1$. Then as long as $\Delta t \leq Mh^{1/4}$ (for any arbitrary $M > 0$, i.e. no ν-dependence), the result of Theorem 3.5 ensures

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 \leq C(h^{2k-1} + h^{2s+1} + h^{-1}\Delta t^4) < \infty$$

as h, $\Delta t \to 0$. Suppose that $\mathbf{u} \in l^\infty(H^2)$, $\partial_t \mathbf{u} \in l^\infty(n_0, N; L^2) \cap L^2(t^{n_0}, T; H^3)$, $\partial_t^{(2)}\mathbf{u} \in L^2(t^{n_0}, T; H^1)$, and $\partial_t^{(3)}\mathbf{u} \in L^2(t^{n_0}, T; L^2)$. Write

$$F_t := C(||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0},T;L^2)} + \dots$$
$$+ ||\mathbf{u}||_{l^\infty(n_0,N;H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0},T;H^1)} + ||\partial_t \mathbf{u}||_{l^\infty(n_0,N;L^2)}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^3)}).$$

Then, (26), (27), and (29) (by Assumption 2.5) gives

$$(\Delta t \sum_{n=n_0}^{N-1} C_1^{n+1})^{1/2} \leq F_t \Delta t^2$$

Apply the above to (144) so that estimate (38) follows with optimal convergence rate as h, $\Delta t \to 0$ under the assumed regularity. Note that F_u and F_p in the estimate are derived from F_1 in (135). □

(Theorem 3.11): Suppose now that the compatibility condition (5) is not satisfied. Suppose further that $n_0 > 0$. Starting with (127), bound C_1 via (24), (25) (28) instead of (26), (27) (29). Then we can replace F_t in Theorem 3.10 with $\sigma(t^{n_0})^{-3/2}\overline{F}_t$ where $\overline{F}_t$ is given in (59). Apply estimates in Assumption 2.6. Then we can replace F_u, F_p in Theorem 3.10 with $\sigma(t^1)^{-(k-1)/2}\overline{F}_u$, $\sigma(t^1)^{-(k-1)/2}\overline{F}_p$ respectively where $\overline{F}_u$, $\overline{F}_p$ are given in (57), (58).

4.3. Proof of $p_h \to p$ in $l^2(L^2)$.

Proof. Fix $n = n_0, n_0+1, \ldots, N-1$. Write

$$R_h^{n+1}(\mathbf{v}) := c_h(\xi^n(\mathbf{e}), \mathbf{u}^{n+1/2}, \mathbf{v}) + (\xi^n(\mathbf{u}) \cdot \nabla \mathbf{e}^{n+1/2}, \mathbf{v}) - c_h(\xi^n(\mathbf{e}) \cdot \nabla \mathbf{e}^{n+1/2}, \mathbf{v}) \tag{147}$$

so that $R_h^{n+1}(\mathbf{v}) = c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}) - c_h(\xi^n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{v})$. Let $\tilde{q}_h \in Q_h$ be the L^2-projection of p. Solve for pressure in (89) to get, for any $\mathbf{v} \in X_h$,

$$\begin{aligned}(\tilde{q}_h^{n+1/2} - p_h^{n+1/2}, \nabla \cdot \mathbf{v}) &= (\tilde{q}_h^{n+1/2} - p^{n+1/2}, \nabla \cdot \mathbf{v}) + (\partial_{\Delta t}^{n+1}\mathbf{e}, \mathbf{v}) \\ &+ \nu(\nabla \mathbf{e}^{n+1/2}, \nabla \mathbf{v}) - R^{n+1}(\mathbf{v}) - R_h^{n+1}(\mathbf{v}). \end{aligned} \tag{148}$$

Application of Hölder's inequality (62) and the duality estimate on $W^{-1,2} \times H_0^1$ gives

$$\begin{aligned}\frac{|(\tilde{q}_h^{n+1/2} - p_h^{n+1/2}, \nabla \cdot \mathbf{v})|}{|\mathbf{v}|_1} &\le \sqrt{d}||\tilde{q}_h^{n+1/2} - p^{n+1/2}|| \\ &+ ||\partial_{\Delta t}^{n+1}\mathbf{e}||_{-1} + \nu|\mathbf{e}^{n+1/2}|_1 + \frac{1}{|\mathbf{v}|_1}|R^{n+1}(\mathbf{v}) + R_h^{n+1}(\mathbf{v})|. \end{aligned} \tag{149}$$

Supposing that $\mathbf{u} \in l^\infty(H^1)$, apply (74)(a) to majorize each term in (149) to get

$$\begin{aligned}|R_h^{n+1}(\mathbf{v})| &\le C((||\xi^n(\mathbf{e})||\ |\xi^n(\mathbf{e})|_1)^{1/2}|\mathbf{u}^{n+1/2}|_1 + \ldots \\ &\ldots + (||\xi^n(\mathbf{u})||\ |\xi^n(\mathbf{u})|_1)^{1/2}|\mathbf{e}^{n+1/2}|_1 + (||\xi^n(\mathbf{e})||\ |\xi^n(\mathbf{e})|_1)^{1/2}|\mathbf{e}^{n+1/2}|_1)|\mathbf{v}|_1 \end{aligned} \tag{150}$$

