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REPRESENTATION OF MATCHED-LAYER KERNELS WITH VISCOELASTIC MECHANICAL MODELS

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Abstract. The Kosloff & Kosloff (KK) absorbing-boundary method is shown to be a particular case of the split-PML method introduced by Bérenger. In its original form, the PML technique has been implemented for Maxwell's electromagnetic equations. On the other hand, the KK method was applied to the Schrödinger and acoustic wave equations. Both techniques have subsequently widely been used in dynamic elasticity, involving different rheological equations, including poroelasticity, and electromagnetism. The coordinate stretching used in the PML method is equivalent to the damping kernel in the KK method, which is based on the Maxwell viscoelastic model. Inside the absorbing strips, the result is a traveling wave which gradually attenuates without changing shape or undergoing dispersion. Moreover, we also show that the recently developed unsplit C-PML method is based on the memory-variable formalism to describe anelasticity introduced by Carcione and co-workers, and that the damping kernel is based on the Zener viscoelastic model. The theoretical reflection coefficients, i.e., before discretization, are obtained and re-interpreted using the theory of viscoelasticity through the acoustic/electromagnetic analogy.

Key words. Absorbing boundaries, viscoelasticity, electromagnetism, reflection of waves.

1. Introduction

The solution of partial differential equations describing several physical processes, mainly related to wave propagation in electromagnetism and dynamic elasticity, are generally solved by using direct methods, based on finite differences, finite elements or pseudospectral methods. In order to avoid reflections and/or wraparound from the edges of the numerical mesh, damping has to be implemented at the boundaries in the form of absorbing strips.

Kosloff and Kosloff [18] introduced a modification of the wave equation inside the absorbing strips, where the solution is a wave traveling without dispersion but whose amplitude decreases with distance at a frequency independent rate. A traveling pulse will thus diminish in amplitude without a change of shape. The method has been applied to the Schrödinger and acoustic wave equations. Subsequenty, this method has been applied to different rheological equations, namely, anisotropy, viscoelasticity and poroelasticity, and to electromagnetism, mainly in algorithms where the spatial derivatives are computed with pseudospectral methods. A review can be found, for instance, in Carcione et al. [9] and Carcione [6]. In particular Carcione et al. [10] and Kosloff et al. [17] applied the method to simulate anelastic wavefields and to the elastic wave equation for modeling surface waves, respectively. They have used the Fourier and Chebyshev pseudospectral operators to compute the spatial derivatives. We note here that the Chebyshev method allows us to use an alternative non-reflecting boundary condition at the edges of the mesh, based on characteristics variables, similar to the paraxial wave equation.

On the other hand, the split-PML method has been proposed by Bérenger as an absorbing boundary condition for electromagnetic waves. The PML method has

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been widely used for finite-difference and finite-element methods. Chew and Liu [13] first proposed the PML method for elastic waves in solids. A recent implementation can be found in Festa and Vilotte [15], where these authors provide a review of the application of the PML method to different stress-strain relations in elastodynamics.

The similarity between the two methods under a practical choice of the attenuation parameter is shown in this paper. The advantage of the split-PML method is that the exact reflection coefficient is zero at all angles of incidence in contrast to the KK method, whose corresponding reflection coefficient is zero only at normal incidence. However, both coefficients differ from zero after the spatial discretization. As shown by Komatitsch and Martin [16], the split nature of the PML method causes spurious events at grazing angles. It is shown by these authors and by Bérenger [2] that a further improvement is obtained by using the C-PML approach. The so-called unsplit C-PML, introduced by Roden and Gedney [19] in electromagnetism, is based on a convolutional relation similar to the stress-strain convolution used by Carcione et al. [10, 11] to simulate anelastic fields in the time domain. In order to overcome the convolutions, additional variables and therefore additional differential equations have to be introduced in the formulation. It is shown in this work that the differential equations used in the C-PML method are the same to the ones obtained by Carcione et al. [10, 11] to model viscoelasticity with the Zener model. The equivalence is also shown in the frequency domain.

2. The KK method

The boundaries of the numerical mesh may generate non-physical artifacts which disturb the physical events. These artifacts consists in field wraparound when using the Fourier method and reflections when using the Chebyshev pseudospectral method.

When we consider the constant-density pressure formulation, the wave equation can be written as a system of coupled equations and modified as

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ c^2\Delta & -\gamma \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{1}$$

where p is the pressure, f are body forces, c is the wave velocity, Δ is the Laplacian, and γ is the absorbing parameter or damping factor. Note that $q = \dot{p}$ if $\gamma = 0$, where the dot above a variable denotes time differentiation. Hereafter, the indices 1, 2 and 3 correspond to the spatial variables x, y and z or x_1, x_2 and x_3, respectively.

