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MULTISCALE COMPUTATION OF A STEKLOV EIGENVALUE PROBLEM WITH RAPIDLY OSCILLATING COEFFICIENTS

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Abstract. In this paper we consider the multiscale computation of a Steklov eigenvalue problem with rapidly oscillating coefficients. The new contribution obtained in this paper is a superapproximation estimate for solving the homogenized Steklov eigenvalue problem and to present a multiscale numerical method. Numerical simulations are then carried out to validate the theoretical results reported in the present paper.

Key Words. Steklov eigenvalue problem, multiscale method, superapproximation estimate.

1. Introduction

In this paper we discuss the multiscale computation of a Steklov eigenvalue problem with rapidly oscillating coefficients given by

$$\begin{cases} \mathcal{L}_\varepsilon u^\varepsilon = 0, & \text{in} \quad \Omega, \\ u^\varepsilon = 0, & \text{on} \quad \Gamma_0, \\ \sigma_\varepsilon(u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, & \text{on} \quad \Gamma_1, \end{cases} \tag{1}$$

where Ω is a bounded Lipschitz polygonal convex domain or a smooth domain in R^n, $n \geq 2$ with a periodic microstructure, and whose boundary is denoted by $\Gamma = \partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Here $\mathcal{L}_\varepsilon$ denotes a second-order partial differential operator with rapidly oscillating coefficients given by

$$\mathcal{L}_\varepsilon \phi \equiv -\frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\frac{\partial \phi}{\partial x_j}\Big) + a_0(\frac{x}{\varepsilon})\phi,$$

and

$$\sigma_\varepsilon(\phi) \equiv \nu_i a_{ij}(\frac{x}{\varepsilon})\frac{\partial \phi}{\partial x_j},$$

where $\vec{\nu} = (\nu_1, \cdots, \nu_n)$ is the outward unit normal to Γ_1, and $\varepsilon > 0$ is a small period parameter. Here and below we use the Einstein summation convention on repeated indices.

We make the following assumptions:

$(\mathbf{A_1})$ Let $\xi = \varepsilon^{-1}x$, and assume that $a_{ij}(\xi)$, $a_0(\xi)$ are 1-periodic functions in ξ.

$(\mathbf{A_2})$ There is a positive constant γ_0 which is independent of ε such that

$$a_{ij}(\frac{x}{\varepsilon})\eta_i\eta_j \geq \gamma_0|\eta|^2$$

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for all $(\eta_1, \cdots, \eta_n) \in R^n$, $|\eta|^2 = \sum_{i=1}^{n} \eta_i^2$ and all $x \in \Omega$, $a_0(\frac{x}{\varepsilon}) \geq 0$.

$(\mathbf{A_3})$ $a_{ij}(\frac{x}{\varepsilon}) = a_{ji}(\frac{x}{\varepsilon})$ for almost every $x \in \Omega$.

$(\mathbf{A_4})$ $a_{ij}^{\varepsilon}, a_0^{\varepsilon} \in L^{\infty}(\overline{\Omega})$.

Problems with an eigenvalue parameter on the boundary appear in many physical situations(see, e.g.[1, 10, 26, 3]). Courant and Hilbert [16] presented early results on the Steklov eigenvalue problems. Osborn [36]developed a general approximation theory for compact operators. Bramble and Osborn [7] presented a Galerkin method for the approximation of the Steklov problem for a non self-adjoint second order differential operator. Andreev and Todorov [2] gave the isoparametric finite element approximation of Steklov eigenvalue problems for second-order, self-adjoint, elliptic differential operators. Several eigenvalue problems arising in physics and engineering, as well as their approximations, are presented in Weinberger [42],Babuska and Osborn [5]. On the other hand, a Steklov eigenvalue problem with constant coefficients can be easily converted into the eigenvalue problem of a boundary integral equation, so the boundary element method is more advantageous in such a case. Han, Guan and He [24] developed the boundary element method for a Steklov eigenvalue problem by means of a boundary integral equation. Huang and Lü [29] used the mechanical quadrature method to obtain the extrapolation formulae for solving the boundary integral equation arising from Steklov eigenvalue problems.

This paper involves Steklov eigenvalue problems arising from structures made of composite materials. In such cases, the direct accurate numerical computation of the solution becomes difficult because of the very fine mesh required. We recall that the homogenization method gives the overall solution behavior by incorporating the fluctuations due to the heterogeneities. Vanninathan [40] obtained a homogenization result for a spectral problem with Steklov boundary conditions on periodically distributed holes inside the domain Ω. There are many other results (see, e.g. [33, 21, 23, 30, 9]). Simulation results (cf. [11], [12]) have shown that the numerical accuracy of the homogenization method may not be satisfactory if ε is not sufficiently small. We also refer to the numerical results presented in Section 5. This is the main motivation for the multiscale asymptotic methods and the associated numerical algorithms.