Apply (99) to get

$$R^{n+1}(\mathbf{v}) \le (C_1^{n+1})^{1/2}|\mathbf{v}|_1 \tag{151}$$

where $C_1 > 0$ is given in (100). Apply estimates (150), (151) to (149). Apply the discrete inf-sup condition (20) (by Assumption 2.4) to get

$$\begin{aligned}||p_h^{n+1/2} - \tilde{q}_h^{n+1/2}|| &\le C(||p^{n+1/2}||_{s+1}h^{s+1} + ||\partial_{\Delta t}^{n+1}\mathbf{e}||_{-1} + \nu|\mathbf{e}^{n+1/2}|_1 + \ldots \\ &\ldots + (||\xi^n(\mathbf{e})||\ |\xi^n(\mathbf{e})|_1)^{1/2}|\mathbf{u}^{n+1/2}|_1 + (||\xi^n(\mathbf{u})||\ |\xi^n(\mathbf{u})|_1)^{1/2}|\mathbf{e}^{n+1/2}|_1 + \ldots \\ &\ldots + (||\xi^n(\mathbf{e})||\ |\xi^n(\mathbf{e})|_1)^{1/2}|\mathbf{e}^{n+1/2}|_1 + (C_1^{n+1})^{1/2}). \end{aligned} \tag{152}$$

Apply the triangle inequality $||p^{n+1/2} - p_h^{n+1/2}|| \le ||p^{n+1/2} - \tilde{q}_h^{n+1/2}|| + ||p_h^{n+1/2} - \tilde{q}_h^{n+1/2}||$ and (152), (21)(b). Square each side of (152), multiply by Δt, and sum from $n = n_0$ to $N-1$ to get

$$\begin{aligned}
&\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2} - p_h^{n+1/2}||^2 \\
&\leq C(\Delta t \sum_{n=n_0}^{N-1} ||p^{n+1/2}||_{s+1}^2 h^{2s+2} + \Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{e}||^2 + \ldots \\
&\quad \ldots + \nu \Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 + ||\mathbf{e}||_{l^\infty(L^2)} ||\nabla \mathbf{e}||_{l^\infty(L^2)} \Delta t \sum_{n=n_0}^{N-1} |\mathbf{u}^{n+1/2}|_1^2 + \ldots \\
&\quad \ldots + ||\mathbf{u}||_{l^\infty(L^2)} ||\nabla \mathbf{u}||_{l^\infty(L^2)} \Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 + \ldots \\
&\quad \ldots + ||\mathbf{e}||_{l^\infty(L^2)} ||\nabla \mathbf{e}||_{l^\infty(L^2)} \Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 + \Delta t \sum_{n=n_0}^{N-1} C_1^{n+1}). \qquad (153)
\end{aligned}$$

For estimate (39): bound $\Delta t \sum_n C_1^{n+1} \leq F_2$ through (146), Theorem 3.8. Bound terms $\Delta t \sum_n |\mathbf{e}^{n+1/2}|_1^2$ and $||\mathbf{e}||_{l^\infty(L^2)}$ via Proposition 3.1 in (153). Bound terms $\Delta t \sum_n ||\partial_{\Delta t}^{n+1}\mathbf{e}||^2$ and $||\nabla \mathbf{e}||_{l^\infty(L^2)}$ via Theorem 3.8 in (153). Then $\mathbf{u} \in (l^2 \cap l^\infty)(H^1)$ proves (39).

For estimate (40): bound $\Delta t \sum_n C_1^{n+1} \leq F_t$ through (**??**), Theorem 3.10. Bound terms $\Delta t \sum_n |\mathbf{e}^{n+1/2}|_1^2$ and $||\mathbf{e}||_{l^\infty(L^2)}$ via Theorem 3.5 in (153). Bound terms $\Delta t \sum_n ||\partial_{\Delta t}^{n+1}\mathbf{e}||^2$ and $||\nabla \mathbf{e}||_{l^\infty(L^2)}$ via Theorem 3.10 in (153). Then $\mathbf{u} \in (l^2 \cap l^\infty)(H^1)$ proves (40). □

5. Conclusions

The analysis in this report was performed for a linearized, fully implicit Crank-Nicolson/finite element method (CNLE) for approximating the NSE. Our analysis includes the general case of arbitrary arbitrary-order extrapolations:

$$\mathbf{u} \cdot \nabla \mathbf{u} \approx \xi^n(\mathbf{u}) \cdot \nabla \frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}, \quad \xi^n(\mathbf{u}) = a_0 \mathbf{u}^{n-1} + a_1 \mathbf{u}^{n-2} + \ldots + a_{n_0} \mathbf{u}^{n-n_0}.$$

We proved a long-outstanding problem: i.e. CNLE converges in the energy norm *without any* Δt*-restriction*. We also proved that the approximating velocity and corresponding discrete time-derivative both converge optimally in $l^\infty(H^1)$ and $l^2(L^2)$ respectively under the mild Δt-restriction $\Delta t \leq M h^{1/4}$ for any arbitrary $M > 0$ (e.g. no ν-dependence). Convergence in these norms is required to derive convergence rates for pressure and drag/lift forces the fluid exerts on embedded obstacles. We prove convergence of pressure in $l^2(L^2)$ under similar conditions.