The absorbing-boundary parameter $\gamma(x, y, z)$ differs from zero only in a strip of nodes surrounding the numerical mesh. Its spatial dependence is chosen to achieve the best amplitude reduction. The following spatial dependence was chosen in Kosloff and Kosloff [18],

$$\gamma = U_0/\cosh^2(d\ n), \tag{2}$$

where U_0 is a constant, d is a decay factor and n denotes the distance in number of grid points from the boundary. In quantum mechanics, the function γ plays a similar role to a complex negative potential added to the Hamiltonian.

A single second-order equation can be obtained after elimination of the variable q. For instance, in the case of homogeneous media and in the absence of the source we obtain

$$\ddot{p} = c^2\Delta p - 2\gamma\dot{p} - \gamma^2 p. \tag{3}$$

Similarly, the acoustic wave equation can be split in a different manner:

$$\frac{\partial}{\partial t}\begin{pmatrix} \dot{p} \\ \partial_1 p \\ \partial_2 p \\ \partial_3 p \end{pmatrix} = \begin{pmatrix} -\gamma & c^2\partial_1 & c^2\partial_2 & c^2\partial_3 \\ \partial_1 & -\gamma & 0 & 0 \\ \partial_2 & 0 & -\gamma & 0 \\ \partial_3 & 0 & 0 & -\gamma \end{pmatrix}\begin{pmatrix} \dot{p} \\ \partial_1 p \\ \partial_2 p \\ \partial_3 p \end{pmatrix} + \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}, \tag{4}$$

where ∂_i denotes the spatial derivative along the i-direction. As shown below, the solution to equation (3) is a wave traveling without dispersion but whose amplitude decreases with distance at a frequency independent rate. A traveling pulse will thus diminish in amplitude without a change of shape.

In the 2-D and 3-D elastic cases, the method has been applied by Kosloff et al. [17] and Tessmer and Kosloff [20]. In 3-D space, the differential equations are

$$\begin{aligned} \dot{v}_i &= \frac{1}{\rho}\partial_j \sigma_{ij} + f_i, \\ \dot{\sigma}_{i(i)} &= (\lambda + 2\mu)\partial_i v_{(i)} + \lambda \partial_j v_j, \quad j \neq i, \\ \dot{\sigma}_{ij} &= \mu(\partial_i v_j + \partial_j v_i), \quad j \neq i, \end{aligned} \tag{5}$$

where $i, j = 1,\ldots 3$, the v's are particle-velocity components, the σ's are stress components, ρ is the mass density, λ and μ are the Lamé constants, and an index between parentheses means that there is no implicit summation. The damped system in this case has been obtained in the same manner as in equations (1) and (4), i.e., by adding the diagonal operator $-\gamma\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, to the right-hand-side of equation (5).

The parameter γ depends on the three spatial coordinates and, depending on the side of the mesh, different parameters to define γ can be chosen. This means that each absorbing strip has a different absorption property according to the particular material properties in that part of the model.

3. Relation between the split-PML and KK methods

The differential equations corresponding to the split-PML method, written in the particle-velocity stress formulation, can be found in many papers, for instance, in Festa and Vilotte [15]. They are

$$\begin{aligned} \dot{v}_i^k + \gamma_k v_i^k &= \frac{1}{\rho}\partial_k \sigma_{i(k)} + \delta_{ik} f_i, \\ \dot{\sigma}_{i(i)}^k + \gamma_k \sigma_{i(i)}^k &= (\lambda + 2\mu)\partial_k v_{(k)}, \quad k = i, \\ \dot{\sigma}_{i(i)}^k + \gamma_k \sigma_{i(i)}^k &= \lambda \partial_k v_{(k)}, \quad k \neq i, \\ \dot{\sigma}_{ij}^k + \gamma_k \sigma_{ij}^k &= \mu \partial_k v_j, \quad k = i \neq j, \end{aligned} \tag{6}$$

where

$$\begin{aligned} v_i &= v_i^1 + v_i^2 + v_i^3, \\ \sigma_{i(i)} &= \sigma_{i(i)}^1 + \sigma_{i(i)}^2 + \sigma_{i(i)}^3, \\ \sigma_{ij} &= \sigma_{ij}^k(\delta_{ik} + \delta_{jk}), \quad i \neq j. \end{aligned} \tag{7}$$

and $\gamma_k(x_k)$ is the absorption parameter for the k-direction.

Equation (6) can be generalized to the case $\gamma_k = \gamma_k(x, y, z)$. If, in particular we make the choice $\gamma_k = \gamma$ for all k, and add triplets of equations (6), according to the split (7), we obtain

$$\begin{aligned} \dot{v}_i + \gamma v_i &= \frac{1}{\rho}\partial_i j \sigma_{ij} + f_i, \\ \dot{\sigma}_{i(i)} + \gamma \sigma_{i(i)} &= (\lambda + 2\mu)\partial_i v_{(i)} + \lambda \partial_j v_j, \quad j \neq i, \\ \dot{\sigma}_{ij} + \gamma \sigma_{ij} &= \mu(\partial_i v_j + \partial_j v_i), \quad j \neq i, \end{aligned} \tag{8}$$

It turns out that this system of equations is equivalent to equation (5) with damping as used in Kosloff et al. [17] and Tessmer and Kosloff [20].

