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STABILITY AND DISPERSION ANALYSIS OF THE STAGGERED DISCONTINUOUS GALERKIN METHOD FOR WAVE PROPAGATION

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Abstract. Staggered discontinuous Galerkin methods have been developed recently and are adopted successfully to many problems such as wave propagation, elliptic equation, convection-diffusion equation and the Maxwell's equations. For wave propagation, the method is proved to have the desirable properties of energy conservation, optimal order of convergence and block-diagonal mass matrices. In this paper, we perform an analysis for the dispersion error and the CFL constant. Our results show that the staggered method provides a smaller dispersion error compared with classical finite element method as well as non-staggered discontinuous Galerkin methods.

Key words. CFL condition, dispersion analysis, dispersion relation, wave propagation, staggered discontinuous Galerkin method

1. Introduction

Discontinuous Galerkin method has become a class of very popular, efficient and highly accurate methodologies for the numerical approximation of wave equations [12, 13, 14, 15]. There are many studies in literature regarding their numerical performance as well as stability and convergence analysis. However, dispersion analysis is rarely seen despite its importance for wave propagation. The first attempt to analyze the numerical dispersion for discontinuous Galerkin method for the scalar wave equation has been carried out in [1], where a complete dispersion analysis for the interior penalty, upwind and central discontinuous Galerkin methods are performed for the numerical approximation of the wave equation in both first order and second order forms. Besides, in [11], dispersion analysis for high order discontinuous Galerkin methods applied to three dimensional Maxwell's equations with both centered and uncentered fluxes are carried out. Some superconvergence results on the dispersion error are also obtained in this work.

Recently, staggered discontinuous Galerkin methods have been developed and are adopted successfully to many problems such as wave propagation [2, 3, 4, 5, 6], elliptic equation [7], convection-diffusion equation [9] and the Maxwell's equations [8]. For the numerical simulation of waves, the method is proved to have the desirable properties of energy conservation, optimal order of convergence and block-diagonal mass matrices. Our aims in this paper are to estimate the CFL stability condition corresponding to the leap-frog time discretization and derive the dispersion relation for the staggered discontinuous Galerkin method developed in [3, 4] for wave propagation. We will show that this method has a better CFL number and a smaller dispersion error compared with the classical conforming finite element for second order wave equation [10] as well as the upwind and central discontinuous Galerkin method for wave equation in first order form [1].

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2. The staggered discontinuous Galerkin method

In this section, we will present the staggered discontinuous Galerkin method developed by Chung and Engquist [3, 4] for the numerical simulation of waves. To facilitate the stability and dispersion analysis, we consider the one-dimensional scalar wave equations

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{\partial p}{\partial t} = \frac{\partial u}{\partial x}, \tag{2}$$

for $(x,t) \in (-\infty,\infty) \times [0,\infty)$, where $c > 0$ is the scalar wave speed. Moreover, we will consider a uniform partition. Let $h > 0$ be the mesh size and let $x_j = jh$, $j = 0, \pm 1, \pm 2, \cdots$, be the nodal points. We define the primal cell $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$. For each primal cell $I_{j+\frac{1}{2}}$, we take a point $x'_{j+\frac{1}{2}} \in I_{j+\frac{1}{2}}$ and define the dual cell I'_j by $I'_j = (x'_{j-\frac{1}{2}}, x'_{j+\frac{1}{2}})$. To simplify the analysis, we will take $x'_{j+\frac{1}{2}}$ to be the mid-point of $I_{j+\frac{1}{2}}$, that is, $x'_{j+\frac{1}{2}} = x_{j+\frac{1}{2}}$ and consequently $I'_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$.

Multiplying both sides of (1) by a test function ϕ, integrating on a primal cell $I_{j+\frac{1}{2}}$ and using integration by parts yields

$$\int_{I_{j+\frac{1}{2}}} \frac{\partial u}{\partial t}\phi \, dx = c^2 \Big\{ p(x_{j+1},t)\phi(x_{j+1}) - p(x_j,t)\phi(x_j) - \int_{I_{j+\frac{1}{2}}} p \frac{d\phi}{dx}\, dx \Big\}.$$

Similarly, multiplying both sides of (2) by a test function ψ, integrating on a dual cell I'_k and using integration by parts yields

$$\int_{I'_k} \frac{\partial p}{\partial t}\psi \, dx = u(x_{k+\frac{1}{2}},t)\psi(x_{k+\frac{1}{2}}) - u(x_{k-\frac{1}{2}},t)\psi(x_{k-\frac{1}{2}}) - \int_{I'_k} u \frac{d\psi}{dx}\, dx.$$

The staggered discontinuous Galerkin method can be described as follows. Find $u_h \in U_h$ and $p_h \in W_h$ such that

$$\int_{I_{j+\frac{1}{2}}} \frac{\partial u_h}{\partial t}\phi \, dx = c^2 \Big\{ p_h(x_{j+1},t)\phi(x_{j+1}) - p_h(x_j,t)\phi(x_j) - \int_{I_{j+\frac{1}{2}}} p_h \frac{d\phi}{dx}\, dx \Big\}, \tag{3}$$

$$\int_{I'_k} \frac{\partial p_h}{\partial t}\psi \, dx = u_h(x_{k+\frac{1}{2}},t)\psi(x_{k+\frac{1}{2}}) - u_h(x_{k-\frac{1}{2}},t)\psi(x_{k-\frac{1}{2}}) - \int_{I'_k} u_h \frac{d\psi}{dx}\, dx, \tag{4}$$

for all $\phi \in U_h$ and $\psi \in W_h$, and for all integers j and k.

We will now discuss the choice of the two finite element spaces U_h and W_h. Let $m \geq 0$ be an integer, that corresponds to the degree of polynomials used for trial and test spaces. For each given primal cell $I_{j+\frac{1}{2}}$, we define $R_m(I_{j+\frac{1}{2}})$ as the space of functions which are polynomials of degree at most m on each of the two sub-cells $(x_j, x_{j+\frac{1}{2}})$ and $(x_{j+\frac{1}{2}}, x_{j+1})$ with continuity at $x_{j+\frac{1}{2}}$. Similarly, for each given dual cell I'_k, we define $R'_m(I'_k)$ as the space of functions which are polynomials of degree at most m on each of the two sub-cells $(x_{k-\frac{1}{2}}, x_k)$ and $(x_k, x_{k+\frac{1}{2}})$ with continuity at x_k. We will state the definitions of U_h and W_h in the following.

Definition 1. *The two finite element spaces U_h and W_h are defined by*

(1) $\phi \in U_h$ *if* $\phi|_{I_{j+\frac{1}{2}}} \in R_m(I_{j+\frac{1}{2}})$.

(2) $\psi \in W_h$ *if* $\psi|_{I'_k} \in R'_m(I'_k)$.

In Figure 1, typical functions in the spaces U_h and W_h are shown for the piecewise linear case, that is $m = 1$. Here, we use solid line to represent a function in U_h and use dotted line to represent a function in W_h. In Figure 2, we show an example of functions in U_h for the piecewise quadratic case, that is $m = 2$.

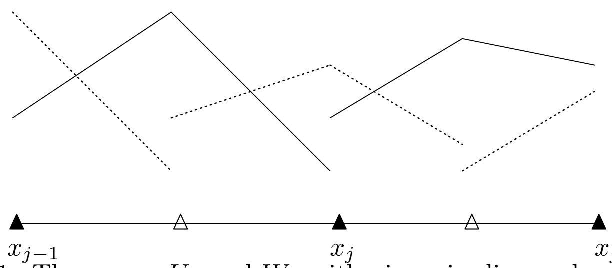

FIGURE 1. The spaces U_h and W_h with piecewise linear elements ($m = 1$). Solid line represents a function in U_h while dotted line represents a function in W_h

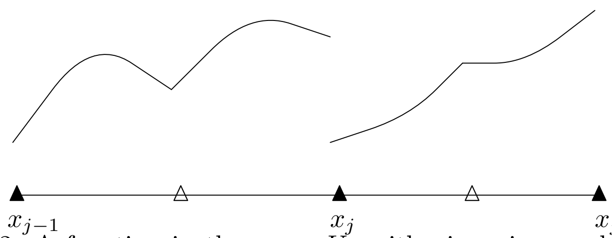

FIGURE 2. A function in the space U_h with piecewise quadratic elements ($m = 2$).

In the following analysis, we will use u and p instead of u_h and p_h in (3)-(4) to simplify notations.

3. Dispersion analysis for piecewise linear elements

In this section, we will perform the dispersion analysis for the scheme (3)-(4) with piecewise linear element, that is, $m = 1$.

First, we recall that on each $I_{j+\frac{1}{2}}$, u is linear on each subinterval $[x_j, x_{j+\frac{1}{2}}]$ and $[x_{j+\frac{1}{2}}, x_{j+1}]$, and is continuous at $x_{j+\frac{1}{2}}$ for each j. We denote the restriction of u on $I_{j+\frac{1}{2}}$ by $u^{(j+\frac{1}{2})}$. We will represent each $u^{(j+\frac{1}{2})}$ by using the basis functions $\varphi_{-1}^{(j+\frac{1}{2})}, \varphi_0^{(j+\frac{1}{2})}$ and $\varphi_1^{(j+\frac{1}{2})}$ defined as follows. We note that $x_{j+\frac{1}{2}} - \frac{h}{3}$ and $x_{j+\frac{1}{2}}$ are the Radau quadrature points on the interval $[x_j, x_{j+\frac{1}{2}}]$ so that the following rule

$$\int_{x_j}^{x_{j+\frac{1}{2}}} g(x)\, dx = \frac{h}{8}\, g(x_{j+\frac{1}{2}}) + \frac{3h}{8}\, g\Big(x_{j+\frac{1}{2}} - \frac{h}{3}\Big)$$

is exact for all quadratic polynomial $g(x)$. Similarly, $x_{j+\frac{1}{2}}$ and $x_{j+\frac{1}{2}} + \frac{h}{3}$ are the Radau quadrature points on the interval $[x_{j+\frac{1}{2}}, x_{j+1}]$ so that the following rule

$$\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} g(x)\, dx = \frac{h}{8}\, g(x_{j+\frac{1}{2}}) + \frac{3h}{8}\, g\Big(x_{j+\frac{1}{2}} + \frac{h}{3}\Big)$$

is exact for all quadratic polynomial $g(x)$.

We observe that the space $R_1(I_{j+\frac{1}{2}})$ has dimension three spanned by $\varphi_{-1}^{(j+\frac{1}{2})}, \varphi_0^{(j+\frac{1}{2})}$ and $\varphi_1^{(j+\frac{1}{2})}$ where

- $\varphi_{-1}^{(j+\frac{1}{2})}$ is the function with support $[x_j, x_{j+\frac{1}{2}}]$ and is linear in $[x_j, x_{j+\frac{1}{2}}]$ with $\varphi_{-1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = 0$ and $\varphi_{-1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \frac{h}{3}) = 1$.
- $\varphi_{0}^{(j+\frac{1}{2})}$ is the function with support $[x_j, x_{j+1}]$ and is piecewise linear in $[x_j, x_{j+\frac{1}{2}}]$ and $[x_{j+\frac{1}{2}}, x_{j+1}]$ with $\varphi_{0}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = 1$ and $\varphi_{0}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \frac{h}{3}) = \varphi_{0}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \frac{h}{3}) = 0$.
- $\varphi_{1}^{(j+\frac{1}{2})}$ is the function with support $[x_{j+\frac{1}{2}}, x_{j+1}]$ and is linear in $[x_{j+\frac{1}{2}}, x_{j+1}]$ with $\varphi_{1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = 0$ and $\varphi_{1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \frac{h}{3}) = 1$.

With the above basis functions, we can write

$$u^{(j+\frac{1}{2})} = u_{-1}^{(j+\frac{1}{2})}\varphi_{-1}^{(j+\frac{1}{2})} + u_{0}^{(j+\frac{1}{2})}\varphi_{0}^{(j+\frac{1}{2})} + u_{1}^{(j+\frac{1}{2})}\varphi_{1}^{(j+\frac{1}{2})}.$$

We remark that there are three degrees of freedom represented by $u_{-1}^{(j+\frac{1}{2})}, u_{0}^{(j+\frac{1}{2})}$ and $u_{1}^{(j+\frac{1}{2})}$ on each primal cell $I_{j+\frac{1}{2}}$, which are the values of u at the points $x_{j+\frac{1}{2}} - \frac{h}{3}$, $x_{j+\frac{1}{2}}$ and $x_{j+\frac{1}{2}} + \frac{h}{3}$ respectively.