The full CN method is believed to be more accurate than CNLE. However, the accuracy of CNLE is easily improvable by increasing the order of extrapolation. Moreover, CNLE methods are linearly implicit (simple to implement and fast to solve). The additional guarantee that CNLE approximations converge *unconditionally* is another important property not proved for full CN methods. Consequently, CNLE methods are of great interest in practical computations in which speed, robustness, ease of implementation, and accuracy are required. We are currently analyzing a *new* extrapolation $\xi^n(\mathbf{u}) = 2\mathbf{u}^{n-1/2} - \mathbf{u}^{n-3/2}$ that avoids Δt-restriction for convergence completely even in $l^\infty(H^1)$ (and hence, corresponding estimates for drag/lift and pressure all without Δt-restrictions)! It is an important open question whether the Δt-restriction for convergence in the energy-norm for fully nonlinear CN and convergence in $l^\infty(H^1)$ for CNLE is strictly necessary (current methods of proof *fail*). It is equally important to formally compare the closely related family of nonlinear and linear CN-variants in practice. Accordingly, we are currently initiating a comparative study of CNLE (e.g. for linear and quadratic extrapolation as well as the new extrapolation of average velocities) against fully nonlinear CN methods and other CN-variants (e.g. Adams-Bashforth linearizations) to determine a baseline for the robustness and accuracy of CNLE.

Appendix A. Derivation of Condition (1)

In this section, we derive (1) for CNFE. Write

$$\kappa^{n+1} := \begin{cases} C\nu^{-\frac{3-2r}{1+2r}} h^{-\frac{4r}{1+2r}} |\mathbf{u}^{n+1}|_1^{\frac{4}{1+2r}}, & 0 \le r \le 3/2, \quad \mathbf{u} \in L^{\frac{4}{1+2r}}(H^1) \\ C\nu^{-\frac{1-r}{1+r}} h^{-\frac{2r}{1+r}} ||\mathbf{u}^{n+1}||_2^{\frac{2}{1+r}} & 0 \le r \le 1, \quad \mathbf{u} \in L^{\frac{2}{1+r}}(H^2) \end{cases} . \tag{154}$$

Theorem A.1. *Let $\xi^n(\mathbf{u}) = \mathbf{u}^{n+1/2}$. Fix $k > 0$, $s > -1$. Suppose that the FE space satisfies Assumption 2.4. Suppose further that Assumptions 2.3, 2.5, and 3.4 (with $n_0 := 0$) are satisfied along with $\partial_t \mathbf{f} \in C^0(W^{-1,2})$, $\mathbf{u} \in l^2(H^{k+1}) \cap C^0(H^k \cap V)$, $\partial_t \mathbf{u} \in C^0(L^3)$, $\partial_t^{(2)} \mathbf{u} \in C^0(W^{-1,2})$, $p \in l^2(H^{s+1})$, $\partial_t p \in C^0(L_0^2)$ are satisfied. If*

$$\Delta t \kappa^{n+1/2} < 1, \quad \forall n = 0, 1, \ldots, N-1 \tag{155}$$

then

$$\begin{aligned} ||\mathbf{e}||_{l^\infty(1,N;L^2)} + \nu^{1/2}(\Delta t \sum_{n=0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} &\le G^N K_p h^{s+1} \\ + (C||\mathbf{u}^N||_k + C\nu^{1/2}(\Delta t \sum_{n=0}^{N-1} ||\mathbf{u}^{n+1/2}||_{k+1}^2)^{1/2} &+ G^N K_u) h^k + G^N K_t \Delta t^2 \end{aligned} \tag{156}$$

where $G^N := \exp(\Delta t \sum_{n=0}^{N-1} \kappa^{n+1/2})$. The constants K_u, K_p, $K_t > 0$ are given in (44), (45), (46) respectively (with $n_0 := 0$) and remain bounded as h, $\Delta t \to 0$.

Remark A.2. *The time-step restriction (155) from the discrete Gronwall Lemma 3.14, exactly leads to condition (1). For example,*

$r = 0$*:*

$$\Delta t \le C \begin{cases} \nu^3 |\mathbf{u}^{n+1}|_1^{-4}, & \text{when } \mathbf{u} \in L^4(H^1) \\ \nu ||\mathbf{u}^{n+1}||_2^{-2}, & \text{when } \mathbf{u} \in L^2(H^2) \end{cases} \tag{157}$$

$r = 3/2,\ 1$ *respectively:*

$$\Delta t \le C \begin{cases} h^{\frac{3}{2}} |\mathbf{u}^{n+1}|_1^{-1}, & \text{when } \mathbf{u} \in L^1(H^1) \\ h ||\mathbf{u}^{n+1}||_2^{-1}, & \text{when } \mathbf{u} \in L^1(H^2) \end{cases} \tag{158}$$