3.1. The equivalence in the frequency domain. Let us assume a kernel $\exp(\mathrm{i}\omega t - \mathrm{i}kx)$ and the 1-D homogeneous case, where ω is the angular frequency and k is the complex wavenumber. Substitution of this kernel into equation (3) gives the following complex velocity

$$v_c = \frac{\omega}{k} = \frac{c}{\sqrt{1 - 2\mathrm{i}\beta - \beta^2}} = \frac{c}{1 - \mathrm{i}\beta}, \quad \beta = \frac{\gamma}{\omega} \tag{9}$$

Since the wavenumber is complex, $k = \kappa - \mathrm{i}\alpha$, where κ is the real wavenumber and α is the attenuation factor, we can obtain from (9) the phase velocity, α and the quality factor:

$$v_p = \frac{\omega}{\kappa} = \left[\mathrm{Re}\left(\frac{1}{v_c}\right)\right]^{-1} = c, \tag{10}$$

and

$$\alpha = -\omega \mathrm{Im}\left(\frac{1}{v_c}\right) = \frac{\gamma}{c} \tag{11}$$

Then, the wave equation (3) possesses a general solution of the form

$$p(x,t) = Af_1(x - ct)\exp\left(-\frac{\gamma}{c}x\right) + Bf_2(x + ct)\exp\left(\frac{\gamma}{c}x\right). \tag{12}$$

This solution represents traveling waves which are exponentially attenuated in space. All frequency components are equally attenuated because the decay factor γ/c is frequency independent. This fact has important significance, as a propagating pulse containing a frequency band will gradually attenuate without changing shape or undergoing dispersion.

On the other hand, the PML method implies a coordinate stretching, which in the frequency domain can be expressed as [15]

$$x \to xs = x\left(1 - \mathrm{i}\frac{\gamma}{\omega}\right). \tag{13}$$

This implies

$$\exp(-\mathrm{i}\kappa xs) = \exp(-\mathrm{i}\kappa x)\exp\left(-\frac{\gamma}{c}x\right), \tag{14}$$

which is equivalent to equation (12).

When the decay factor γ is spatially variable, the 1-D equation can be solved by the propagator matrix method and the effectiveness of the absorbing region can thus be evaluated numerically [18]. Consider the region $-\infty < x < \infty$. The acoustic velocity is uniform and the absorbing coefficient differs significantly from zero only in the region $A < x < B$. A sinosoidal wave $\exp[\mathrm{i}(\omega t - \kappa x)]$ in the region $-\infty < x < A$, and with $\omega/\kappa = c$ creates a reflected wave $R\exp[\mathrm{i}(\omega t + \kappa x)]$ for $x < A$, and a transmitted wave $T\exp[\mathrm{i}(\omega t - \kappa x)]$ for $B < x$. When the spatial dependence of γ is chosen properly, the magnitude of T and R can be kept small thus effectively ensuring that no energy is reflected or transmitted from the absorbing region.

We obtain in the appendix the exact electromagnetic reflection coefficients between vacuum and a PML layer, corresponding to both damping formulations. The differential equations are those of the original paper of Bérenger [1], who developed the method for electromagnetic waves. In view of the acoustic/electromagnetic analogy [7, 12], the TM equations are mathematically equivalent to the equations describing SH viscoelastic and viscoacoustic (P) waves, so the derivation is rather general. We then re-derive Bérenger's reflection coefficient using the mathematics of viscoelasticity.

4. The C-PML method and the memory-variable formalism

The last version of the PML method to absorb unphysical reflections from the edges of the mesh is the unsplit convolutional PML method or C-PML method, also known as the non-split complex frequency-shifted convolution PML [14]. Komatitsch and Martin [16] implement the technique to solve the elastic wave equation and note that the convolution approach is effectively similar to that of Carcione et al. [11] to simulate viscoelastic fields. We show in this section that, in fact, the resulting equations corresponding to the additional field variables are identical. Note that in this new method, which improves the split-PML approach, the field variables are not split.

Inside the PML strips, each of the spatial derivatives in equations (5) are replaced by a time convolution. Let f be a particle-velocity or a stress component. We have

$$\partial_i f \to s * \partial_i f, \quad i = 1, 2, 3, \tag{15}$$

where

$$s(t) = \frac{\delta(t)}{\epsilon} + a \exp(-bt) H(t), \tag{16}$$

"$*$" denotes time convolution, H is the step function, δ is the Dirac function, and ϵ, a and b are absorbing parameters.