Similarly, we let $p^{(k)}$ be the restriction of p on I'_k. For each k, we can represent $p^{(k)}$ as

$$p^{(k)} = p_{-1}^{(k)}\varphi_{-1}^{(k)} + p_{0}^{(k)}\varphi_{0}^{(k)} + p_{1}^{(k)}\varphi_{1}^{(k)}$$

where the basis functions are defined as follows:

- $\varphi_{-1}^{(k)}$ is the function whose support is $[x_{k-\frac{1}{2}}, x_k]$ such that it is linear in $[x_{k-\frac{1}{2}}, x_k]$ with $\varphi_{-1}^{(k)}(x_k) = 0$ and $\varphi_{-1}^{(k)}(x_k - \frac{h}{3}) = 1$.
- $\varphi_{0}^{(k)}$ is the function with support $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$ and is piecewise linear in $[x_{k-\frac{1}{2}}, x_k]$ and $[x_k, x_{k+\frac{1}{2}}]$ with $\varphi_{0}^{(k)}(x_k) = 1$ and $\varphi_{0}^{(k)}(x_k - \frac{h}{3}) = \varphi_{0}^{(k)}(x_k + \frac{h}{3}) = 0$.
- $\varphi_{1}^{(k)}$ is the function whose support is $[x_k, x_{k+\frac{1}{2}}]$ such that it is linear in $[x_k, x_{k+\frac{1}{2}}]$ with $\varphi_{1}^{(k)}(x_k) = 0$ and $\varphi_{1}^{(k)}(x_k + \frac{h}{3}) = 1$.

Here we remark that the x_k and $x_k - \frac{h}{3}$ are the Radau quadrature points for $[x_{k-\frac{1}{2}}, x_k]$ while x_k and $x_k + \frac{h}{3}$ are the Radau quadrature points for $[x_k, x_{k+\frac{1}{2}}]$.

Now we will show that the scheme (3)-(4) can be written as follows.

$$\begin{aligned}
\frac{du_{-1}^{(j+\frac{1}{2})}}{dt} &= \frac{3c^2}{h}\Big(-p_0^{(j)} + p_1^{(j)}\Big),\\
\frac{du_{0}^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{2h}\Big(p_0^{(j)} - 9p_1^{(j)} - p_0^{(j+1)} + 9p_{-1}^{(j+1)}\Big),\\
\frac{du_{1}^{(j+\frac{1}{2})}}{dt} &= \frac{3c^2}{h}\Big(-p_{-1}^{(j+1)} + p_0^{(j+1)}\Big),\\
\frac{dp_{-1}^{(j)}}{dt} &= \frac{3}{h}\Big(-u_0^{(j-\frac{1}{2})} + u_1^{(j-\frac{1}{2})}\Big),\\
\frac{dp_{0}^{(j)}}{dt} &= \frac{1}{2h}\Big(u_0^{(j-\frac{1}{2})} - 9u_1^{(j-\frac{1}{2})} - u_0^{(j+\frac{1}{2})} + 9u_{-1}^{(j+\frac{1}{2})}\Big),\\
\frac{dp_{1}^{(j)}}{dt} &= \frac{3}{h}\Big(-u_{-1}^{(j+\frac{1}{2})} + u_0^{(j+\frac{1}{2})}\Big).
\end{aligned} \tag{5}$$

To derive the first equation in (5), we take $\phi = \varphi_{-1}^{(j+\frac{1}{2})}$ in (3) and evaluate the integrals by the Gauss-Radau quadrature rule. Since

$$\int \varphi_{-1}^{(j+\frac{1}{2})}\varphi_{-1}^{(j+\frac{1}{2})}dx = \frac{3h}{8} \quad \text{and} \quad \int \varphi_{-1}^{(j+\frac{1}{2})}\varphi_{1}^{(j+\frac{1}{2})}dx = \int \varphi_{-1}^{(j+\frac{1}{2})}\varphi_{0}^{(j+\frac{1}{2})}dx = 0,$$

the integral of the left hand side of (3) can be computed by the following:

$$\int_{x_j}^{x_{j+\frac{1}{2}}} \frac{du^{(j+\frac{1}{2})}}{dt}\varphi_{-1}^{(j+\frac{1}{2})}dx = \frac{3h}{8}\frac{du_{-1}^{(j+\frac{1}{2})}}{dt}.$$

The right hand side of (3) can be computed as follows:

$$\begin{aligned}
&-\Big(p^{(j)}\varphi_{-1}^{(j+\frac{1}{2})}\Big)(x_j) - \int_{x_j}^{x_{j+\frac{1}{2}}} p^{(j)}\Big(\varphi_{-1}^{(j+\frac{1}{2})}\Big)'dx \\
&= -\frac{3}{2}p_0^{(j)} + \frac{3}{h}\Big(\frac{1}{4}\frac{h}{2}p_0^{(j)} + \frac{3}{4}\frac{h}{2}p_1^{(j)}\Big) \\
&= -\frac{9}{8}p_0^{(j)} + \frac{9}{8}p_1^{(j)}.
\end{aligned}$$

Thus, we have

$$\frac{du_{-1}^{(j+\frac{1}{2})}}{dt} = \frac{3c^2}{h}\Big(-p_0^{(j)} + p_1^{(j)}\Big).$$

Similarly, we take $\phi = \varphi_0^{(j+\frac{1}{2})}$ in (3) to derive the second equation in (5). First, we have

$$\int_{x_j}^{x_{j+1}} \frac{du^{(j+\frac{1}{2})}}{dt}\varphi_0^{(j+\frac{1}{2})}dx = \frac{h}{4}\frac{du_0^{(j+\frac{1}{2})}}{dt}$$

for the left hand side of (3). For the right hand side,

$$\begin{aligned}
&\Big(p\varphi_0^{(j+\frac{1}{2})}\Big)(x_{j+1}) - \Big(p\varphi_0^{(j+\frac{1}{2})}\Big)(x_j) - \int_{x_j}^{x_{j+1}} p^{(j)}\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'dx \\
&= \Big(p\varphi_0^{(j+\frac{1}{2})}\Big)(x_{j+1}) - \Big(p\varphi_0^{(j+\frac{1}{2})}\Big)(x_j) \\
&\quad - \int_{x_j}^{x_{j+\frac{1}{2}}} p^{(j)}\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'dx - \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p^{(j+1)}\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'dx \\
&= -\frac{1}{2}p_0^{(j+1)} + \frac{1}{2}p_0^{(j)} - \frac{3}{h}\Big(\frac{1}{4}\frac{h}{2}p_0^{(j)} + \frac{3}{4}\frac{h}{2}p_1^{(j)}\Big) + \frac{3}{h}\Big(\frac{3}{4}\frac{h}{2}p_{-1}^{(j+1)} + \frac{1}{4}\frac{h}{2}p_0^{(j+1)}\Big) \\
&= \frac{1}{8}p_0^{(j)} - \frac{9}{8}p_1^{(j)} + \frac{9}{8}p_{-1}^{(j+1)} - \frac{1}{8}p_0^{(j+1)}.
\end{aligned}$$

Therefore,

$$\frac{du_0^{(j+\frac{1}{2})}}{dt} = \frac{c^2}{2h}\Big(p_0^{(j)} - 9p_1^{(j)} + 9p_{-1}^{(j+1)} - p_0^{(j+1)}\Big).$$

To derive the third equation in (5), we take $\phi = \varphi_1^{(j+\frac{1}{2})}$ in (3). The left hand side can be computed as

$$\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} \frac{du^{(j+\frac{1}{2})}}{dt}\varphi_1^{(j+\frac{1}{2})}dx = \frac{3h}{8}\frac{du_1^{(j+\frac{1}{2})}}{dt}$$

on the left hand side. On the right hand side,

$$\begin{aligned}&\Big(p^{(j+1)}\varphi_1^{(j+\frac{1}{2})}\Big)(x_{j+1})-\int_{x_{j+\frac{1}{2}}}^{x_{j+1}}p^{(j+1)}\Big(\varphi_1^{(j+\frac{1}{2})}\Big)'dx\\&=\frac{3}{2}p_0^{(j+1)}-\frac{3}{h}\Big(\frac{3}{4}\frac{h}{2}p_{-1}^{(j+1)}+\frac{1}{4}\frac{h}{2}p_0^{(j+1)}\Big)\\&=-\frac{9}{8}p_{-1}^{(j+1)}+\frac{9}{8}p_0^{(j+1)}.\end{aligned}$$

Therefore,

$$\frac{du_1^{(j+\frac{1}{2})}}{dt}=\frac{3c^2}{h}\Big(-p_{-1}^{(j+1)}+p_0^{(j+1)}\Big).$$

The last three equations in (5) can be derived in the same way.

To perform the dispersion analysis for (5), we consider solution of the form

$$\begin{aligned}u_{-1}^{(j+\frac{1}{2})}&=\alpha_{-1}e^{i(x_{j+\frac{1}{6}}k-\omega t)},\\u_0^{(j+\frac{1}{2})}&=\alpha_0e^{i(x_{j+\frac{1}{2}}k-\omega t)},\\u_1^{(j+\frac{1}{2})}&=\alpha_1e^{i(x_{j+\frac{5}{6}}k-\omega t)},\end{aligned}\tag{6}$$

where $\alpha_{-1},\alpha_0,\alpha_1\in\mathbb{C}$. Now we substitute the formula (6) in (5). From the first equation of (5), we have

$$-i\omega u_{-1}^{(j+\frac{1}{2})}=\frac{3c^2}{h}\Big(-p_0^{(j)}+p_1^{(j)}\Big).$$

Taking time derivative, we have

$$\begin{aligned}-\omega^2u_{-1}^{(j+\frac{1}{2})}&=\frac{3c^2}{h}\Big(-\frac{dp_0^{(j)}}{dt}+\frac{dp_1^{(j)}}{dt}\Big)\\&=\frac{3c^2}{h}\Big\{-\frac{1}{2h}\Big((u_0-9u_1)^{(j-\frac{1}{2})}+(-u_0+9u_{-1})^{(j+\frac{1}{2})}\Big)\\&\qquad+\frac{3}{h}\Big((-u_{-1}+u_0)^{(j+\frac{1}{2})}\Big)\Big\}\\&=\frac{3c^2}{2h^2}\Big(-u_0^{(j-\frac{1}{2})}+9u_1^{(j-\frac{1}{2})}-15u_{-1}^{(j+\frac{1}{2})}+7u_0^{(j+\frac{1}{2})}\Big).\end{aligned}$$

Using the definition of u from (6),

$$\begin{aligned}-\omega^2\alpha_{-1}e^{ix_{j+\frac{1}{6}}k}&=\frac{3c^2}{2h^2}\Big(-\alpha_0e^{ix_{j-\frac{1}{2}}k}+9\alpha_1e^{ix_{j-\frac{1}{6}}k}-15\alpha_{-1}e^{ix_{j+\frac{1}{6}}k}+7\alpha_0e^{ix_{j+\frac{1}{2}}k}\Big)\\&=\frac{3c^2}{2h^2}e^{ix_{j+\frac{1}{6}}k}\Big(-\alpha_0e^{ihk(-\frac{4}{6})}+9\alpha_1e^{ihk(-\frac{2}{6})}-15\alpha_{-1}+7\alpha_0e^{ihk(\frac{2}{6})}\Big).\end{aligned}$$

Dividing common factors,

$$\omega^2\alpha_{-1}=-\frac{3c^2}{2h^2}\Big\{-15\alpha_{-1}+\Big(-e^{ihk(-\frac{4}{6})}+7e^{ihk(\frac{2}{6})}\Big)\alpha_0+9e^{ihk(-\frac{2}{6})}\alpha_1\Big\}.\tag{7}$$

From the second equation of (5), we have

$$-i\omega u_0^{(j+\frac{1}{2})}=\frac{c^2}{2h}\Big(p_0^{(j)}-9p_1^{(j)}-p_0^{(j+1)}+9p_{-1}^{(j+1)}\Big).$$