$r = 1/4,\ 1/2$ *respectively:*

$$\Delta t \le C \begin{cases} \nu^{\frac{5}{3}} h^{\frac{2}{3}} |\mathbf{u}^{n+1}|_1^{-\frac{8}{3}}, & \text{when } \mathbf{u} \in L^{\frac{8}{3}}(H^1) \\ \nu^{\frac{1}{3}} h^{\frac{2}{3}} ||\mathbf{u}^{n+1}||_2^{-\frac{4}{3}} & \text{when } \mathbf{u} \in L^{\frac{4}{3}}(H^2) \end{cases} . \tag{159}$$

CNFE is reported to converge with $\Delta t \le F$ in [20] (Theorem 4.1 and following remarks) where $F > 0$ depends on problem data including ν, but not necessarily h. It *is not clear, however, whether (157), (158), (159) is the best choice for* $\Delta t < F(h,\nu)$ *since we do not have a priori estimates for* $\mathbf{u}$ *beyond* $L^\infty(L^2)\cap L^2(H^1)$.

Proof. Fix $n = 0, 1, \ldots, N-1$. Set $\tilde{\mathbf{v}}_h^n = P_e(\mathbf{u}^n)$ defined by (77) in (87). Set $\mathbf{v} = \mathbf{U}_h^{n+1/2} \in V_h$ in (89). Fix $\tilde{q}_h^{n+1} \in Q_h$ so that $(\tilde{q}_h^{n+1}, \nabla\cdot\mathbf{U}_h^{n+1/2}) = 0$ and recall (65). Then

$$\frac{1}{2\Delta t}(||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2) + \nu|\mathbf{U}_h^{n+1/2}|_1^2 = -R^{n+1}(\mathbf{U}_h^{n+1/2}) - R_h^{n+1}(\mathbf{U}_h^{n+1/2}) \\ + (\partial_{\Delta t}^{n+1}\eta, \mathbf{U}_h^{n+1/2}) - (p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla\cdot\mathbf{U}_h^{n+1/2}). \tag{160}$$

Lemma A.3. *Suppose that the FE space satisfies Assumption 2.4. Fix either* $s=0$ *or* 1. *Suppose further that* $\mathbf{u}(\cdot,t)\in V$ *for any* $t\in[0,T]$ *and* $\mathbf{u}(\cdot,t)\in H^s$ *and for any* $t\in[0,T]$. *Then, for each* $n = 0,1,\ldots,N-1$,

$$|R_h^{n+1}(\mathbf{U}_h^{n+1/2})| \le C(|\mathbf{u}^{n+1/2}|_1|\eta^{n+1/2}|_1 + |\mathbf{u}^{n+1/2}|_1|\eta^{n+1/2}|_1 + \ldots \\ \ldots + ||\mathbf{u}^{n+1/2}||_s||\mathbf{U}_h^{n+1/2}||^{s/2}|\mathbf{U}^{n+1/2}|_1^{\frac{2-s}{2}})|\mathbf{U}^{n+1/2}|_1 \tag{161}$$

Proof. Apply (74)(a) along with $\mathbf{u}\in l^\infty(H^1)$ and $\mathbf{U}_h^{n+1/2}\in H_0^1$ to get

$$|c_h(\eta^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C|\mathbf{u}^{n+1/2}|_1|\eta^{n+1/2}|_1|\mathbf{U}_h^{n+1/2}|_1. \tag{162}$$

Apply (74) to get

$$|c_h(\mathbf{U}_h^{n+1/2}, \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C\left\{\begin{array}{l} ||\mathbf{u}^{n+1/2}||_2||\mathbf{U}_h^{n+1/2}||\ |\mathbf{U}_h^{n+1/2}|_1 \\ |\mathbf{u}^{n+1/2}|_1||\mathbf{U}_h^{n+1/2}||^{1/2}|\mathbf{U}_h^{n+1/2}|_1^{3/2} \end{array}\right. . \tag{163}$$

Apply (73) along with $\mathbf{u}\in V$ and $\mathbf{U}_h^{n+1/2}\in H_0^1$ to rewrite the remaining trilinear term:

$$c_h(\mathbf{u}_h^{n+1/2}, \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) = (\mathbf{u}^{n+1/2}\cdot\nabla\eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\ - c_h(\eta^{n+1/2}, \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) + c_h(\mathbf{U}_h^{n+1/2}, \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}).$$

Estimate (66)(a) gives

$$|(\mathbf{u}^{n+1/2}\cdot\nabla\eta^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C|\mathbf{u}^{n+1/2}|_1|\eta^{n+1/2}|_1\ |\mathbf{U}_h^{n+1/2}|_1. \tag{164}$$

Recall that (78), (21)(a) give $|\eta|_1 \le Ch^k||\mathbf{u}||_{k+1}$. Then (74)(a) and (78), (21)(a) with $k=0$ gives

$$|c_h(\eta^{n+1/2}, \eta^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C|\mathbf{u}^{n+1/2}|_1|\eta^{n+1/2}|_1\ |\mathbf{U}_h^{n+1/2}|_1. \tag{165}$$