Sarkis and Versieux [41] presented the numerical boundary corrector for elliptic equations with rapidly oscillating periodic coefficients and derived the convergence results of their method in [39]. Hou and Wu[27] and Hou, Wu and Cai [28] provided an interesting multiscale finite element method (MsFEM) based on the first order asymptotic expansion. The basic idea of MsFEM is to find new finite element space; i.e., the set of basis functions consists of two parts, the first part being the set of piecewise polynomials and the second part the set of some oscillatory functions obtained by simultaneously solving locally partial differential equations in subdomains. Efendiev and Hou [19] gave a comprehensive survey of MsFEM. E and Engquist [17] proposed the overall framework of an important heterogeneous multi-scale method (HMM). A review of HMM was presented in [18]. In [13], authors presented recently the multiscale asymptotic method for a Steklov problem with rapidly oscillating coefficients.

The new contribution obtained in the present paper is a superapproximation estimate for solving the homogenized Steklov eigenvalue problem and to present a multiscale finite element method on the basis of the theoretical results of [13]. The key steps in this approach are the application of an adaptive finite element method to solve the cell problems and considerations of the boundary layer equation, followed by a numerical algorithm for solving the algebraic eigenvalue problem. Numerical simulations are carried out at the end to validate the theoretical results reported earlier in the paper.

The paper is organized as follows. In Section 2, we present the multiscale asymptotic expansions for the eigenvalues and the eigenfunctions of the Steklov eigenvalue problem (1) and define the boundary layer solution. Section 3 is devoted to the finite element computations of the related problems. In particular, we obtain a superapproximation estimate for solving the homogenized Steklov eigenvalue problem. In Section 4, we present the multiscale finite element method for problem (1) based on multiscale asymptotic expansions. Finally, in Section 5, we give some numerical case studies as validation for the numerical results.

2. Multiscale Asymptotic Method

In this section, we introduce the multiscale asymptotic method for problem (1), also see [13].

Let V be the closed subspace of $H^1(\Omega)$ given by

$$V = H^1(\Omega, \Gamma_0) = \{v \in H^1(\Omega), \quad | \, v = 0 \quad \text{on} \quad \Gamma_0\}.$$

Obviously $H_0^1(\Omega) \subset V \subset H^1(\Omega)$.

Assume that the space $L^2(\Gamma_1)$ is equipped with the scalar product

$$\langle \phi, \psi \rangle = \int\limits_{\Gamma_1} \phi \psi d\sigma.$$

The bilinear form on $V \times V$ associated with $\mathcal{L}_\varepsilon$ is given by

$$a_\varepsilon(\phi, \psi) = \int\limits_{\Omega} \Big(a_{ij}(\frac{x}{\varepsilon}) \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + a_0(\frac{x}{\varepsilon}) \phi \psi \Big) dx.$$

Let $(\lambda^\varepsilon, u^\varepsilon)$ be the exact Steklov eigenpair of problem (1) as given in the weak formulation:

$$a_\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon \langle u^\varepsilon, v \rangle, \quad \forall v \in V. \tag{2}$$

From assumptions $(A_2) - (A_4)$, we can easily infer that

$$\begin{aligned} &\beta_0 \|v\|_{1,\Omega}^2 \le a_\varepsilon(v, v), \\ &|a_\varepsilon(u, v)| \le \beta_1 \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad \forall u, v \in V \end{aligned}$$

where β_0, β_1 are positive constants independent of ε.

Then from the classical theory of abstract elliptic eigenvalue problems (see, e.g. [5], [36]), we obtain

Lemma 2.1 ([42, 5]) Problem (2) has a countably infinite set of eigenvalues, all of finite multiplicity, without finite accumulation point. If $\Gamma_0 = \emptyset$, then it follows that

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots \to \infty.$$

If $\Gamma_0 \ne \emptyset$, then

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty,$$

where each eigenvalue occurs as many times as given by its multiplicity. Furthermore, the orthonormal eigenfunctions u_k^ε, $k \ge 1$ form a basis of the Hilbert space $L^2(\overline{\Omega})$ with respect to the inner product $\langle u, v \rangle$ given in (2.1).

Remark 2.1 Throughout the paper we assume that all eigenfunctions $u(x)$ for the Steklov eigenvalue problem are canonical, i.e. $\|u\|_{0,\Gamma_1} = 1$, where $\|u\|^2_{0,\Gamma_1} = \langle u, u\rangle$ without loss of generality. Otherwise, we replace u by $u/\|u\|_{0,\Gamma_1}$.