Taking time derivative, we have

$$\begin{aligned}
-\omega^2 u_0^{(j+\frac{1}{2})} &= \frac{c^2}{2h}\Big(\frac{dp_0^{(j)}}{dt} - 9\frac{dp_1^{(j)}}{dt} - \frac{dp_0^{(j+1)}}{dt} + 9\frac{dp_{-1}^{(j+1)}}{dt}\Big)\\
&= \frac{c^2}{2h}\Big\{\frac{1}{2h}\Big((u_0-9u_1)^{(j-\frac{1}{2})} + (-u_0+9u_{-1})^{(j+\frac{1}{2})}\Big) - \frac{27}{h}(-u_{-1}+u_0)^{(j+\frac{1}{2})}\\
&\quad - \frac{1}{2h}\Big((u_0-9u_1)^{(j+\frac{1}{2})} + (-u_0+9u_{-1})^{(j+\frac{3}{2})}\Big) + \frac{27}{h}(-u_0+u_1)^{(j+\frac{1}{2})}\Big\}\\
&= \frac{c^2}{2h^2}\Big(\frac{1}{2}u_0^{(j-\frac{1}{2})} - \frac{9}{2}u_1^{(j-\frac{1}{2})} - 55u_0^{(j+\frac{1}{2})} + \frac{63}{2}u_{-1}^{(j+\frac{1}{2})} + \frac{63}{2}u_1^{(j+\frac{1}{2})}\\
&\qquad + \frac{1}{2}u_0^{(j+\frac{3}{2})} - \frac{9}{2}u_{-1}^{(j+\frac{3}{2})}\Big).
\end{aligned}$$

Using the definition of u from (6),

$$\begin{aligned}
-\omega^2\alpha_0 e^{ix_{j+\frac{1}{2}}k} &= \frac{c^2}{2h^2}\Big(\frac{1}{2}\alpha_0 e^{ix_{j-\frac{1}{2}}k} - \frac{9}{2}\alpha_1 e^{ix_{j-\frac{1}{6}}k} - 55\alpha_0 e^{ix_{j+\frac{1}{2}}k} + \frac{63}{2}\alpha_{-1}e^{ix_{j+\frac{1}{6}}k}\\
&\quad + \frac{63}{2}\alpha_1 e^{ix_{j+\frac{5}{6}}k} + \frac{1}{2}\alpha_0 e^{ix_{j+\frac{3}{2}}k} - \frac{9}{2}\alpha_{-1}e^{ix_{j+\frac{7}{6}}k}\Big)\\
&= \frac{c^2}{2h^2}e^{ix_{j+\frac{1}{2}}k}\Big(\frac{1}{2}\alpha_0 e^{ihk(-1)} - \frac{9}{2}\alpha_1 e^{ihk(-\frac{4}{6})} - 55\alpha_0 + \frac{63}{2}\alpha_{-1}e^{ihk(-\frac{2}{6})}\\
&\quad + \frac{63}{2}\alpha_1 e^{ihk(\frac{2}{6})} + \frac{1}{2}\alpha_0 e^{ihk} - \frac{9}{2}\alpha_{-1}e^{ihk(\frac{4}{6})}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega^2\alpha_0 = &-\frac{c^2}{2h^2}\Big\{\Big(\frac{63}{2}e^{ihk(-\frac{2}{6})} - \frac{9}{2}e^{ihk(\frac{4}{6})}\Big)\alpha_{-1} + \Big(\frac{1}{2}e^{-ihk} - 55 + \frac{1}{2}e^{ihk}\Big)\alpha_0\\
&+ \Big(-\frac{9}{2}e^{ihk(-\frac{4}{6})} + \frac{63}{2}e^{ihk(\frac{2}{6})}\Big)\alpha_1\Big\}.
\end{aligned} \tag{8}$$

From the third equation of (5), we have

$$-i\omega u_1^{(j+\frac{1}{2})} = \frac{3c^2}{h}\Big(-p_{-1}^{(j+1)} + p_0^{(j+1)}\Big).$$

Taking time derivative, we have

$$\begin{aligned}
-\omega^2 u_1^{(j+\frac{1}{2})} &= \frac{3c^2}{h}\Big(-\frac{dp_{-1}^{(j+1)}}{dt} + \frac{dp_0^{(j+1)}}{dt}\Big)\\
&= \frac{3c^2}{h}\Big\{-\frac{3}{h}(-u_0+u_1)^{(j+\frac{1}{2})}\\
&\qquad + \frac{1}{2h}\Big((u_0-9u_1)^{(j+\frac{1}{2})} + (-u_0+9u_{-1})^{(j+\frac{3}{2})}\Big)\Big\}\\
&= \frac{3c^2}{2h^2}\Big(7u_0^{(j+\frac{1}{2})} - 15u_1^{(j+\frac{1}{2})} - u_0^{(j+\frac{3}{2})} + 9u_{-1}^{(j+\frac{3}{2})}\Big).
\end{aligned}$$

Using the definition of u in (6), we get

$$\begin{aligned}
-\omega^2\alpha_1 e^{ix_{j+\frac{5}{6}}k} &= \frac{3c^2}{2h^2}\Big(7\alpha_0 e^{ix_{j+\frac{1}{2}}k} - 15\alpha_1 e^{ix_{j+\frac{5}{6}}k} - \alpha_0 e^{ix_{j+\frac{3}{2}}k} + 9\alpha_{-1}e^{ix_{j+\frac{7}{6}}k}\Big)\\
&= \frac{3c^2}{2h^2}e^{ix_{j+\frac{5}{6}}k}\Big(7\alpha_0 e^{ihk(-\frac{2}{6})} - 15\alpha_1 - \alpha_0 e^{ihk(\frac{4}{6})} + 9\alpha_{-1}e^{ihk(\frac{2}{6})}\Big).
\end{aligned}$$

Consequently, we obtain

$$\omega^2\alpha_1 = -\frac{3c^2}{2h^2}\Big\{9e^{ihk(\frac{2}{6})}\alpha_{-1} + \Big(7e^{ihk(-\frac{2}{6})} - e^{ihk(\frac{4}{6})}\Big)\alpha_0 - 15\alpha_1\Big\}. \tag{9}$$

Combining the above three equations (7), (8) and (9), we obtain the following eigenvalue problem:

$$-\frac{c^2}{2h^2} M \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} = \omega^2 \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix}, \tag{10}$$

where

$$M = \begin{pmatrix} -45 & -3e^{ihk(-\frac{4}{6})} + 21e^{ihk(\frac{2}{6})} & 27e^{ihk(-\frac{2}{6})} \\ \frac{63}{2}e^{ihk(-\frac{2}{6})} - \frac{9}{2}e^{ihk(\frac{4}{6})} & \frac{1}{2}e^{-ihk} - 55 + \frac{1}{2}e^{ihk} & -\frac{9}{2}e^{ihk(-\frac{4}{6})} + \frac{63}{2}e^{ihk(\frac{2}{6})} \\ 27e^{ihk(\frac{2}{6})} & 21e^{ihk(-\frac{2}{6})} - 3e^{ihk(\frac{4}{6})} & -45 \end{pmatrix}.$$

With the help of a mathematics software (such as Mathematica), we find that the characteristic polynomial of the following eigenvalue problem

$$M_1 \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} = (\omega')^2 \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix}$$

is

$$x^3 + \{145 - \cos(hk)\}x^2 + \{4896 + 288\cos(hk)\}x + \{20736 - 20736\cos(hk)\} = 0.$$

Thus, the eigenvalues for (10) are

$$\begin{aligned} \omega_{h,1}^2 &= -\frac{c^2}{2h^2}\Big(\frac{1}{3}\{-145 + \cos(hk)\} + \frac{a}{3b} - \frac{b}{3}\Big), \\ \omega_{h,2}^2 &= -\frac{c^2}{2h^2}\Big(\frac{1}{3}\{-145 + \cos(hk)\} - \frac{(1 + i\sqrt{3})a}{6b} + \frac{(1 - i\sqrt{3})b}{6}\Big), \\ \omega_{h,3}^2 &= -\frac{c^2}{2h^2}\Big(\frac{1}{3}\{-145 + \cos(hk)\} - \frac{(1 - i\sqrt{3})a}{6b} + \frac{(1 + i\sqrt{3})b}{6}\Big), \end{aligned}$$

where

$$\begin{aligned} a &= -6337 + 1154\cos(hk) - \cos^2(hk), \\ b &= \Big(133921 - 508899\cos(hk) + 1731\cos^2(hk) - \cos^3(hk) \\ &\quad + 7776\sqrt{3}\sqrt{-1304 + 15\cos(hk) + 1290\cos^2(hk) - \cos^3(hk)}\Big)^{\frac{1}{3}}. \end{aligned}$$

Hence, by using the Taylor's expansion, we have

$$\begin{aligned} \frac{\omega_{h,1}^2}{\omega^2} &= \frac{36}{(hk)^2} + \frac{9}{hk} - \frac{3}{8} - \frac{31(hk)}{128} - \frac{(hk)^2}{96} + \frac{4027(hk)^3}{1474560} + \frac{7(hk)^4}{17280} \\ &\quad - \frac{246397(hk)^5}{11890851840} - \frac{17(hk)^6}{4354560} - \frac{48433199(hk)^7}{54793045278720} + \frac{451(hk)^8}{1567641600} + O\Big((hk)^9\Big), \\ \frac{\omega_{h,2}^2}{\omega^2} &= 1 - \frac{(hk)^4}{8640} - \frac{(hk)^6}{217728} - \frac{49(hk)^8}{111974400} + O\Big((hk)^9\Big), \\ \frac{\omega_{h,3}^2}{\omega^2} &= \frac{36}{(hk)^2} - \frac{9}{hk} - \frac{3}{8} + \frac{31(hk)}{128} - \frac{(hk)^2}{96} - \frac{4027(hk)^3}{1474560} + \frac{7(hk)^4}{17280} \\ &\quad + \frac{246397(hk)^5}{11890851840} - \frac{17(hk)^6}{4354560} + \frac{48433199(hk)^7}{54793045278720} + \frac{451(hk)^8}{1567641600} + O\Big((hk)^9\Big). \end{aligned}$$

Thus, we see that $\omega_{h,2}^2$ is the physically correct eigenvalue, while $\omega_{h,1}^2$ and $\omega_{h,3}^2$ are spurious modes with small amplitude [10].

4. Dispersion analysis for piecewise quadratic elements

Now we will perform the dispersion analysis for (3)-(4) with piecewise quadratic elements, that is, $m = 2$.

We recall that $u^{(j+\frac{1}{2})}$ is the restriction of u on $I_{j+\frac{1}{2}}$, and recall that $u^{(j+\frac{1}{2})}$ is quadratic on each subinterval $[x_j, x_{j+\frac{1}{2}}]$ and $[x_{j+\frac{1}{2}}, x_{j+1}]$, and is continuous at $x_{j+\frac{1}{2}}$ for each j. Since the space $R_2(I_{j+\frac{1}{2}})$ has dimension five, each $u^{(j+\frac{1}{2})}$ can be represented by a linear combination of the five basis functions $\varphi_{-2}^{(j+\frac{1}{2})}, \varphi_{-1}^{(j+\frac{1}{2})}, \varphi_0^{(j+\frac{1}{2})}, \varphi_1^{(j+\frac{1}{2})}$ and $\varphi_2^{(j+\frac{1}{2})}$, that is

$$u^{(j+\frac{1}{2})} = u_{-2}^{(j+\frac{1}{2})}\varphi_{-2}^{(j+\frac{1}{2})} + u_{-1}^{(j+\frac{1}{2})}\varphi_{-1}^{(j+\frac{1}{2})} + u_0^{(j+\frac{1}{2})}\varphi_0^{(j+\frac{1}{2})} + u_1^{(j+\frac{1}{2})}\varphi_1^{(j+\frac{1}{2})} + u_2^{(j+\frac{1}{2})}\varphi_2^{(j+\frac{1}{2})}.$$

To define the basis functions, we first recall that the points $x_{j+\frac{1}{2}}$, $x_{j+\frac{1}{2}} - \gamma_1$ and $x_{j+\frac{1}{2}} - \gamma_2$, where $\gamma_1 = \frac{(6-\sqrt{6})h}{20}$ and $\gamma_2 = \frac{(6+\sqrt{6})h}{20}$, are the Radau quadrature points so that the following rule

$$\int_{x_j}^{x_{j+\frac{1}{2}}} g(x)\, dx = \frac{h}{18} g(x_{j+\frac{1}{2}}) + \frac{h}{72}(16+\sqrt{6}) g(x_{j+\frac{1}{2}} - \gamma_1) + \frac{h}{72}(16-\sqrt{6}) g(x_{j+\frac{1}{2}} - \gamma_2)$$

is exact for all polynomials of degree less than or equal to 4. Similarly, the three points $x_{j+\frac{1}{2}}$, $x_{j+\frac{1}{2}} + \gamma_1$ and $x_{j+\frac{1}{2}} + \gamma_2$ are the Radau quadrature points so that the following rule

$$\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} g(x)\, dx = \frac{h}{18} g(x_{j+\frac{1}{2}}) + \frac{h}{72}(16+\sqrt{6}) g(x_{j+\frac{1}{2}} + \gamma_1) + \frac{h}{72}(16-\sqrt{6}) g(x_{j+\frac{1}{2}} + \gamma_2)$$

is exact for all polynomials of degree less than or equal to 4. Now we give the definitions of the basis functions.