Similarly (74)(a) and (78), (21)(a) with $k = 1$, 0, and inverse estimate (22) give

$$|c_h(\mathbf{U}_h^{n+1/2}, \eta^{n+1/2}, \mathbf{U}_h^{n+1/2})| \le C\left\{\begin{array}{l} h^{1/2}||\mathbf{u}^{n+1/2}||_2||\mathbf{U}_h^{n+1/2}||\ |\mathbf{U}_h^{n+1/2}|_1 \\ |\mathbf{u}^{n+1/2}|_1||\mathbf{U}_h^{n+1/2}||^{1/2}|\mathbf{U}_h^{n+1/2}|_1^{3/2} \end{array}\right. . \tag{166}$$

Estimates (162), (163), (164), (165), (166) imply (161). □

We focus now on majorizing the 3rd term on the RHS of (161). Fix $\varepsilon > 0$.

Case 1: Fix $0 \le r \le 3/2$. Suppose that $\mathbf{u} \in L^{4/(1+2r)}(H^1)$. Fix $p = 4/(3-2r)$ and $1/p + 1/q = 1$ so that $q = 4/(1+2r)$. First apply the inverse estimate (22) to get $|\mathbf{U}_h^{n+1/2}|_1^{3/2} \le Ch^{-r}||\mathbf{U}_h^{n+1/2}||^r|\mathbf{U}_h^{n+1/2}|_1^{3/2-r}$. Then apply Young (61) to get

$$|\mathbf{u}^{n+1/2}|_1||\mathbf{U}_h^{n+1/2}||^{1/2}|\mathbf{U}^{n+1/2}|_1^{\frac{3}{2}} \le \frac{\nu}{\varepsilon}|\mathbf{U}_h^{n+1/2}|_1^2 + C\nu^{-(3-2r)/(1+2r)}h^{-4r/(1+2r)}|\mathbf{u}^{n+1/2}|_1^{4/(1+2r)}||\mathbf{U}_h^{n+1/2}||^2 \tag{167}$$

Case 2: Fix $0 \le r \le 1$. Suppose that $\mathbf{u} \in L^{2/(1+r)}(H^2)$. Fix $p = 2/(1-r)$ and $1/p + 1/q = 1$ so that $q = 2/(1+r)$. First apply the inverse estimate (22) to get $|\mathbf{U}_h^{n+1/2}||_1 \le Ch^{-r}||\mathbf{U}_h^{n+1/2}||^r|\mathbf{U}_h^{n+1/2}|_1^{1-r}$. Then apply Young (61) to get

$$||\mathbf{u}^{n+1/2}||_2||\mathbf{U}_h^{n+1/2}|||\mathbf{U}^{n+1/2}|_1 \le \frac{\nu}{\varepsilon}|\mathbf{U}_h^{n+1/2}|_1^2 + C\nu^{-(1-r)/(1+r)}h^{-2r/(1+r)}||\mathbf{u}^{n+1/2}||_2^{2/(1+r)}||\mathbf{U}_h^{n+1/2}||^2 \tag{168}$$

Bound each term on the RHS of (160) with (91), (92), (161), (99). Then, successive application of Young (61) with either (167) or (168) gives

$$||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2 + \nu\Delta t|\mathbf{U}_h^{n+1/2}|_1^2 \le \nu^{-1}\Delta t(C_2^{n+1} + C_1^{n+1} + \kappa^{n+1/2}||\mathbf{U}_h^{n+1/2}||^2) \tag{169}$$

where

$$C_2^{n+1} := C(||\nabla\mathbf{u}||_{l^\infty(L^2)}^2|\eta^{n+1/2}|_1^2 + |\mathbf{u}^{n+1/2}|_1^2|\eta^{n+1/2}|_1^2 + \dots \dots + ||\partial_{\Delta t}^{n+1}\eta||_{-1}^2 + ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2) \tag{170}$$

$$\kappa^{n+1} := C\begin{cases} \nu^{-(3-2r)/(1+2r)}h^{-4r/(1+2r)}|\mathbf{u}^{n+1}|_1^{4/(1+2r)} & 0 \le r \le 3/2 \\ \nu^{-(1-r)/(1+r)}h^{-2r/(1+r)}||\mathbf{u}^{n+1}||_2^{2/(1+r)} & 0 \le r \le 1 \end{cases}.$$