Let us transform the convolution into a differential equation, according to the memory-variable formulation of Carcione et al. [10]. Note that

$$s = \frac{\delta}{\epsilon} + gH, \quad g = a \exp(-bt), \quad \dot{g} = -bg. \tag{17}$$

We then have

$$s * \partial_i f = \frac{\partial_i f}{\epsilon} + e_f, \quad e_f = gH * \partial_i f, \tag{18}$$

where e_f is a memory variable.

The time derivative of the memory variable is

$$\dot{e}_f = (\delta g + \dot{g} H) * \partial_i f = g(0)\partial_i f - bgH * \partial_i f = g(0)\partial_i f - be_f, \tag{19}$$

or

$$\dot{e}_f = a\partial_i f - be_f. \tag{20}$$

Each derivative $\partial_i f$ is replaced by $(\partial_i f)/\epsilon + e_f$ and the additional differential equation $\dot{e}_f = a\partial_i f - be_f$ has to be solved. This equation is slightly different from that obtained by Komatitsch and Martin [16]. However, both equations are equivalent for the time step used in the numerical calculations (dt small enough) and the performance is the same. This is shown in the following. Using our notation, equation (26) of Komatitsch and Martin [16] can be written as

$$e_f^{n+1} = \left[e_f^n - \frac{a}{b}(\partial_i f)^{n+1/2} \right] \exp(-bdt) + \frac{a}{b}(\partial_i f)^{n+1/2}, \tag{21}$$

where n denotes the n-th time step. Since $\exp(-bdt) \approx 1 - bdt$, we obtain

$$e_f^{n+1} = e_f^n + dt \left[a(\partial_i f)^{n+1/2} - be_f^n \right], \tag{22}$$

which is precisely the first-order time discretization of equation (20).

5. The damping kernels in terms of viscoelastic mechanical models

Let us consider for simplicity the 1-D wave equation in the particle-velocity/stress formulation,

$$\begin{aligned} \dot{v} &= \frac{1}{\rho}\partial_x \sigma, \\ \dot{\sigma} &= M \partial_x v, \end{aligned} \tag{23}$$

where v is the particle velocity, σ is the stress and $M = \rho c^2$. The KK or PML equations in the absorbing strips are

$$\begin{aligned} \dot{v} + \gamma v &= \frac{1}{\rho}\partial_x \sigma, \\ \dot{\sigma} + \gamma \sigma &= M \partial_x v. \end{aligned} \tag{24}$$

In the frequency domain these equations simplify to

$$\omega^2 \sigma + c^2 s^2 \partial_{xx} \sigma = 0, \tag{25}$$

where

$$s(\omega) = \frac{\mathrm{i}\omega}{\mathrm{i}\omega + \gamma}. \tag{26}$$

This is the kernel of a Maxwell viscoelastic solid, where $1/\gamma$ is the relaxation time and c^2/γ is the viscosity of the dashpot [6].

On the other hand, the C-PML equations are

$$\begin{aligned} \dot{v} &= \frac{1}{\rho} s * \partial_x \sigma, \\ \dot{\sigma} &= M s * \partial_x v, \end{aligned} \tag{27}$$

where s is given by equation (16). In the frequency domain we obtain an equation similar to (25), with

$$s(\omega) = s_R \left(\frac{1 + \mathrm{i}\omega\tau_\epsilon}{1 + \mathrm{i}\omega\tau_\sigma} \right), \tag{28}$$

where

$$s_R = \frac{1}{\epsilon} + \frac{a}{b}, \quad \tau_\sigma = \frac{1}{b}, \quad \tau_\epsilon = \frac{1}{b + a\epsilon}, \tag{29}$$

and the subindex R denotes relaxed. This is the kernel of a Zener viscoelastic model [6]. (Note: s as defined here is the time Fourier transform of $\bar{s}_x(t)$ given in equation (18) in Komatitsch and Martin [16]).

Note that equation (25) is a Helmholtz equation, where the complex velocity is given by

$$v_c = cs. \tag{30}$$

In the split PML case we obtain equation (9) and in the C-PML case we obtain

$$v_c = c_R \left(\frac{1 + \mathrm{i}\omega\tau_\epsilon}{1 + \mathrm{i}\omega\tau_\sigma} \right), \quad c_R = c s_R \tag{31}$$

i.e., the complex velocity of a Zener viscoelastic medium, where c_R is the relaxed (low-frequency) velocity.

6. Conclusions

We have re-interpreted known absorbing-boundary methods in terms of mechanical models and showed their relationships in the time and frequency domain. The split-PML and KK methods are based on a Maxwell viscoelastic model. The new modification of the PML method, developed at the end of nineties and called C-PML, is based on the well-known memory-variable equations used to model wave propagation in anelastic media. A representation in terms of mechanical models shows that the C-PML method is based on a kernel given by the Zener model. A re-derivation of Bérenger's reflection coefficient of a vacuum/PML layer is performed by using the mathematics of viscoelasticity, which shows how the two methods are related.