- $\varphi_{-2}^{(j+\frac{1}{2})}$ is the function with support $[x_j, x_{j+\frac{1}{2}}]$ and is quadratic in $[x_j, x_{j+\frac{1}{2}}]$ with $\varphi_{-2}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = \varphi_{-2}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \gamma_1) = 0$ and $\varphi_{-2}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \gamma_2) = 1$.
- $\varphi_{-1}^{(j+\frac{1}{2})}$ is the function with support $[x_j, x_{j+\frac{1}{2}}]$ and is quadratic in $[x_j, x_{j+\frac{1}{2}}]$ with $\varphi_{-1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = \varphi_{-1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \gamma_2) = 0$ and $\varphi_{-1}^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \gamma_1) = 1$.
- $\varphi_0^{(j+\frac{1}{2})}$ is the function with support $[x_j, x_{j+1}]$ and is piecewise quadratic in $[x_j, x_{j+\frac{1}{2}}]$ and $[x_{j+\frac{1}{2}}, x_{j+1}]$ with $\varphi_0^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = 1$ and $\varphi_0^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \gamma_2) = \varphi_0^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} - \gamma_1) = \varphi_0^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \gamma_1) = \varphi_0^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \gamma_2) = 0$.
- $\varphi_1^{(j+\frac{1}{2})}$ is the function with support $[x_{j+\frac{1}{2}}, x_{j+1}]$ and is quadratic in $[x_{j+\frac{1}{2}}, x_{j+1}]$ with $\varphi_1^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = \varphi_1^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \gamma_2) = 0$ and $\varphi_1^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \gamma_1) = 1$.
- $\varphi_2^{(j+\frac{1}{2})}$ is the function with support $[x_{j+\frac{1}{2}}, x_{j+1}]$ and is quadratic in $[x_{j+\frac{1}{2}}, x_{j+1}]$ with $\varphi_2^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}}) = \varphi_2^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \gamma_1) = 0$ and $\varphi_2^{(j+\frac{1}{2})}(x_{j+\frac{1}{2}} + \gamma_2) = 1$.

Let $p^{(j)}$ be the restriction of p on I'_j. We recall that $p^{(j)}$ is quadratic on each subinterval $[x_{j-\frac{1}{2}}, x_j]$ and $[x_j, x_{j+\frac{1}{2}}]$, and is continuous at x_j for each j. For each

$p^{(j)}$, we can write

$$p^{(j)} = p_{-2}^{(j)}\varphi_{-2}^{(j)} + p_{-1}^{(j)}\varphi_{-1}^{(j)} + p_0^{(j)}\varphi_0^{(j)} + p_1^{(j)}\varphi_1^{(j)} + p_2^{(j)}\varphi_2^{(j)}$$

where the basis functions $\varphi_{-2}^{(j)}, \varphi_{-1}^{(j)}, \varphi_0^{(j)}, \varphi_1^{(j)}$ and $\varphi_2^{(j)}$ are defined similarly.

Next, we will show that the scheme (3)-(4) can be written in the following way.

$$\begin{aligned}
\frac{du_{-2}^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{3h}\Big((-8-4\sqrt{6})p_0^{(j)} + (16+\sqrt{6})p_1^{(j)} + (-8+3\sqrt{6})p_2^{(j)}\Big)\\
\frac{du_{-1}^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{3h}\Big((-8+4\sqrt{6})p_0^{(j)} + (-8-3\sqrt{6})p_1^{(j)} + (16-\sqrt{6})p_2^{(j)}\Big)\\
\frac{du_{0}^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{3h}\Big(-p_0^{(j)} + (-13+7\sqrt{6})p_1^{(j)} + (-13-7\sqrt{6})p_2^{(j)} + p_0^{(j+1)}\\
&\qquad + (13-7\sqrt{6})p_{-1}^{(j+1)} + (13+7\sqrt{6})p_{-2}^{(j+1)}\Big)\\
\frac{du_{1}^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{3h}\Big((8-4\sqrt{6})p_0^{(j+1)} + (8+3\sqrt{6})p_{-1}^{(j+1)} + (-16+\sqrt{6})p_{-2}^{(j+1)}\Big)\\
\frac{du_{2}^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{3h}\Big((8+4\sqrt{6})p_0^{(j+1)} + (-16-\sqrt{6})p_{-1}^{(j+1)} + (8-3\sqrt{6})p_{-2}^{(j+1)}\Big)\\
\frac{dp_{-2}^{(j)}}{dt} &= \frac{1}{3h}\Big((-8-4\sqrt{6})u_0^{(j-\frac{1}{2})} + (16+\sqrt{6})u_1^{(j-\frac{1}{2})} + (-8+3\sqrt{6})u_2^{(j-\frac{1}{2})}\Big)\\
\frac{dp_{-1}^{(j)}}{dt} &= \frac{1}{3h}\Big((-8+4\sqrt{6})u_0^{(j-\frac{1}{2})} + (-8-3\sqrt{6})u_1^{(j-\frac{1}{2})} + (16-\sqrt{6})u_2^{(j-\frac{1}{2})}\Big)\\
\frac{dp_{0}^{(j)}}{dt} &= \frac{1}{3h}\Big(-u_0^{(j-\frac{1}{2})} + (-13+7\sqrt{6})u_1^{(j-\frac{1}{2})} + (-13-7\sqrt{6})u_2^{(j-\frac{1}{2})} + u_0^{(j+\frac{1}{2})}\\
&\qquad + (13-7\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (13+7\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\\
\frac{dp_{1}^{(j)}}{dt} &= \frac{1}{3h}\Big((8-4\sqrt{6})u_0^{(j+\frac{1}{2})} + (8+3\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (-16+\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\\
\frac{dp_{2}^{(j)}}{dt} &= \frac{1}{3h}\Big((8+4\sqrt{6})u_0^{(j+\frac{1}{2})} + (-16-\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (8-3\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big).
\end{aligned}\tag{11}$$

To do so, we first calculate mass integrals as follows:

$$\begin{aligned}
\int \Big(\varphi_{-2}^{(j+\frac{1}{2})}\Big)^2 dx &= \frac{(16-\sqrt{6})h}{72},\\
\int \Big(\varphi_{-1}^{(j+\frac{1}{2})}\Big)^2 dx &= \frac{(16+\sqrt{6})h}{72},\\
\int \Big(\varphi_{0}^{(j+\frac{1}{2})}\Big)^2 dx &= \frac{h}{9},\\
\int \Big(\varphi_{1}^{(j+\frac{1}{2})}\Big)^2 dx &= \frac{(16+\sqrt{6})h}{72},\\
\int \Big(\varphi_{2}^{(j+\frac{1}{2})}\Big)^2 dx &= \frac{(16-\sqrt{6})h}{72}.
\end{aligned}$$

To derive the first equation in (11), we take

$$\phi = \varphi_{-2}^{(j+\frac{1}{2})}$$

in the equation (3). Then we have

$$\int_{x_j}^{x_{j+\frac{1}{2}}} \frac{du^{(j+\frac{1}{2})}}{dt}\varphi_{-2}^{(j+\frac{1}{2})}dx = \frac{(16-\sqrt{6})h}{72}\frac{du_{-2}^{(j+\frac{1}{2})}}{dt}$$

for the left hand side of (3). For the right hand side, we need to compute the following.

$$-\left(p^{(j)}\varphi_{-2}^{(j+\frac{1}{2})}\right)(x_j) - \int_{x_j}^{x_{j+\frac{1}{2}}} p^{(j)}\left(\varphi_{-2}^{(j+\frac{1}{2})}\right)'dx.$$

By Gauss-Radau quadrature rule, it suffices to compute

$$\begin{aligned}
\varphi_{-2}^{(j+\frac{1}{2})}(x_j) &= \frac{2+3\sqrt{6}}{6},\\
\left(\varphi_{-2}^{(j+\frac{1}{2})}\right)'(x_j) &= \frac{8-13\sqrt{6}}{3h},\\
\left(\varphi_{-2}^{(j+\frac{1}{2})}\right)'(x_j+\gamma_1) &= \frac{-16+\sqrt{6}}{3h},\\
\left(\varphi_{-2}^{(j+\frac{1}{2})}\right)'(x_j+\gamma_2) &= \frac{8-3\sqrt{6}}{3h}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&-\left(p^{(j)}\varphi_{-2}^{(j+\frac{1}{2})}\right)(x_j) - \int_{x_j}^{x_{j+\frac{1}{2}}} p^{(j)}\left(\varphi_{-2}^{(j+\frac{1}{2})}\right)'dx\\
&= -p_0^{(j)}\frac{2+3\sqrt{6}}{6}\\
&\quad -\left(p_0^{(j)}\frac{8-13\sqrt{6}}{3h}\frac{h}{18} + p_1^{(j)}\frac{-16+\sqrt{6}}{3h}\frac{(16+\sqrt{6})h}{72} + p_2^{(j)}\frac{8-3\sqrt{6}}{3h}\frac{(16-\sqrt{6})h}{72}\right)\\
&= \frac{-13-7\sqrt{6}}{27}p_0^{(j)} + \frac{125}{108}p_1^{(j)} + \frac{-73+28\sqrt{6}}{108}p_2^{(j)}.
\end{aligned}$$

Thus,

$$\frac{du_{-2}^{(j+\frac{1}{2})}}{dt} = \frac{c^2}{3h}\Big((-8-4\sqrt{6})p_0^{(j)} + (16+\sqrt{6})p_1^{(j)} + (-8+3\sqrt{6})p_2^{(j)}\Big).$$

For the second equation in (11), we first compute

$$\begin{aligned}
\varphi_{-1}^{(j+\frac{1}{2})}(x_j) &= \frac{2-3\sqrt{6}}{6},\\
\left(\varphi_{-1}^{(j+\frac{1}{2})}\right)'(x_j) &= \frac{8+13\sqrt{6}}{3h},\\
\left(\varphi_{-1}^{(j+\frac{1}{2})}\right)'(x_j+\gamma_1) &= \frac{8+3\sqrt{6}}{3h},\\
\left(\varphi_{-1}^{(j+\frac{1}{2})}\right)'(x_j+\gamma_2) &= \frac{-16-\sqrt{6}}{3h}.
\end{aligned}$$

Then, by taking $\phi = \varphi_{-1}^{(j+\frac{1}{2})}$ in (3), we have

$$
\begin{aligned}
\frac{du_{-1}^{(j+\frac{1}{2})}}{dt}\frac{(16+\sqrt{6})h}{72} &= c^2\Big\{-\Big(p^{(j)}\varphi_{-1}^{(j+\frac{1}{2})}\Big)(x_j) - \int_{x_j}^{x_{j+\frac{1}{2}}} p^{(j)}\Big(\varphi_{-1}^{(j+\frac{1}{2})}\Big)' dx\Big\} \\
&= c^2\Big\{-p_0^{(j)}\frac{2-3\sqrt{6}}{6} \\
&\qquad - \Big(p_0^{(j)}\frac{8+13\sqrt{6}}{3h}\frac{h}{18} + p_1^{(j)}\frac{8+3\sqrt{6}}{3h}\frac{(16+\sqrt{6})h}{72} \\
&\qquad + p_2^{(j)}\frac{-16-\sqrt{6}}{3h}\frac{(16-\sqrt{6})h}{72}\Big)\Big\} \\
&= c^2\Big(\frac{-13+7\sqrt{6}}{27}p_0^{(j)} - \frac{73+28\sqrt{6}}{108}p_1^{(j)} + \frac{125}{108}p_2^{(j)}\Big).
\end{aligned}
$$