Lemma A.4. *Suppose that the FE space satisfies Assumption 2.4. Fix $k \ge 0$, $k* \ge 0$, $s \ge -1$. Suppose further that Assumption 2.3 is satisfied and $\mathbf{u} \in l^\infty(H^k) \cap l^2(H^{k+1} \cap H^2)$, $\partial_t\mathbf{u} \in L^2(H^{k*+1}) \cap l^2(W^{-1,2})$, and $p \in l^2(H^{s+1})$. Then,*

$$||\mathbf{U}_h^N||^2 + \nu\Delta t\sum_{n=0}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2 \le C||\mathbf{e}^0||^2 + K_1^2 + \Delta t\sum_{n=0}^{N-1}(\nu^{-1}C_1^{n+1} + \kappa^{n+1/2}||\mathbf{U}_h^{n+1/2}||^2) \tag{171}$$

where

$$K_1 := (C\nu^{-1/2}||\nabla\mathbf{u}||_{l^\infty(L^2)}||\mathbf{u}||_{l^2(H^{k+1})} + C||\mathbf{u}^0||_k)h^k + C\nu^{-1/2}||p||_{l^2(H^{s+1})}h^{s+1} + C\nu^{-1/2}||\partial_t\mathbf{u}||_{L^2(H^{k*+1})}h^{k*+2}. \tag{172}$$

Proof. Recall that (78), (21)(a) give $|\eta|_1 \le Ch^k||\mathbf{u}||_{k+1}$. Fix $k* \ge 0$. Then (78), (21)(a) along with (23) gives

$$||\partial_{\Delta t}^{n+1}\eta||_{-1}^2 \le Ch^{2k*+4}\Delta t^{-1}\int_{t^n}^{t^{n+1}}||\partial_t\mathbf{u}(\cdot,t)||_{k*+1}^2dt. \tag{173}$$

Estimate (21)(b) gives

$$\inf_{\tilde{q}_h \in Q_h} ||p^{n+1/2} - \tilde{q}_h|| \leq C h^{s+1} ||p^{n+1/2}||_{s+1}. \tag{174}$$

Write

$$\kappa_1^{n+1} := C(||\nabla \mathbf{u}||_{l^\infty(L^2)} ||\mathbf{u}^{n+1/2}||_{k+1}^2 h^{2k} + \dots$$
$$\dots + \Delta t^{-1} ||\partial_t \mathbf{u}||^2_{L^2(t^n, t^{n+1}; H^{k*+1})} h^{2k*+4} + ||p^{n+1/2}||_{s+1}^2 h^{2s+2}). \tag{175}$$

Application of (78), (21)(a), (173), (174) to (169), (170) proves $C_2^{n+1} \leq \kappa_1^{n+1}$ so that

$$||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2 + \nu \Delta t |\mathbf{U}_h^{n+1/2}|_1^2$$
$$\leq \nu^{-1} \Delta t (\kappa_1^{n+1} + C_1^{n+1} + \kappa^{n+1/2} ||\mathbf{U}_h^{n+1/2}||^2). \tag{176}$$

Sum from $n = 0$ to $n = N - 1$ to prove (171). □

Write

$$G^N := \exp(\Delta t \sum_{n=0}^{N-1} \kappa^{n+1/2}). \tag{177}$$

Suppose that

$$\Delta t \kappa^{n+1/2} < 1, \quad \forall n = 0, 1, \dots, N-1.$$

Then Gronwall Lemma 3.14 applied to (171) gives

$$||\mathbf{U}_h^N||^2 + \nu \Delta t \sum_{n=0}^{N-1} |\mathbf{U}_h^{n+1/2}|_1^2 \leq G^N (C ||\mathbf{e}^0||^2 + K_1^2 + \nu^{-1} \Delta t \sum_{n=0}^{N-1} C_1^{n+1}) \tag{178}$$

Application of the triangle inequality $||\mathbf{e}|| \leq ||\mathbf{U}_h|| + ||\eta||$ along with (78), (21)(a), and (178) gives

$$||\mathbf{e}^N|| + \nu^{1/2} (\Delta t \sum_{n=0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2}$$
$$\leq C(||\mathbf{u}^N||_k + \nu^{1/2} (\Delta t \sum_{n=0}^{N-1} ||\mathbf{u}^{n+1/2}||_{k+1}^2)^{1/2}) h^k$$
$$+ G^N (C ||\mathbf{e}^0|| + K_1 + \nu^{-1/2} (\Delta t \sum_{n=0}^{N-1} C_1^{n+1})^{1/2}) \tag{179}$$

The rest of the proof follows the proof of Theorem 3.5 as presented in Section 4.1. □

Appendix B. Detailed derivation of intermediate estimates

Proof of Estimate (23). Fix $n \geq n_0$. Then, for $k \geq 0$

$$|\partial_{\Delta t}^{n+1} \mathbf{u}|_k^2 = \int \left| \Delta t^{-1} \int_{t^n}^{t^{n+1}} D^k \partial_t \mathbf{u}(\cdot, t) dt \right|^2$$
$$\leq \Delta t^{-2} \int \left(\int_{t^n}^{t^{n+1}} dt \int_{t^n}^{t^{n+1}} \left| D^k \partial_t \mathbf{u}(\cdot, t) \right|^2 dt \right) \leq \Delta t^{-1} \int_{t^n}^{t^{n+1}} |\partial_t \mathbf{u}(\cdot, t)|_k^2 dt.$$