References

[1] J. P. Bérenger. A perfectly matched layer for the absorption of electromagnetic waves. *Journal of Computational Physics*, 114: 185-200, 1994.

[2] J.P. Bérenger. Numerical reflection from FDTD-PMLs: A comparison of the split PML with the unsplit and CFS PMLs. *IEEE Transactions on Antennas and Propagation*, 50: 258-265, 2002.

[3] P. W. Buchen. Reflection, transmission and diffraction of SH-waves in linear viscoelastic solids. *Geophysical Journal of the Royal Astronomical Society*, 25: 97-113, 1971.

[4] J. M. Carcione. The effects of vector attenuation on AVO of off-shore reflections. *Geophysics*, 64: 815-819, 1999.

[5] J. M. Carcione. Vector attenuation: elliptical polarization, raypaths and the Rayleigh-window effect. *Geophysical Prospecting*, 54: 399-407, 2006.

[6] J. M. Carcione. Wave Fields in Real Media. Theory and numerical simulation of wave propagation in anisotropic, anelastic, porous and electromagnetic media, 2nd edition, revised and extended, Elsevier Science, 2007.

[7] J. M. Carcione, and F. Cavallini. On the acoustic-electromagnetic analogy. *Wave Motion*, 21: 149-162, 1995.

[8] J. M. Carcione, and F. Cavallini. Forbidden directions for TEM waves in anisotropic conducting media. *IEEE Transactiosn on Antennas and Propagation*, 45: 133-139, 1997.

[9] J. M. Carcione, G. Herman, and F. P. E. ten Kroode. Seismic modeling. *Geophysics*, 67: 1304-1325, 2002.

[10] J. M. Carcione, D. Kosloff, and R. Kosloff. Viscoacoustic wave propagation simulation in the earth. *Geophysics*, 53: 769-777, 1988a.

[11] J. M. Carcione, D. Kosloff, and R. Kosloff. Wave propagation simulation in a linear viscoelastic medium. *Geophysical Journal International*, 95: 597-611, 1988b.

[12] J. M. Carcione, and E. Robinson. On the acoustic-electromagnetic analogy for the reflection-refraction problem. *Studia Geophysica et Geodetica*, 46: 321-345, 2002. (Special issue dedicated to V. Červený.)

[13] W. C. Chew, and Q. H. Liu. Perfectly matched layers for elastodynamics: A new absorbing boundary condition. *Journal of Computational Acoustics*, 4: 72-79., 1996.

[14] F. H. Drossaert, and A. Giannopoulos. A nonsplit complex frequency-shifted PML based on recursive integration for FDTD modeling of elastic waves. *Geophysics*, 72: T9-T17, 2007.

[15] G. Festa, and J.-P. Vilotte. The Newmark scheme as a velocity-stress time staggering: An efficient PML for spectral element simulations of elastodynamics. *Geophysical Journal International*, 161: 789-812, 2005.

[16] D. Komatitsch, and R. Martin. An unsplit convolutional perfectly matched layer improved at grazing incidence for the seismic wave equation. *Geophysics*, 72: SM155-SM167, 2007.

[17] D. Kosloff, D. Kessler, A. Queiroz Filho, E. Tessmer, A. Behle, and R. Strahilevitz. Solution of the equations of dynamic elasticity by a Chebychev spectral method. *Geophysics*, 55: 734-748, 1990.

[18] D. Kosloff, and R. Kosloff. Absorbing boundaries for wave propagation problems. *Journal of Computational Physics*, 63: 363-376., 1986.

[19] J. A. Roden, and S. D. Gedney. Convolutional PML (CPML): An efficient FDTD implementation of the CFS-PML for arbitrary media. *Microwave and Optical Technology Letters*, 27: 334-339, 2000.

[20] E. Tessmer, and D. Kosloff. 3-D elastic modeling with surface topography by a Chebyshev spectral method. *Geophysics*, 59: 464-473, 1994.
[21] L. Wennerberg. Snell's law for viscoelastic materials. *Geophysical Journal of the Royal Astronomical Society*, 81: 13-18, 1985.

Appendix A. Electromagnetic reflection coefficient between vacuum and a PML layer

We denote by PML layer the medium describing the absorbing strips and obtain the reflection coefficient corresponding to the Kosloff & Kosloff [18] and split-PML [1] methods, in order to better understand the relation between the two approaches in terms of reflection coefficients.