Therefore, we have

$$
\frac{du_{-1}^{(j+\frac{1}{2})}}{dt} = \frac{c^2}{h}\Big(\frac{-8+4\sqrt{6}}{3}p_0^{(j)} + \frac{-8-3\sqrt{6}}{3}p_1^{(j)} + \frac{16-\sqrt{6}}{3}p_2^{(j)}\Big).
$$

To derive the third equation in (11), we first compute

$$
\begin{aligned}
&\varphi_0^{(j+\frac{1}{2})}(x_j) = \frac{1}{3}, \\
&\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'(x_j) = \frac{-16}{3h}, \\
&\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'(x_j+\gamma_1) = \frac{8-4\sqrt{6}}{3h}, \\
&\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'(x_j+\gamma_2) = \frac{8+4\sqrt{6}}{3h}, \\
&\varphi_0^{(j+\frac{1}{2})}(x_{j+1}) = \frac{1}{3}, \\
&\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'(x_{j+1}) = \frac{16}{3h}, \\
&\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'(x_{j+1}-\gamma_1) = \frac{-8+4\sqrt{6}}{3h}, \\
&\Big(\varphi_0^{(j+\frac{1}{2})}\Big)'(x_{j+1}-\gamma_2) = \frac{-8-4\sqrt{6}}{3h}.
\end{aligned}
$$

Then, by taking $\phi = \varphi_0^{(j+\frac{1}{2})}$ in (3), we have

$$
\begin{aligned}
\frac{du_0^{(j+\frac{1}{2})}}{dt}\frac{h}{9} &= c^2\Big\{\Big(p^{(j+1)}\varphi_0^{(j+\frac{1}{2})}\Big)(x_{j+1}) - \Big(p^{(j)}\varphi_0^{(j+\frac{1}{2})}\Big)(x_j) \\
&\quad - \int_{x_j}^{x_{j+\frac{1}{2}}} p^{(j)}\Big(\varphi_0^{(j+\frac{1}{2})}\Big)' dx - \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p^{(j+1)}\Big(\varphi_0^{(j+\frac{1}{2})}\Big)' dx\Big\} \\
&= c^2\Big\{p_0^{(j+1)}\frac{1}{3} - p_0^{(j)}\frac{1}{3} \\
&\quad - \Big(p_0^{(j)}\frac{-16}{3h}\frac{h}{18} + p_1^{(j)}\frac{8-4\sqrt{6}}{3h}\frac{(16+\sqrt{6})h}{72} + p_2^{(j)}\frac{8+4\sqrt{6}}{3h}\frac{(16-\sqrt{6})h}{72}\Big) \\
&\quad - \Big(p_0^{(j+1)}\frac{16}{3h}\frac{h}{18} + p_{-1}^{(j+1)}\frac{-8+4\sqrt{6}}{3h}\frac{(16+\sqrt{6})h}{72} \\
&\qquad + p_{-2}^{(j+1)}\frac{-8-4\sqrt{6}}{3h}\frac{(16-\sqrt{6})h}{72}\Big)\Big\} \\
&= c^2\Big(\frac{-1}{27}p_0^{(j)} + \frac{-13+7\sqrt{6}}{27}p_1^{(j)} + \frac{-13-7\sqrt{6}}{27}p_2^{(j)} \\
&\quad + \frac{1}{27}p_0^{(j+1)} + \frac{13-7\sqrt{6}}{27}p_{-1}^{(j+1)} + \frac{13+7\sqrt{6}}{27}p_{-2}^{(j+1)}\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{du_0^{(j+\frac{1}{2})}}{dt} &= \frac{c^2}{h}\Big(\frac{-1}{3}p_0^{(j)} + \frac{-13+7\sqrt{6}}{3}p_1^{(j)} + \frac{-13-7\sqrt{6}}{3}p_2^{(j)} \\
&\quad + \frac{1}{3}p_0^{(j+1)} + \frac{13-7\sqrt{6}}{3}p_{-1}^{(j+1)} + \frac{13+7\sqrt{6}}{3}p_{-2}^{(j+1)}\Big).
\end{aligned}
$$

For the fourth equation in (11), we first compute

$$
\begin{aligned}
\varphi_1^{(j+\frac{1}{2})}(x_{j+1}) &= \frac{2-3\sqrt{6}}{6}, \\
\Big(\varphi_1^{(j+\frac{1}{2})}\Big)'(x_{j+1}) &= \frac{-8-13\sqrt{6}}{3h}, \\
\Big(\varphi_1^{(j+\frac{1}{2})}\Big)'(x_{j+1}-\gamma_1) &= \frac{-8-3\sqrt{6}}{3h}, \\
\Big(\varphi_1^{(j+\frac{1}{2})}\Big)'(x_{j+1}-\gamma_2) &= \frac{16+\sqrt{6}}{3h}.
\end{aligned}
$$

Then, by taking $\phi = \varphi_1^{(j+\frac{1}{2})}$ in (3), we have

$$
\begin{aligned}
\frac{du_1^{(j+\frac{1}{2})}}{dt}\frac{(16+\sqrt{6})h}{72} &= c^2\Big\{\Big(p^{(j+1)}\varphi_1^{(j+\frac{1}{2})}\Big)(x_{j+1}) - \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p^{(j+1)}\Big(\varphi_1^{(j+\frac{1}{2})}\Big)' dx\Big\} \\
&= c^2\Big\{p_0^{(j+1)}\frac{2-3\sqrt{6}}{6} \\
&\quad - \Big(p_0^{(j+1)}\frac{-(8+13\sqrt{6})}{3h}\frac{h}{18} + p_{-1}^{(j+1)}\frac{-(8+3\sqrt{6})}{3h}\frac{(16+\sqrt{6})h}{72} \\
&\qquad + p_{-2}^{(j+1)}\frac{16+\sqrt{6}}{3h}\frac{(16-\sqrt{6})h}{72}\Big)\Big\} \\
&= c^2\Big(\frac{13-7\sqrt{6}}{27}p_0^{(j+1)} + \frac{73+28\sqrt{6}}{108}p_{-1}^{(j+1)} - \frac{125}{108}p_{-2}^{(j+1)}\Big).
\end{aligned}
$$

Thus,

$$\frac{du_1^{(j+\frac{1}{2})}}{dt} = \frac{c^2}{h}\Big(\frac{8-4\sqrt{6}}{3}p_0^{(j+1)} + \frac{8+3\sqrt{6}}{3}p_{-1}^{(j+1)} + \frac{-16+\sqrt{6}}{3}p_{-2}^{(j+1)}\Big).$$

For the fifth equation in (11), we first compute

$$\begin{aligned}
\varphi_2^{(j+\frac{1}{2})}(x_{j+1}) &= \frac{2+3\sqrt{6}}{6},\\
\Big(\varphi_2^{(j+\frac{1}{2})}\Big)'(x_{j+1}) &= \frac{-8+13\sqrt{6}}{3h},\\
\Big(\varphi_2^{(j+\frac{1}{2})}\Big)'(x_{j+1}-\gamma_1) &= \frac{16-\sqrt{6}}{3h},\\
\Big(\varphi_2^{(j+\frac{1}{2})}\Big)'(x_{j+1}-\gamma_2) &= \frac{-8+3\sqrt{6}}{3h}.
\end{aligned}$$

Then, by taking $\phi = \varphi_2^{(j+\frac{1}{2})}$ in (3), we have

$$\begin{aligned}
\frac{du_2^{(j+\frac{1}{2})}}{dt}\frac{(16-\sqrt{6})h}{72} &= c^2\Big\{\Big(p^{(j+1)}\varphi_2^{(j+\frac{1}{2})}\Big)(x_{j+1}) - \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p^{(j+1)}\Big(\varphi_2^{(j+\frac{1}{2})}\Big)'dx\Big\}\\
&= c^2\Big\{\frac{2+3\sqrt{6}}{6}p_0^{(j+1)}\\
&\quad - \Big(p_0^{(j+1)}\frac{-8+13\sqrt{6}}{3h}\frac{h}{18} + p_{-1}^{(j+1)}\frac{16-\sqrt{6}}{3h}\frac{(16+\sqrt{6})h}{72}\\
&\quad + p_{-2}^{(j+1)}\frac{-8+3\sqrt{6}}{3h}\frac{(16-\sqrt{6})h}{72}\Big)\Big\}\\
&= c^2\Big(\frac{13+7\sqrt{6}}{27}p_0^{(j+1)} - \frac{125}{108}p_{-1}^{(j+1)} + \frac{73-28\sqrt{6}}{108}p_{-2}^{(j+1)}\Big).
\end{aligned}$$

Therefore, we have

$$\frac{du_2^{(j+\frac{1}{2})}}{dt} = \frac{c^2}{h}\Big(\frac{8+4\sqrt{6}}{3}p_0^{(j+1)} + \frac{-16-\sqrt{6}}{3}p_{-1}^{(j+1)} + \frac{8-3\sqrt{6}}{3}p_{-2}^{(j+1)}\Big).$$

The last 5 equations in (11) can be obtained in a similar way.

For the dispersion analysis of the scheme (11), we consider solution of the form

$$\begin{aligned}
u_{-2}^{(q)} &= \alpha_{-2}e^{i[(q-a)hk-\omega t]},\\
u_{-1}^{(q)} &= \alpha_{-1}e^{i[(q-b)hk-\omega t]},\\
u_{0}^{(q)} &= \alpha_{0}e^{i(qhk-\omega t)},\\
u_{1}^{(q)} &= \alpha_{1}e^{i[(q+b)hk-\omega t]},\\
u_{2}^{(q)} &= \alpha_{2}e^{i[(q+a)hk-\omega t]},\\
\text{where } a &= \frac{6+\sqrt{6}}{20}, b = \frac{6-\sqrt{6}}{20},\\
\text{and } &\alpha_{-2},\alpha_{-1},\alpha_0,\alpha_1,\alpha_2 \in \mathbb{C}.
\end{aligned}$$

We will use these formula in (11).

From the first equation in (11), we have

$$-i\omega u_{-2}^{(j+\frac{1}{2})} = \frac{c^2}{3h}\Big((-8-4\sqrt{6})p_0^{(j)} + (16+\sqrt{6})p_1^{(j)} + (-8+3\sqrt{6})p_2^{(j)}\Big).$$

Taking time derivatives,

$$
\begin{aligned}
&-\omega^2 u_{-2}^{(j+\frac{1}{2})}\\
&= \frac{c^2}{3h}\Big((-8-4\sqrt{6})\frac{dp_0^{(j)}}{dt} + (16+\sqrt{6})\frac{dp_1^{(j)}}{dt} + (-8+3\sqrt{6})\frac{dp_2^{(j)}}{dt}\Big)\\
&= \frac{c^2}{9h^2}\Big\{(-8-4\sqrt{6})\Big(-u_0^{(j-\frac{1}{2})} + (-13+7\sqrt{6})u_1^{(j-\frac{1}{2})} + (-13-7\sqrt{6})u_2^{(j-\frac{1}{2})}\\
&\quad + u_0^{(j+\frac{1}{2})} + (13-7\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (13+7\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\\
&\quad + (16+\sqrt{6})\Big((8-4\sqrt{6})u_0^{(j+\frac{1}{2})} + (8+3\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (-16+\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\\
&\quad + (-8+3\sqrt{6})\Big((8+4\sqrt{6})u_0^{(j+\frac{1}{2})} + (-16-\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (8-3\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\Big\}\\
&= \frac{c^2}{9h^2}\Big((8+4\sqrt{6})u_0^{(j-\frac{1}{2})} + (-64-4\sqrt{6})u_1^{(j-\frac{1}{2})} + (272+108\sqrt{6})u_2^{(j-\frac{1}{2})}\\
&\quad + (104-68\sqrt{6})u_0^{(j+\frac{1}{2})} + (320+20\sqrt{6})u_{-1}^{(j+\frac{1}{2})} + (-640-60\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)
\end{aligned}
$$