Similar proof for $k = -1$ applied to definition of $W^{-1,2}$-norm. □

Proof of Estimate (24). Fix $n \geq n_0$ and $k \geq 0$. A Taylor-expansion with integral remainder gives

$$|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})|_k^2 \leq C \int \left| \int_{t^n}^{t^{n+1/2}} (t - t^n) D^k \partial_t^{(2)} \mathbf{u}(\cdot, t) dt \right|^2$$
$$+ C \int \left| \int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t) D^k \partial_t^{(2)} \mathbf{u}(\cdot, t) dt \right|^2 \tag{180}$$

where, for any $r \in \mathbb{R}$,

$$\begin{aligned}
&\int \left| \int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t) D^k \partial_t^{(2)} \mathbf{u}(\cdot, t) dt \right|^2 \\
&\leq C \int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)^2 dt \int_{t^{n+1/2}}^{t^{n+1}} ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt \\
&\leq C \Delta t^3 \int_{t^{n+1/2}}^{t^{n+1}} \frac{1}{t^r} \left(t^r \ ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 \right) dt \\
&\leq \frac{C \Delta t^3}{(t^{n+1/2})^r} \int_{t^{n+1/2}}^{t^{n+1}} t^r \ ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt.
\end{aligned} \tag{181}$$

and similarly on the time interval $(t^n, t^{n+1/2})$ when $n > 0$. If $n = n_0 = 0$, then

$$\begin{aligned}
\int \left| \int_0^{\Delta t/2} t \ D^k \mathbf{u}(\cdot, t) dt \right|^2 &\leq C \int_0^{\Delta t/2} dt \int_0^{\Delta t/2} t^2 \ ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt \\
&\leq C \Delta t \int_0^{\Delta t/2} t^2 ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt.
\end{aligned} \tag{182}$$

Note that $\sqrt{t^{n+1/2}} = \sqrt{\Delta t/2}$ when $n = 0$. Then estimates (181), (182) applied to (180) give

$$|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})|_k^2 \leq \frac{C \Delta t^3}{(t^{n+1/2})^2} \int_{t^n}^{t^{n+1}} t^2 ||\partial_t^{(2)} \mathbf{u}(\cdot, t)||_k^2 dt. \tag{183}$$

□

Proof of Estimate (25). Fix $n \geq n_0$. First add/subtract $\partial_t \mathbf{u}(\cdot, t^{n+1/2})$ and apply the triangle inequality to get

$$\begin{aligned}
&||\partial_{\Delta t}^{n+1} \mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_k^2 \\
&\leq ||\partial_{\Delta t}^{n+1} \mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2})||_k^2 + ||\partial_t \mathbf{u}(\cdot, t^{n+1/2}) - (\partial_t \mathbf{u})^{n+1/2}||_k^2
\end{aligned} \tag{184}$$

Following a similar method used to derive (180), we get

$$||\partial_t \mathbf{u}(\cdot, t^{n+1/2}) - (\partial_t \mathbf{u})^{n+1/2}||_k^2 \leq \frac{C \Delta t^3}{(t^{n+1/2})^2} \int_{t^n}^{t^{n+1}} t^2 \ ||\partial_t^{(3)} \mathbf{u}(\cdot, t)||_k^2 dt. \tag{185}$$

Additionally,

$$
\begin{aligned}
&||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2})||_k^2 \\
&= \left\| \int_{t^n}^{t^{n+1/2}} (t - t^n)\partial_t^{(3)}\mathbf{u}(\cdot, t)dt + \int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)\partial_t^{(3)}\mathbf{u}(\cdot, t)dt \right\|_k^2 \\
&\leq \frac{C\Delta t^3}{(t^{n+1/2})^2} \int_{t^n}^{t^{n+1}} t^2 ||\partial_t^{(3)}\mathbf{u}(\cdot, t)||_k^2 dt. \qquad (186)
\end{aligned}
$$

Apply (185) and (186) to (187) to get

$$
||\partial_{\Delta t}^{n+1}\mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}||_k^2 \leq \frac{C\Delta t^3}{(t^{n+1/2})^2} \int_{t^n}^{t^{n+1}} t^2 ||\partial_t^{(3)}\mathbf{u}(\cdot, t)||_k^2 dt. \qquad (187)
$$

□

Proof of Estimate (28) for particular $\xi^n(\cdot)$. Let $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$ so that Taylor-expansion with integral remainder gives

$$
\xi^n(\mathbf{u}) = \mathbf{u}(\cdot, t^{n+1/2}) + \sum_{i=0}^{1} \int_{t^{n-i}}^{t^{n+1/2}} (t - t^i)\partial_t^{(2)}\mathbf{u}(\cdot, t)dt.
$$

Fix $n \geq n_0$. Then

$$
|\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})|_k^2 \leq C \sum_{i=0}^{1} \int \left| \int_{t^{n-i}}^{t^{n+1/2}} (t - t^{n-i}) D^k \partial_t^{(2)}\mathbf{u}(\cdot, t)dt \right|^2
$$

Following a similar method used to derive (180) we get

$$
||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_k^2 \leq \frac{C\Delta t^3}{(t^{n+1/2})^2} \int_{t^{n-1}}^{t^{n+1}} t^2 \ |\partial_t^{(2)}\mathbf{u}(\cdot, t)|_k^2 dt \qquad (188)
$$