Let us consider the TM equation, called TE equation by Bérenger [1] (his eq. (1)), and its extension at the absorbing strips according to the method proposed by Kosloff and Kosloff [18]. We have

$$\begin{aligned} \dot{E}_x + \gamma E_x &= \frac{1}{\epsilon_0}\partial_y H_z, \\ \dot{E}_y + \gamma E_y &= -\frac{1}{\epsilon_0}\partial_x H_z, \\ \dot{H}_z + \gamma H_z &= \frac{1}{\mu_0}(\partial_x E_y - \partial_y E_x). \end{aligned} \tag{32}$$

where E_x and E_y are electric-field components, H_z is the magnetic-field component, ϵ_0 is the permittivity of vacuum, μ_0 is the magnetic permeability of vacuum, and γ is the absorbing parameter. Equation (32) is similar to eq. (1) of Bérenger [1], with the choice

$$\gamma = \frac{\sigma}{\epsilon_0} = \frac{\sigma^*}{\mu_0}, \tag{33}$$

where σ and σ^* are conductivity parameters. Hence, equation (32) satisfies by construction the matching condition given by eq. (2) of Bérenger [1], and the fact that the impedance of the PML layers is equal to that of vacuum and no reflection occurs at normal incidence, as we shall show in the following. We note here that the KK method is referred as to matched-layer (ML) method by Bérenger [1] (see his Table I).

It is easy to show that the complex permittivity and magnetic permeability associated with equation (1) are $\bar{\epsilon} = \epsilon_0(1-\mathrm{i}\gamma/\omega)$ and $\bar{\mu} = \mu_0(1-\mathrm{i}\gamma/\omega)$, respectively. Thus, the impedance is $\sqrt{\bar{\mu}/\bar{\epsilon}} = \sqrt{\mu_0/\epsilon_0} \equiv Z_0$, i.e., that of vacuum. On the other hand, the complex velocity is $v_c = 1/\sqrt{\bar{\mu}\bar{\epsilon}} = c/(1-\mathrm{i}\gamma/\omega)$, where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the light velocity. Note that this velocity is equivalent to the acoustic complex velocity (9) in view of the acoustic/electromagnetic analogy [7, 12].

Let us denote by subscript 1 the upper layer (vacuum) and by subscript 2 the lower medium (the PML layer). The complex reflection coefficient of the vacuum/PML layer is [12]

$$R_{\mathrm{TM}} = \frac{\sqrt{\bar{\mu}/\bar{\epsilon}}\cos\theta_I - \sqrt{\mu_0/\epsilon_0}\cos\theta_T}{\sqrt{\bar{\mu}/\bar{\epsilon}}\cos\theta_I + \sqrt{\mu_0/\epsilon_0}\cos\theta_T}, \tag{34}$$

where θ_I and θ_T are the incidence and transmission angles. These angles are complex quantities in general (In this particular case, θ_I is real since the incidence medium is vacuum). Bérenger [1] uses the notation θ_1 and θ_2 for the (real) incidence and transmission phase angles. The mathematical form given by equation (34) can also be used to obtain the reflection coefficient of SH waves [6]. Since the two media have the same impedance, we have

$$R_{\mathrm{TM}} = \frac{\cos\theta_I - \cos\theta_T}{\cos\theta_I + \cos\theta_T}. \tag{35}$$

Now, Snell's law at an interface separating vacuum from a lossy medium is [21, 12, 6]

$$\frac{\sin\theta_I}{c} = \frac{\sin\theta_T}{v_c}. \tag{36}$$

Using this equation and replacing the complex velocity $v_c = c/(1 - \mathrm{i}\gamma/\omega)$ into equation (35) yields

$$R_{\mathrm{TM}} = \frac{\sqrt{1-\sin^2\theta_I} - \mathrm{pv}\ \sqrt{1-\sin^2\theta_I/(1-\mathrm{i}\beta)^2}}{\sqrt{1-\sin^2\theta_I} + \mathrm{pv}\ \sqrt{1-\sin^2\theta_I/(1-\mathrm{i}\beta)^2}}, \qquad \beta = \gamma/\omega, \tag{37}$$

where pv denotes the principal value. The reflection coefficient is equal to zero only if $\theta_I = 0$, i.e., at normal incidence. The value of β can be chosen in order to minimize the reflection coefficient at non-normal incidence.

Let us obtain the physical phase angle of the transmitted wave as a function of the incidence angle. By physical phase angle we mean the angle made by the real wavevector (or slowness vector) with a line perpendicular to the interface. Since the incidence medium is lossless, the attenuation vector in the transmission medium is perpendicular to the interface, indicating that the transmitted wave is the analogous to an inhomogeneous viscoelastic wave [4]. This is a consequence of Snell's law in lossy media [21]. Let us assume an interface parallel to x-axis and perpendicular to the y-axis. The physical transmission phase angle θ_2 is given by

$$\tan\theta_2 = \frac{s_1}{\mathrm{Re}(s_2^T)} \tag{38}$$

(eq. (6.47) in Carcione [6]), where $s_1 = \sin\theta_I/c$ is the horizontal slowness component parallel to the interface in the transmission medium, which by Snell's law equals that of the incidence medium, and s_2^T is the complex slowness component perpendicular to the interface in the transmission medium. This component is given by [3, 12, 6]

$$s_2^T = \ \mathrm{pv}\sqrt{\frac{1}{v_c^2} - \sin\theta_I^2}. \tag{39}$$

Therefore, the transmission angle can be obtained from

$$\tan\theta_2 = \frac{\sin\theta_I}{\mathrm{Re}\left(\mathrm{pv}\sqrt{\dfrac{c^2}{v_c^2} - \sin\theta_I^2}\right)}. \tag{40}$$