Using the definition of u,

$$
\begin{aligned}
&-\omega^2\alpha_{-2}e^{ihk(j+\frac{1}{2}-a)}\\
&= \frac{c^2}{9h^2}\Big((8+4\sqrt{6})\alpha_0 e^{ihk(j-\frac{1}{2})} + (-64-4\sqrt{6})\alpha_1 e^{ihk(j-\frac{1}{2}+b)}\\
&\quad + (272+108\sqrt{6})\alpha_2 e^{ihk(j-\frac{1}{2}+a)} + (104-68\sqrt{6})\alpha_0 e^{ihk(j+\frac{1}{2})}\\
&\quad + (320+20\sqrt{6})\alpha_{-1}e^{ihk(j+\frac{1}{2}-b)} + (-640-60\sqrt{6})\alpha_{-2}e^{ihk(j+\frac{1}{2}-a)}\Big)\\
&= \frac{c^2}{9h^2}e^{ihk(j+\frac{1}{2}-a)}\Big((8+4\sqrt{6})\alpha_0 e^{ihk(a-1)} + (-64-4\sqrt{6})\alpha_1 e^{ihk(-1+b+a)}\\
&\quad + (272+108\sqrt{6})\alpha_2 e^{ihk(2a-1)} + (104-68\sqrt{6})\alpha_0 e^{ihka}\\
&\quad + (320+20\sqrt{6})\alpha_{-1}e^{ihk(a-b)} + (-640-60\sqrt{6})\alpha_{-2}\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\omega^2\alpha_{-2}\\
&= -\frac{c^2}{9h^2}\Big\{(-640-60\sqrt{6})\alpha_{-2} + (320+20\sqrt{6})e^{ihk(a-b)}\alpha_{-1}\\
&\quad + \Big((8+4\sqrt{6})e^{ihk(a-1)} + (104-68\sqrt{6})e^{ihka}\Big)\alpha_0\\
&\quad + (-64-4\sqrt{6})e^{ihk(a+b-1)}\alpha_1 + (272+108\sqrt{6})e^{ihk(2a-1)}\alpha_2\Big\}.
\end{aligned}
$$

From the second equation in (11), we have

$$
-\omega^2 u_{-1}^{(j+\frac{1}{2})} = \frac{c^2}{3h}\Big((-8+4\sqrt{6})\frac{dp_0^{(j)}}{dt} + (-8-3\sqrt{6})\frac{dp_1^{(j)}}{dt} + (16-\sqrt{6})\frac{dp_2^{(j)}}{dt}\Big)
$$

Then

$$
\begin{aligned}
&-\omega^2 u_{-1}^{(j+\frac{1}{2})}\\
=&\frac{c^2}{9h^2}\Big\{(-8+4\sqrt{6})\Big(-u_0^{(j-\frac{1}{2})}+(-13+7\sqrt{6})u_1^{(j-\frac{1}{2})}+(-13-7\sqrt{6})u_2^{(j-\frac{1}{2})}\\
&+u_0^{(j+\frac{1}{2})}+(13-7\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(13+7\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\\
&+(-8-3\sqrt{6})\Big((8-4\sqrt{6})u_0^{(j+\frac{1}{2})}+(8+3\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(-16+\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\\
&+(16-\sqrt{6})\Big((8+4\sqrt{6})u_0^{(j+\frac{1}{2})}+(-16-\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(8-3\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big)\Big\}\\
=&\frac{c^2}{9h^2}\Big((8-4\sqrt{6})u_0^{(j-\frac{1}{2})}+(272-108\sqrt{6})u_1^{(j-\frac{1}{2})}+(-64+4\sqrt{6})u_2^{(j-\frac{1}{2})}\\
&+(104+68\sqrt{6})u_0^{(j+\frac{1}{2})}+(-640+60\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(320-20\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big).
\end{aligned}
$$

Using the definition of u,

$$
\begin{aligned}
&-\omega^2\alpha_{-1}e^{ihk(j+\frac{1}{2}-b)}\\
=&\frac{c^2}{9h^2}\Big((8-4\sqrt{6})\alpha_0 e^{ihk(j-\frac{1}{2})}+(272-108\sqrt{6})\alpha_1 e^{ihk(j-\frac{1}{2}+b)}\\
&+(-64+4\sqrt{6})\alpha_2 e^{ihk(j-\frac{1}{2}+a)}+(104+68\sqrt{6})\alpha_0 e^{ihk(j+\frac{1}{2})}\\
&+(-640+60\sqrt{6})\alpha_{-1}e^{ihk(j+\frac{1}{2}-b)}+(320-20\sqrt{6})\alpha_{-2}e^{ihk(j+\frac{1}{2}-a)}\Big)\\
=&\frac{c^2}{9h^2}e^{ihk(j+\frac{1}{2}-b)}\Big((8-4\sqrt{6})\alpha_0 e^{ihk(b-1)}+(272-108\sqrt{6})\alpha_1 e^{ihk(2b-1)}\\
&+(-64+4\sqrt{6})\alpha_2 e^{ihk(a+b-1)}+(104+68\sqrt{6})\alpha_0 e^{ihkb}\\
&+(-640+60\sqrt{6})\alpha_{-1}+(320-20\sqrt{6})\alpha_{-2}e^{ihk(b-a)}\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\omega^2\alpha_{-1}=&-\frac{c^2}{9h^2}\Big\{(320-20\sqrt{6})e^{ihk(b-a)}\alpha_{-2}+(-640+60\sqrt{6})\alpha_{-1}\\
&+\Big((8-4\sqrt{6})e^{ihk(b-1)}+(104+68\sqrt{6})e^{ihkb}\Big)\alpha_0\\
&+(272-108\sqrt{6})e^{ihk(2b-1)}\alpha_1+(-64+4\sqrt{6})e^{ihk(a+b-1)}\alpha_2\Big\}.
\end{aligned}
$$

From the third equation in (11), we have

$$
\begin{aligned}
-\omega^2 u_0^{(j+\frac{1}{2})}=&\frac{c^2}{3h}\Big(-\frac{dp_0^{(j)}}{dt}+(-13+7\sqrt{6})\frac{dp_1^{(j)}}{dt}+(-13-7\sqrt{6})\frac{dp_2^{(j)}}{dt}\\
&+\frac{dp_0^{(j+1)}}{dt}+(13-7\sqrt{6})\frac{dp_{-1}^{(j+1)}}{dt}+(13+7\sqrt{6})\frac{dp_{-2}^{(j+1)}}{dt}\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
& -\omega^2 u_0^{(j+\frac{1}{2})} \\
={}& \frac{c^2}{9h^2}\Big\{-\Big(-u_0^{(j-\frac{1}{2})}+(-13+7\sqrt{6})u_1^{(j-\frac{1}{2})}+(-13-7\sqrt{6})u_2^{(j-\frac{1}{2})} \\
& +u_0^{(j+\frac{1}{2})}+(13-7\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(13+7\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big) \\
& +(-13+7\sqrt{6})\Big((8-4\sqrt{6})u_0^{(j+\frac{1}{2})}+(8+3\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(-16+\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big) \\
& +(-13-7\sqrt{6})\Big((8+4\sqrt{6})u_0^{(j+\frac{1}{2})}+(-16-\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(8-3\sqrt{6})u_{-2}^{(j+\frac{1}{2})}\Big) \\
& +\Big(-u_0^{(j+\frac{1}{2})}+(-13+7\sqrt{6})u_1^{(j+\frac{1}{2})}+(-13-7\sqrt{6})u_2^{(j+\frac{1}{2})} \\
& +u_0^{(j+\frac{3}{2})}+(13-7\sqrt{6})u_{-1}^{(j+\frac{3}{2})}+(13+7\sqrt{6})u_{-2}^{(j+\frac{3}{2})}\Big) \\
& +(13-7\sqrt{6})\Big((-8+4\sqrt{6})u_0^{(j+\frac{1}{2})}+(-8-3\sqrt{6})u_1^{(j+\frac{1}{2})}+(16-\sqrt{6})u_2^{(j+\frac{1}{2})}\Big) \\
& +(13+7\sqrt{6})\Big((-8-4\sqrt{6})u_0^{(j+\frac{1}{2})}+(16+\sqrt{6})u_1^{(j+\frac{1}{2})}+(-8+3\sqrt{6})u_2^{(j+\frac{1}{2})}\Big)\Big\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& -\omega^2 u_0^{(j+\frac{1}{2})} \\
={}& \frac{c^2}{9h^2}\Big(u_0^{(j-\frac{1}{2})}+(13-7\sqrt{6})u_1^{(j-\frac{1}{2})}+(13+7\sqrt{6})u_2^{(j-\frac{1}{2})} \\
& +(-1090)u_0^{(j+\frac{1}{2})}+(259+149\sqrt{6})u_{-1}^{(j+\frac{1}{2})}+(259-149\sqrt{6})u_{-2}^{(j+\frac{1}{2})} \\
& +(259+149\sqrt{6})u_1^{(j+\frac{1}{2})}+(259-149\sqrt{6})u_2^{(j+\frac{1}{2})} \\
& +u_0^{(j+\frac{3}{2})}+(13-7\sqrt{6})u_{-1}^{(j+\frac{3}{2})}+(13+7\sqrt{6})u_{-2}^{(j+\frac{3}{2})}\Big).
\end{aligned}
$$

Using the definition of u,

$$
\begin{aligned}
& -\omega^2\alpha_0 e^{ihk(j+\frac{1}{2})} \\
={}& \frac{c^2}{9h^2}\Big(\alpha_0 e^{ihk(j-\frac{1}{2})}+(13-7\sqrt{6})\alpha_1 e^{ihk(j-\frac{1}{2}+b)} \\
& +(13+7\sqrt{6})\alpha_2 e^{ihk(j-\frac{1}{2}+a)}+(-1090)\alpha_0 e^{ihk(j+\frac{1}{2})} \\
& +(259+149\sqrt{6})\alpha_{-1}e^{ihk(j+\frac{1}{2}-b)}+(259-149\sqrt{6})\alpha_{-2}e^{ihk(j+\frac{1}{2}-a)} \\
& +(259+149\sqrt{6})\alpha_1 e^{ihk(j+\frac{1}{2}+b)}+(259-149\sqrt{6})\alpha_2 e^{ihk(j+\frac{1}{2}+a)} \\
& +\alpha_0 e^{ihk(j+\frac{3}{2})}+(13-7\sqrt{6})\alpha_{-1}e^{ihk(j+\frac{3}{2}-b)} \\
& +(13+7\sqrt{6})\alpha_{-2}e^{ihk(j+\frac{3}{2}-a)}\Big) \\
={}& \frac{c^2}{9h^2}e^{ihk(j+\frac{1}{2})}\Big(\alpha_0 e^{ihk(-1)}+(13-7\sqrt{6})\alpha_1 e^{ihk(b-1)} \\
& +(13+7\sqrt{6})\alpha_2 e^{ihk(a-1)}+(-1090)\alpha_0 \\
& +(259+149\sqrt{6})\alpha_{-1}e^{ihk(-b)}+(259-149\sqrt{6})\alpha_{-2}e^{ihk(-a)} \\
& +(259+149\sqrt{6})\alpha_1 e^{ihkb}+(259-149\sqrt{6})\alpha_2 e^{ihka} \\
& +\alpha_0 e^{ihk}+(13-7\sqrt{6})\alpha_{-1}e^{ihk(1-b)}+(13+7\sqrt{6})\alpha_{-2}e^{ihk(1-a)}\Big).
\end{aligned}
$$

Consequently,

$$\begin{aligned}
\omega^2\alpha_0 = &-\frac{c^2}{9h^2}\Big\{\Big((259-149\sqrt{6})e^{ihk(-a)}+(13+7\sqrt{6})e^{ihk(1-a)}\Big)\alpha_{-2}\\
&+\Big((259+149\sqrt{6})e^{ihk(-b)}+(13-7\sqrt{6})e^{ihk(1-b)}\Big)\alpha_{-1}\\
&+\Big(e^{ihk(-1)}-1090+e^{ihk}\Big)\alpha_0\\
&+\Big((13-7\sqrt{6})e^{ihk(b-1)}+(259+149\sqrt{6})e^{ihkb}\Big)\alpha_1\\
&+\Big((13+7\sqrt{6})e^{ihk(a-1)}+(259-149\sqrt{6})e^{ihka}\Big)\alpha_2\Big\}.
\end{aligned}$$