□

Proof of Estimate (106) and (131). Fix $n \geq n_0$. Then, for any $r \in \mathbb{R}$, and for either $i = 0$ or 1,

$$
\begin{aligned}
&\int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)(\partial_t^{(2)}\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v})dt \leq C \int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)||\mathbf{u}||_2 ||\partial_t^{(2)}\mathbf{u}||_{1-i} |\mathbf{v}|_i \\
&\leq C||\mathbf{u}||_{L^\infty(t^n, t^{n+1}; H^2)} (\int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)^2 dt)^{1/2} (\int_{t^{n+1/2}}^{t^{n+1}} ||\partial_t^{(2)}\mathbf{u}||_{1-i}^2 dt)^{1/2} |\mathbf{v}|_i \\
&\leq \frac{C\Delta t^{3/2}}{(t^{n+1/2})^{r/2}} ||\mathbf{u}||_{L^\infty(t^n, t^{n+1}; H^2)} (\int_{t^{n+1/2}}^{t^{n+1}} t^r \ ||\partial_t^{(2)}\mathbf{u}||_{1-i}^2 dt)^{1/2} |\mathbf{v}|_i. \qquad (189)
\end{aligned}
$$

A similar estimate holds when time interval is shifted to (t^n, t^{n+1}) except when $n = n_0 = 0$ (note that for Δt^2 extrapolations, $n_0 > 0$). In this case

$$
\begin{aligned}
&\int_0^{\Delta t/2} t(\partial_t^{(2)}\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v})dt \leq C \int_0^{\Delta t/2} t||\mathbf{u}||_2 ||\partial_t^{(2)}\mathbf{u}||_{1-i} |\mathbf{v}|_i \\
&\leq C||\mathbf{u}||_{L^\infty(0, \Delta t; H^2)} (\int_0^{\Delta t/2} dt)^{1/2} (\int_0^{\Delta t/2} t^2 \ ||\partial_t^{(2)}\mathbf{u}||_{1-i}^2 dt)^{1/2} |\mathbf{v}|_i \\
&\leq C\Delta t^{1/2} ||\mathbf{u}||_{L^\infty(0, \Delta t; H^2)} (\int_0^{\Delta t/2} t^2 \ ||\partial_t^{(2)}\mathbf{u}||_{1-i}^2 dt)^{1/2} |\mathbf{v}|_i. \qquad (190)
\end{aligned}
$$

Note that $\sqrt{t^{n+1/2}} = \sqrt{\Delta t/2}$ when $n = 0$. Therefore, (189), (190) combine to give, for $n \geq 0$

$$\int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)(\partial_t^{(2)} \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) dt + \int_{t^n}^{t^{n+1/2}} (t - t^n)(\partial_t^{(2)} \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) dt$$
$$\leq \frac{C\Delta t^{3/2}}{t^{n+1/2}} ||\mathbf{u}||_{L^\infty(t^n,t^{n+1};H^2)} (\int_{t^n}^{t^{n+1}} t^2 \; ||\partial_t^{(2)} \mathbf{u}||_{1-i}^2 dt)^{1/2} |\mathbf{v}|_i. \tag{191}$$

Now recall that $(\mathbf{u} \cdot \nabla \partial_t^{(2)} \mathbf{u}, \mathbf{v}) = -(\mathbf{u} \cdot \nabla \mathbf{v}, \partial_t^{(2)} \mathbf{u})$ since $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{v} = 0$. Then again a similar argument used to derive (190) proves

$$\int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)(\mathbf{u} \cdot \nabla \partial_t^{(2)} \mathbf{u}, \mathbf{v}) dt + \int_{t^n}^{t^{n+1/2}} (t - t^n)(\mathbf{u} \cdot \nabla \partial_t^{(2)} \mathbf{u}, \mathbf{v}) dt$$
$$\leq \frac{C\Delta t^{3/2}}{t^{n+1/2}} ||\mathbf{u}||_{L^\infty(t^n,t^{n+1};H^2)} (\int_{t^n}^{t^{n+1}} t^2 \; ||\partial_t^{(2)} \mathbf{u}||_{1-i}^2 dt)^{1/2} |\mathbf{v}|_i. \tag{192}$$

Once again, following a similar argument used to derive (190) proves, for $n \geq n_0$,

$$\int_{t^{n+1/2}}^{t^{n+1}} (t^{n+1} - t)(\partial_t \mathbf{u} \cdot \nabla \partial_t \mathbf{u}, \mathbf{v}) dt + \int_{t^n}^{t^{n+1/2}} (t - t^n)(\partial_t \mathbf{u} \cdot \nabla \partial_t \mathbf{u}, \mathbf{v}) dt$$
$$\leq \frac{C\Delta t^{3/2}}{t^{n+1/2}} ||\partial_t \mathbf{u}||_{L^\infty(t^n,t^{n+1};L^2)} (\int_{t^n}^{t^{n+1}} t^2 \; ||\partial_t \mathbf{u}||_{3-i}^2 dt)^{1/2} |\mathbf{v}|_i. \tag{193}$$

□

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