In the case of impedance matching we have $v_c = c/(1 - \mathrm{i}\beta)$ and

$$\tan\theta_2 = \frac{\sin\theta_I}{\mathrm{Re}\left(\mathrm{pv}\sqrt{(1-\mathrm{i}\beta)^2 - \sin\theta_I^2}\right)}. \tag{41}$$

We obtain $\theta_2 = \theta_I = \theta_1$ at normal incidence and in the lossless case ($\beta = 0$), but in the latter case there is no damping at all in the PML layer.

We develop in the following the demonstration of the reflection and transmission coefficient at a vacuum/lossy medium interface, including anisotropy and the field splitting introduced by Bérenger [1], and show that the reflection coefficient is zero at non-normal incidence for a given choice of the conductivity parameters. This is a re-derivation of Bérenger's reflection coefficient by using elements of viscoelasticity theory [12]. First, we obtain the dispersion equation and the polarizations of the lossy anisotropic medium. The split PML equations are

$$\begin{aligned} &\epsilon_0 \dot{E}_x + \sigma_y E_x = \partial_y (H_{zx} + H_{zy}), \\ &\epsilon_0 \dot{E}_y + \sigma_x E_y = -\partial_x (H_{zx} + H_{zy}), \\ &\mu_0 \dot{H}_{zx} + \sigma_x^* H_{zx} = -\partial_x E_y, \\ &\mu_0 \dot{H}_{zy} + \sigma_y^* H_{zy} = \partial_y E_x, \end{aligned} \tag{42}$$

where $H_{zx} + H_{zy} = H_z$. Let us assume the plane-wave solution

$$\mathbf{P} = [E_x, E_y, H_{zx}, H_{zy}]^\top \exp[\mathrm{i}\omega(t - s_1 x - s_2 y)], \tag{43}$$

where s_1 and s_2 are the slowness components, and every quantity is complex except t, ω and the spatial variables. Substituting this solution into equation (42) gives

$$\begin{aligned} &a_y E_x = -s_2 (H_{zx} + H_{zy}), \\ &a_x E_y = s_1 (H_{zx} + H_{zy}), \\ &b_x H_{zx} = s_1 E_y, \\ &b_y H_{zy} = -s_2 E_x, \end{aligned} \tag{44}$$

where

$$a_x = \epsilon_0 - \mathrm{i}\frac{\sigma_x}{\omega}, \quad a_y = \epsilon_0 - \mathrm{i}\frac{\sigma_y}{\omega}, \quad b_x = \mu_0 - \mathrm{i}\frac{\sigma_x^*}{\omega}, \quad b_y = \mu_0 - \mathrm{i}\frac{\sigma_y^*}{\omega}, \tag{45}$$

Note that the matching condition (33) implies $b_x = Z_0^2 a_x$ and $b_y = Z_0^2 a_y$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of vacuum. Eliminating the magnetic-field components gives

$$\begin{aligned} &\left(a_y - \frac{s_2^2}{b_y}\right) E_x + \frac{s_1 s_2}{b_x} E_y = 0, \\ &\frac{s_1 s_2}{b_y} E_x + \left(a_x - \frac{s_1^2}{b_x}\right) E_y = 0. \end{aligned} \tag{46}$$

Setting the determinant of this system equal to zero gives the dispersion equation:

$$\frac{s_1^2}{a_x b_x} + \frac{s_2^2}{a_y b_y} = 1. \tag{47}$$

This functional form is an ellipse with complex coefficients. Note that to obtain the physical dispersion equations, a relations between $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ is necessary and can be found from this complex equation, since these real quantities are the physical components of the slowness vector. This relation has been found by Carcione [6] in the case of homogeneous viscoelastic SH waves, which are equivalent to TM waves. If the matching condition (33) holds, we have

$$\frac{s_1^2}{a_x^2} + \frac{s_2^2}{a_y^2} = Z_0^2. \tag{48}$$

Let us obtain the eigenvector. Assume that $E_x = E_0 \sin\phi$ and $E_y = E_0 \cos\phi$, where E_0 and ϕ are complex. This substitution is a change of variables performed for convenience. From equations (44) and (46) the eigenvector is

$$\mathbf{E} \equiv [E_x, E_z, H_{zx}, H_{zy}]^\top = E_0 \left[\sin\phi, \cos\phi, \frac{s_1}{b_x}\cos\phi, -\frac{s_2}{b_y}\sin\phi\right]^\top, \tag{49}$$

where

$$\tan\phi = -\frac{s_1 s_2}{b_x(a_y - s_2^2/b_y)} = -(a_x - s_1^2/b_x)\frac{b_y}{s_1 s_2}. \tag{50}$$