From the fourth equation in (11), we have

$$\begin{aligned}
&-\omega^2 u_1^{(j+\frac{1}{2})}\\
=&\frac{c^2}{3h}\Big((8-4\sqrt{6})\frac{dp_0^{(j+1)}}{dt}+(8+3\sqrt{6})\frac{dp_{-1}^{(j+1)}}{dt}+(-16+\sqrt{6})\frac{dp_{-2}^{(j+1)}}{dt}\Big)\\
=&\frac{c^2}{9h^2}\Big\{(8-4\sqrt{6})\Big(-u_0^{(j+\frac{1}{2})}+(-13+7\sqrt{6})u_1^{(j+\frac{1}{2})}+(-13-7\sqrt{6})u_2^{(j+\frac{1}{2})}\\
&+u_0^{(j+\frac{3}{2})}+(13-7\sqrt{6})u_{-1}^{(j+\frac{3}{2})}+(13+7\sqrt{6})u_{-2}^{(j+\frac{3}{2})}\Big)\\
&+(8+3\sqrt{6})\Big((-8+4\sqrt{6})u_0^{(j+\frac{1}{2})}+(-8-3\sqrt{6})u_1^{(j+\frac{1}{2})}+(16-\sqrt{6})u_2^{(j+\frac{1}{2})}\Big)\\
&+(-16+\sqrt{6})\Big((-8-4\sqrt{6})u_0^{(j+\frac{1}{2})}+(16+\sqrt{6})u_1^{(j+\frac{1}{2})}+(-8+3\sqrt{6})u_2^{(j+\frac{1}{2})}\Big)\Big\}\\
=&\frac{c^2}{9h^2}\Big((104+68\sqrt{6})u_0^{(j+\frac{1}{2})}+(-640+60\sqrt{6})u_1^{(j+\frac{1}{2})}+(320-20\sqrt{6})u_2^{(j+\frac{1}{2})}\\
&+(8-4\sqrt{6})u_0^{(j+\frac{3}{2})}+(272-108\sqrt{6})u_{-1}^{(j+\frac{3}{2})}+(-64+4\sqrt{6})u_{-2}^{(j+\frac{3}{2})}\Big).
\end{aligned}$$

Then

$$\begin{aligned}
&-\omega^2\alpha_1 e^{ihk(j+\frac{1}{2}+b)}\\
=&\frac{c^2}{9h^2}\Big((104+68\sqrt{6})\alpha_0 e^{ihk(j+\frac{1}{2})}+(-640+60\sqrt{6})\alpha_1 e^{ihk(j+\frac{1}{2}+b)}\\
&+(320-20\sqrt{6})\alpha_2 e^{ihk(j+\frac{1}{2}+a)}+(8-4\sqrt{6})\alpha_0 e^{ihk(j+\frac{3}{2})}\\
&+(272-108\sqrt{6})\alpha_{-1}e^{ihk(j+\frac{3}{2}-b)}+(-64+4\sqrt{6})\alpha_{-2}e^{ihk(j+\frac{3}{2}-a)}\Big)\\
=&\frac{c^2}{9h^2}e^{ihk(j+\frac{1}{2}+b)}\Big((104+68\sqrt{6})\alpha_0 e^{ihk(-b)}+(-640+60\sqrt{6})\alpha_1\\
&+(320-20\sqrt{6})\alpha_2 e^{ihk(a-b)}+(8-4\sqrt{6})\alpha_0 e^{ihk(1-b)}\\
&+(272-108\sqrt{6})\alpha_{-1}e^{ihk(1-2b)}+(-64+4\sqrt{6})\alpha_{-2}e^{ihk(1-a-b)}\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\omega^2\alpha_1\\
=&-\frac{c^2}{9h^2}\Big\{(-64+4\sqrt{6})e^{ihk(1-a-b)}\alpha_{-2}+(272-108\sqrt{6})e^{ihk(1-2b)}\alpha_{-1}\\
&+\Big((104+68\sqrt{6})e^{ihk(-b)}+(8-4\sqrt{6})e^{ihk(1-b)}\Big)\alpha_0\\
&+(-640+60\sqrt{6})\alpha_1+(320-20\sqrt{6})e^{ihk(a-b)}\alpha_2\Big\}.
\end{aligned}$$

From the fifth equation in (11), we have

$$
\begin{aligned}
&-\omega^2 u_2^{(j+\frac{1}{2})}\\
=&\frac{c^2}{3h}\Big((8+4\sqrt{6})\frac{dp_0^{(j+1)}}{dt}+(-16-\sqrt{6})\frac{dp_{-1}^{(j+1)}}{dt}+(8-3\sqrt{6})\frac{dp_{-2}^{(j+1)}}{dt}\Big)\\
=&\frac{c^2}{9h^2}\Big\{(8+4\sqrt{6})\Big(-u_0^{(j+\frac{1}{2})}+(-13+7\sqrt{6})u_1^{(j+\frac{1}{2})}+(-13-7\sqrt{6})u_2^{(j+\frac{1}{2})}\\
&+u_0^{(j+\frac{3}{2})}+(13-7\sqrt{6})u_{-1}^{(j+\frac{3}{2})}+(13+7\sqrt{6})u_{-2}^{(j+\frac{3}{2})}\Big)\\
&+(-16-\sqrt{6})\Big((-8+4\sqrt{6})u_0^{(j+\frac{1}{2})}+(-8-3\sqrt{6})u_1^{(j+\frac{1}{2})}+(16-\sqrt{6})u_2^{(j+\frac{1}{2})}\Big)\\
&+(8-3\sqrt{6})\Big((-8-4\sqrt{6})u_0^{(j+\frac{1}{2})}+(16+\sqrt{6})u_1^{(j+\frac{1}{2})}+(-8+3\sqrt{6})u_2^{(j+\frac{1}{2})}\Big)\Big\}\\
=&\frac{c^2}{9h^2}\Big((104-68\sqrt{6})u_0^{(j+\frac{1}{2})}+(320+20\sqrt{6})u_1^{(j+\frac{1}{2})}+(-640-60\sqrt{6})u_2^{(j+\frac{1}{2})}\\
&+(8+4\sqrt{6})u_0^{(j+\frac{3}{2})}+(-64-4\sqrt{6})u_{-1}^{(j+\frac{3}{2})}+(272+108\sqrt{6})u_{-2}^{(j+\frac{3}{2})}\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\omega^2\alpha_2 e^{ihk(j+\frac{1}{2}+a)}\\
=&\frac{c^2}{9h^2}\Big((104-68\sqrt{6})\alpha_0 e^{ihk(j+\frac{1}{2})}+(320+20\sqrt{6})\alpha_1 e^{ihk(j+\frac{1}{2}+b)}\\
&+(-640-60\sqrt{6})\alpha_2 e^{ihk(j+\frac{1}{2}+a)}+(8+4\sqrt{6})\alpha_0 e^{ihk(j+\frac{3}{2})}\\
&+(-64-4\sqrt{6})\alpha_{-1}e^{ihk(j+\frac{3}{2}-b)}+(272+108\sqrt{6})\alpha_{-2}e^{ihk(j+\frac{3}{2}-a)}\Big)\\
=&\frac{c^2}{9h^2}e^{ihk(j+\frac{1}{2}+a)}\Big((104-68\sqrt{6})\alpha_0 e^{ihk(-a)}+(320+20\sqrt{6})\alpha_1 e^{ihk(b-a)}\\
&+(-640-60\sqrt{6})\alpha_2+(8+4\sqrt{6})\alpha_0 e^{ihk(1-a)}\\
&+(-64-4\sqrt{6})\alpha_{-1}e^{ihk(1-a-b)}+(272+108\sqrt{6})\alpha_{-2}e^{ihk(1-2a)}\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\omega^2\alpha_2=&-\frac{c^2}{9h^2}\Big\{(272+108\sqrt{6})e^{ihk(1-2a)}\alpha_{-2}+(-64-4\sqrt{6})e^{ihk(1-a-b)}\alpha_{-1}\\
&+\Big((104-68\sqrt{6})e^{ihk(-a)}+(8+4\sqrt{6})e^{ihk(1-a)}\Big)\alpha_0\\
&+(320+20\sqrt{6})e^{ihk(b-a)}\alpha_1+(-640-60\sqrt{6})\alpha_2\Big\}.
\end{aligned}
$$

Combining the above 5 equations, we obtain the following eigenvalue problem:

$$
-\frac{c^2}{9h^2}A\begin{pmatrix}\alpha_{-2}\\ \alpha_{-1}\\ \alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix}=\omega^2\begin{pmatrix}\alpha_{-2}\\ \alpha_{-1}\\ \alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix}
$$

where the 5×5 matrix $A = (a_{ij})$ is defined by

$$
\begin{aligned}
a_{11} &= -20(32 + 3\sqrt{6}) \\
a_{12} &= 20(16 + \sqrt{6})e^{ihk(\frac{1}{5}\sqrt{\frac{3}{2}})} \\
a_{13} &= 4(2 + \sqrt{6})e^{ihk(\frac{1}{20}(-14+\sqrt{6}))} + (104 - 68\sqrt{6})e^{ihk(\frac{1}{20}(6+\sqrt{6}))} \\
a_{14} &= -4(16 + \sqrt{6})e^{ihk(\frac{-2}{5})} \\
a_{15} &= 4(68 + 27\sqrt{6})e^{ihk(\frac{1}{10}(-4+\sqrt{6}))} \\
a_{21} &= -20(-16 + \sqrt{6})e^{ihk(\frac{-1}{5}\sqrt{\frac{3}{2}})} \\
a_{22} &= (-640 + 60\sqrt{6}) \\
a_{23} &= 4(26 + 17\sqrt{6})e^{ihk(\frac{-1}{20}(-6+\sqrt{6}))} - 4(-2 + \sqrt{6})e^{ihk(\frac{-1}{20}(14+\sqrt{6}))} \\
a_{24} &= -4(-68 + 27\sqrt{6})e^{ihk(\frac{-1}{10}(4+\sqrt{6}))} \\
a_{25} &= 4(-16 + \sqrt{6})e^{ihk(\frac{-2}{5})} \\
a_{31} &= e^{ihk(\frac{-1}{20}(6+\sqrt{6}))}(259 - 149\sqrt{6} + (13 + 7\sqrt{6})e^{ihk}) \\
a_{32} &= e^{ihk(\frac{1}{20}(-6+\sqrt{6}))}(259 + 149\sqrt{6} + (13 - 7\sqrt{6})e^{ihk}) \\
a_{33} &= -1090 + e^{-ihk} + e^{ihk} \\
a_{34} &= (259 + 149\sqrt{6})e^{ihk(\frac{-1}{20}(-6+\sqrt{6}))} + (13 - 7\sqrt{6})e^{ihk(\frac{-1}{20}(14+\sqrt{6}))} \\
a_{35} &= (13 + 7\sqrt{6})e^{ihk(\frac{1}{20}(-14+\sqrt{6}))} + (259 - 149\sqrt{6})e^{ihk(\frac{1}{20}(6+\sqrt{6}))} \\
a_{41} &= 4(-16 + \sqrt{6})e^{ihk(\frac{2}{5})} \\
a_{42} &= -4(-68 + 27\sqrt{6})e^{ihk(\frac{1}{10}(4+\sqrt{6}))} \\
a_{43} &= e^{ihk(\frac{1}{20}(-6+\sqrt{6}))}\{104 + 68\sqrt{6} - 4(-2 + \sqrt{6})e^{ihk}\}) \\
a_{44} &= -640 + 60\sqrt{6} \\
a_{45} &= -20(-16 + \sqrt{6})e^{ihk(\frac{1}{5}\sqrt{\frac{3}{2}})} \\
a_{51} &= 4(68 + 27\sqrt{6})e^{ihk(\frac{-1}{10}(-4+\sqrt{6}))} \\
a_{52} &= -4(16 + \sqrt{6})e^{ihk(\frac{2}{5})} \\
a_{53} &= 4e^{ihk(\frac{-1}{20}(6+\sqrt{6}))}\{26 - 17\sqrt{6} + (2 + \sqrt{6})e^{ihk}\} \\
a_{54} &= 20(16 + \sqrt{6})e^{ihk(\frac{-1}{5}\sqrt{\frac{3}{2}})} \\
a_{55} &= -20(32 + 3\sqrt{6})
\end{aligned}
$$

By using a computer package, we find that the characteristic polynomial of the following problem

$$
A \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = (\omega')^2 \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}
$$

is

$$
\begin{aligned}
&- 4837294080000 \\
&- x\Big\{257989017600 + x\Big(1903315968 + x\{4356864 + x(3650 + x)\}\Big)\Big\} \\
&+ 2\{1555200 + (-1728 + x)x\}^2 \cos(hk) = 0.
\end{aligned}
$$

Using Taylor's expansion for the physically correct eigenvalue, we have

$$
\frac{w_h^2}{w^2} = 1 - \frac{(hk)^6}{8064000} - \frac{(hk)^8}{2073600000} - \frac{229(hk)^{10}}{6082560000000} - \frac{333121(hk)^{12}}{105670656000000000} + O\Big((hk)^{13}\Big).
$$

5. CFL stability analysis

In this section, we will analyze the stability of the staggered discontinuous Galerkin method (3)-(4) with the leap-frog time discretization. More precisely, let $\alpha = c\Delta t/h$, we will find α_M such that the method is stable for all $\alpha \leq \alpha_M$. We will, in the next section, compare the results for our staggered discontinuous Galerkin method with the classical conforming finite element method, which has $\alpha_M = 1$ for piecewise linear elements and $\alpha_M = 0.4082$ for piecewise quadratic elements [10].