In vacuum, $a_x = a_y = \epsilon_0$ and $b_x = b_y = \mu_0$, and the eigenvector is

$$\mathbf{E} = E_0 \left[\cos\theta_I, \sin\theta_I, -\frac{\sin^2\theta_I}{Z_0}, -\frac{\cos^2\theta_I}{Z_0}\right]^\top, \tag{51}$$

where θ_I is the angle of incidence (a real quantity), with $s_1 = \sin\theta_I/c$ and $s_2 = \cos\theta_I/c$, since it is a lossless medium.

The electromagnetic field for a wave of unit intensity incident from vacuum (medium 1) is

$$\mathbf{E}_1 = \mathbf{E}_I + \mathbf{E}_R, \quad \text{and} \quad \mathbf{E}_2 = \mathbf{E}_T, \tag{52}$$

where

$$\begin{aligned}
\mathbf{E}_I &= \left[\cos\theta_I, \sin\theta_I, -\frac{\sin^2\theta_I}{Z_0}, -\frac{\cos^2\theta_I}{Z_0}\right]^\top \exp[\mathrm{i}\omega(t - s_1x - s_2y)],\\
\mathbf{E}_R &= R_{\mathrm{TM}} \left[\cos\theta_I, -\sin\theta_I, -\frac{\sin^2\theta_I}{Z_0}, -\frac{\cos^2\theta_I}{Z_0}\right]^\top \exp[\mathrm{i}\omega(t - s_1x + s_2y)],\\
\mathbf{E}_T &= T_{\mathrm{TM}} \left[\sin\phi, \cos\phi, \frac{s_1}{b_x}\cos\phi, -\frac{s_2^T}{b_y}\sin\phi\right]^\top \exp[\mathrm{i}\omega(t - s_1x - s_2^T y)],
\end{aligned} \tag{53}$$

where R_{TM} and T_{TM} are the electric-field reflection and transmission coefficients, and from equation (47)

$$s_2^T = \mathrm{pv}\sqrt{a_y b_y \left(1 - \frac{s_1^2}{a_x b_x}\right)}. \tag{54}$$

Note in equation (53) the sign differences between the incidence and reflected electric-field components and vertical slowness. The boundary conditions require continuity of

$$E_x \quad \text{and} \quad H_{zx} + H_{zy} \tag{55}$$

at the interface. This yields two equations with two unknowns, the reflection and transmission coefficients. Using (50), we obtain

$$R_{\mathrm{TM}} = \frac{s_2^T - Z_0 a_y \cos\theta_I}{s_2^T + Z_0 a_y \cos\theta_I}. \tag{56}$$

This is a general reflection coefficient, when there are four conductivity parameters defining the lower medium (medium 2), i.e., σ_x, σ_x^*, σ_y and σ_y^*.

The interest here is to match the layers and therefore we use condition (33). In this case, $b_x = Z_0^2 a_x$ and $b_y = Z_0^2 a_y$,

$$s_2^T = \mathrm{pv}\sqrt{a_y^2\left(Z_0^2 - \frac{s_1^2}{a_x^2}\right)}. \tag{57}$$

and

$$R_{\mathrm{TM}} = \frac{\sqrt{1-\sin^2\theta_I} - \mathrm{pv}\sqrt{1 - \sin^2\theta_I/(1-\mathrm{i}\beta_x)^2}}{\sqrt{1-\sin^2\theta_I} + \mathrm{pv}\sqrt{1 - \sin^2\theta_I/(1-\mathrm{i}\beta_x)^2}}, \quad \beta_x = \frac{\sigma_x}{\epsilon_0\omega}, \tag{58}$$

where we have used equation (45) and $s_1 = \sin\theta_I/c$. Note that the reflection coefficient does not depend on $\beta_y = \sigma_y/(\epsilon_0\omega)$, and no matter the value of β_y, $R_{\mathrm{TM}} = 0$ if $\sigma_x = 0$.

The different cases are: i) $\sigma_x = \sigma_y = 0$; the lower medium is vacuum and there is no damping; ii) $\sigma_x = \sigma_y$; this case, which has been already discussed and its solution is equation (37), corresponds to the KK method; iii) $\sigma_x = 0$ and

$\sigma_y \neq 0$; this is Bérenger's case $R_{\mathrm{TM}} = 0$, and there is perfect transmission – before spatial discretization – at all angles of incidence; v) $\sigma_x \neq 0$ and $\sigma_y = 0$; this case corresponds to zero reflection coefficient when the interface is parallel to the y-axis.

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