5.1. Stability for piecewise linear elements. By using the leap-frog scheme for time discretization, we have the following relations for the piecewise linear method:

$$
\begin{aligned}
\frac{4}{\Delta t^2}\sin^2\Big(\frac{\omega_{h,1}\Delta t}{2}\Big) &= -\frac{c^2}{2h^2}\Big(\frac{1}{3}\{-145 + \cos(hk)\} + \frac{a}{3b} - \frac{b}{3}\Big), \\
\frac{4}{\Delta t^2}\sin^2\Big(\frac{\omega_{h,2}\Delta t}{2}\Big) &= -\frac{c^2}{2h^2}\Big(\frac{1}{3}\{-145 + \cos(hk)\} - \frac{(1+i\sqrt{3})a}{6b} + \frac{(1-i\sqrt{3})b}{6}\Big), \\
\frac{4}{\Delta t^2}\sin^2\Big(\frac{\omega_{h,3}\Delta t}{2}\Big) &= -\frac{c^2}{2h^2}\Big(\frac{1}{3}\{-145 + \cos(hk)\} - \frac{(1-i\sqrt{3})a}{6b} + \frac{(1+i\sqrt{3})b}{6}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= -6337 + 1154\cos(hk) - \cos^2(hk), \\
b &= \Big(133921 - 508899\cos(hk) + 1731\cos^2(hk) - \cos^3(hk) \\
&\quad + 7776\sqrt{3}\sqrt{-1304 + 15\cos(hk) + 1290\cos^2(hk) - \cos^3(hk)}\Big)^{\frac{1}{3}}.
\end{aligned}
$$

By the method in [10], it suffices to consider the characteristic polynomial

$$
x^3 + \{145 - \cos(hk)\}x^2 + \{4896 + 288\cos(hk)\}x + \{20736 - 20736\cos(hk)\} = 0.
$$

Notice that this is the characteristic polynomial of the eigenvalue problem without the factor $-\frac{c^2}{2h^2}$ in the matrix. Writing $hk = 2\pi K$, we have

$$
x^3 + \{145 - \cos(2\pi K)\}x^2 + \{4896 + 288\cos(2\pi K)\}x + \{20736 - 20736\cos(2\pi K)\} = 0. \tag{12}
$$

Differentiating the characteristic polynomial with respect to K and setting it to zero, we have

$$
2\pi\sin(2\pi K)(x - 144)^2 = 0.
$$

Hence, we have either $x = 144$ or $K \in \mathbb{Z}$ or $K + \frac{1}{2} \in \mathbb{Z}$. Note that the solutions of (12) are always non-positive, thus $x \neq 144$. For the case $K \in \mathbb{Z}$, the equation (12) becomes

$$
x^3 + 144x^2 + 5184x = 0.
$$

The solutions are 0 and -72. Note -72 is a double root. For the case $K + \frac{1}{2} \in \mathbb{Z}$, the equation (12) becomes

$$x^3 + 146x^2 + 4608x + 41472 = 0.$$

The solutions are -18 and $16(-4 \pm \sqrt{7})$. Comparing the five solutions, the one with greatest absolute value is $16(-4 - \sqrt{7})$. Thus, the method is stable if

$$\alpha \leq \alpha_M = \frac{2\sqrt{2}}{\sqrt{16(4+\sqrt{7})}} = \frac{1}{\sqrt{2(4+\sqrt{7})}} \approx 0.2743.$$

5.2. Stability for piecewise quadratic elements. By the calculations from the previous section, the characteristic polynomial without the factor $-\frac{c^2}{9h^2}$ is

$$\begin{aligned} &-4837294080000 \\ &-x\{257989017600 + x\{1903315968 + x(4356864 + x(3650 + x))\}\} \\ &+2\{1555200 + (-1728 + x)x\}^2 \cos(hk) = 0. \end{aligned}$$

Writing $hk = 2\pi K$, we have

$$\begin{aligned} &-4837294080000 \\ &-x\{257989017600 + x\{1903315968 + x(4356864 + x(3650 + x))\}\} \\ &+2\{1555200 + (-1728 + x)x\}^2 \cos(2\pi K) = 0. \end{aligned} \tag{13}$$

Differentiating the characteristic polynomial with respect to K and setting it to zero, we have

$$-4\pi\{1555200 + (-1728 + x)x\}^2 \sin(2\pi K) = 0.$$

Since the equation $1555200 + (-1728 + x)x = 0$ has no real solution, we have either $K \in \mathbb{Z}$ or $K + \frac{1}{2} \in \mathbb{Z}$. For the case $K \in \mathbb{Z}$, the solutions of (13) are 0 and $48(-19 \pm 2\sqrt{34})$. Note each $48(-19 \pm 2\sqrt{34})$ is a double root. For the case $K + \frac{1}{2} \in \mathbb{Z}$, the solutions of (13) are $-1734.45, -1092.94, -646.97, -89.0305$ and -88.6033. Comparing the solutions, the one with greatest absolute value is -1734.45. Hence, the method is stable if

$$\alpha \leq \alpha_M = \frac{\sqrt{36}}{\sqrt{1734.45}} \approx 0.1441.$$

6. Comparison

In this section, we will compare the stability and dispersion results for the staggered method [3, 4] with those for the classical finite element method [10] and discontinuous Galerkin method [1, 11].

First, we see that numerical methods for the first order form of the wave equation typically give better dispersion error. More precisely, for the same degree of approximation polynomial, the staggered method (3)-(4) gives more accurate dispersion relation than that of the standard conforming finite element applied to the second order wave equation. In particular, for piecewise linear element, the error of the wave speed for the staggered method is $O((hk)^4)$ while that of conforming method is $O((hk)^2)$ [10]. Moreover, for piecewise quadratic element, the error of the wave speed for the staggered method is $O((hk)^6)$ while that of conforming method is $O((hk)^4)$. Similar comparison holds for interior penalty discontinuous Galerkin method for the wave equation in second order form [1].

Second, we compare the dispersion errors of staggered DG and non-staggered DG methods. Consider a closed interval with an even number of subintervals and the wave equation with periodic boundary condition. We use h to represent the length of primal and dual cells for the staggered method. Assume also that the upwind and central discontinuous Galerkin methods [1, 11] are defined so that the numerical solutions are piecewise polynomial on subinterval with length $h/2$. In this sense, one can see that the finite element spaces for our method (3)-(4) are a restriction of the finite element spaces for classical upwind and central discontinuous Galerkin methods. Note that we need to replace h by $\frac{h}{2}$ in the dispersion relations from [1, 11]. For piecewise linear elements, the leading term for upwind and central discontinuous Galerkin methods [1] are $1+\dfrac{(hk)^4}{2880}$ and $1+O\Big((hk)^6\Big)$ (with an optimal choice of parameters) respectively while the leading term for staggered method is $1-\dfrac{(hk)^4}{8640}$. Thus we see that the dispersion error for staggered DG method is smaller than that of upwind DG method. For the central DG method, with a special choice of parameter, has better dispersion error than that of staggered method. On the other hand, for the piecewise linear DG method in [11], the dispersion error is $O(h^2)$ with the choice of centered flux, and is $O(h^4)$ with the choice of uncentered flux. Therefore, we see that the staggered method has a smaller dispersion error than the centered DG method of [11], and has similar dispersion error to that of the uncentered DG method of [11]. Finally, we note that only the centered DG method of [11] has energy conservation.

Finally we will compare the CFL number α_M for the staggered method and the classical conforming finite element method. From the above calculation, we see that $\alpha_M = 0.2743$ for the linear staggered method while $\alpha_M = 0.1441$ for the quadratic staggered method. Moreover, for the conforming FEM, we have $\alpha_M = 0.5$ for piecewise linear elements and $\alpha_M = 0.2041$ for quadratic elements. Notice that we have divided the α_M for the conforming method by 2 since we are considering a mesh size of $h/2$. Thus we see that the staggered methods have a little bit more restrictive stability condition.

7. Conclusion

In this paper, we present dispersion analysis for the staggered DG method for wave propagation. Both the linear and the quadratic elements are considered. In addition, we derive stability conditions on the time step for the leap-frog time discretization. The results of the paper provide a better understanding of the dispersion and stability properties of the staggered DG method developed in [3, 4].

References

[1] M. Ainsworth, P. Monk and W. Muniz. *Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second-order wave equation*. J. Sci. Comput., 27 (2006), pp. 5–40.

[2] Eric T. Chung and Patrick Ciarlet, Jr. *Scalar transmission problems between dielectrics and metamaterials: T-coercivity and the discontinuous Galerkin approach*. Proceeding for the 10th international conference on mathematical and numerical aspects of waves, Simon Fraser University, 2011.

[3] Eric T. Chung and Bjorn Engquist. *Optimal discontinuous galerkin methods for wave propagation*. SIAM J. Numer. Anal., 44 (2006), pp. 2131–2158.

[4] Eric T. Chung and Bjorn Engquist. *Optimal discontinuous Galerkin methods for the acoustic wave equation in higher dimensions*. SIAM J. Numer. Anal., 47 (2009), pp. 3820–3848.

[5] Eric T. Chung, Yalchin Efendiev and Richard Gibson. *Multiscale finite element modeling of acoustic wave equation*. SEG Annual meeting, Expanded Abstract 2011.

[6] Eric T. Chung, Yalchin Efendiev and Richard Gibson. *An energy-conserving discontinuous multiscale finite element method for the wave equation in heterogeneous media*. Advances in Adaptive Data Analysis, 3 (2011), pp. 251–268.

[7] Eric T. Chung, Hyea Hyun Kim and Olof Widlund. *Two-level overlapping Schwarz algorithms for a staggered discontinuous Galerkin method*. Submitted.

[8] Eric T. Chung and Chak Shing Lee. *A staggered discontinuous Galerkin method for the curl curl operator*. To appear in IMA J. Numer. Anal.

[9] Eric T. Chung and Chak Shing Lee. *A staggered discontinuous Galerkin method for the convection diffusion equation*. J. Numer. Math., 20 (2012), pp. 1–31.

[10] G. C. Cohen. *Higher-Order Numerical Methods for Transient Wave Equations*. Springer, Berlin, 2002.

[11] G. Cohen, X. Ferrieres and S. Pernet. *A spatial high-order hexahedral discontinuous Galerkin method to solve Maxwells equations in time domain*. J. Comput. Phys., 217 (2006), pp. 340–363.

[12] L. Fezoui, S. Lanteri, S. Lohrengel and S. Piperno. *Convergence and Stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes*. M2AN, 39 (2005), pp. 1149–1176.

[13] Marcus J. Grote and Dominik Schötzau. *Optimal error estimates for the fully discrete interior penalty DG Method for the wave equation*. J. Sci. Comput., 40 (2009), pp. 257–272.

[14] J. S. Hesthaven and T. Warburton. *Nodal high-order methods on unstructured grids - I. Time-domain solution of Maxwells equations*. J. Comput. Phys., 181 (2002), pp. 186–221.

[15] B. Riviere and M. Wheeler. *Discontinuous finite element methods for acoustic and elastic wave problem*. In ICM2002-Beijing Satellite Conference on Scientific Computing, volume 329 of Contemporary Mathematics, 2002, AMS, pp. 271–282.

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