In [13], we presented the multiscale asymptotic expansions of the eigenvalues and the eigenfunctions of problem (1). Setting $\xi = \varepsilon^{-1}x$ and following the terminology of [6], x, ξ are called as "slow" and "fast" variables, respectively. We define the function $u^\varepsilon_{s,k}(x)$ by

$$u^\varepsilon_{s,k}(x) = u^0_k(x) + \sum_{l=1}^{s} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi)\frac{\partial^l u^0_k(x)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}, \quad s = 1, 2. \tag{3}$$

The cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\alpha_1, \alpha_2 = 1, 2, \cdots, n$ are defined as follows

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi)\Big), \quad \xi \in Q, \\ N_{\alpha_1}(\xi) \quad \text{is 1-periodic} \quad \text{in} \quad \xi, \\ \int_Q N_{\alpha_1}(\xi)d\xi = 0, \end{cases} \tag{4}$$

and

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}(a_{ij}(\xi)\dfrac{\partial N_{\alpha_1\alpha_2}(\xi)}{\partial \xi_j}) = -\dfrac{\partial}{\partial \xi_i}(a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi)) \\ \quad -a_{\alpha_1 j}(\xi)\dfrac{\partial N_{\alpha_2}(\xi)}{\partial \xi_j} - a_{\alpha_1\alpha_2}(\xi) + \hat{a}_{\alpha_1\alpha_2}, \quad \xi \in Q, \\ N_{\alpha_1\alpha_2}(\xi) \quad \text{is 1-periodic} \quad \text{in} \quad \xi, \\ \int_Q N_{\alpha_1\alpha_2}(\xi)d\xi = 0, \end{cases} \tag{5}$$

where $\hat{a}_{ij} = \int_Q (a_{ij}(\xi) + a_{ip}(\xi)\dfrac{\partial N_j(\xi)}{\partial \xi_p})d\xi$, and the reference cell $Q = (0, 1)^n$.

The homogenized Steklov eigenvalue problem associated with problem (1) is then given by

$$\begin{cases} \mathcal{L}u^0_k \equiv -\dfrac{\partial}{\partial x_i}(\hat{a}_{ij}\dfrac{\partial u^0_k(x)}{\partial x_j}) + \langle a_0\rangle u^0_k(x) = 0, \quad \text{in} \quad \Omega, \\ u^0_k(x) = 0, \quad \text{on} \quad \Gamma_0, \\ \nu_i \hat{a}_{ij}\dfrac{\partial u^0_k(x)}{\partial x_j} = \lambda^{(0)}_k u^0_k(x) \quad \text{on} \quad \Gamma_1, \quad k \geq 1, \end{cases} \tag{6}$$

where $\vec{\nu} = (\nu_1, \cdots, \nu_n)$ is the outward unit normal to the boundary Γ_1, $(\hat{a}_{ij})$ is the homogenized coefficients matrix and $\langle a_0\rangle = \int_Q a_0(\xi)d\xi$.

If Ω is a bounded domain in R^n with Lipschitz-continuous piecewise C^{s+2} boundary $\partial\Omega$, then we can prove that the k-th eigenfunction $u^0_k \in H^{s+2}(\Omega)$ of the homogenized Steklov eigenvalue problem. However, generally speaking, for a general bounded Lipschitz polygonal domain, the condition $u^0_k \in H^{s+2}(\Omega)$, $s = 1, 2$, is invalid. To overcome this difficulty, we need to define the boundary layer solution. To begin, let us introduce the notation: We construct an interior subdomain $\overline{\Omega}_0 = \bigcup_{z\in\widehat{T}_\varepsilon} \varepsilon(z + \overline{Q}) \subset \Omega$ as illustrated in Figure 1 such that $dist(\partial\Omega_0, \partial\Omega) \geq \frac{\varepsilon}{2}$, where the index set $\widehat{T}_\varepsilon = \{z = (z_1, \cdots, z_n) \in Z^n, \quad \varepsilon(z + Q) \subset \Omega\}$, and the unit cube $Q = (0, 1)^n$. The boundary layer $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$ is as shown in Figure 2.

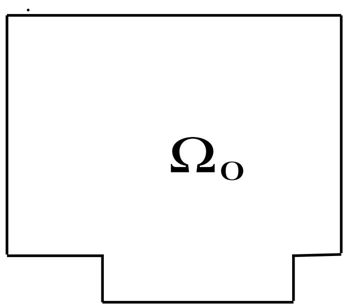


FIGURE 1. Interior subdomain Ω_0

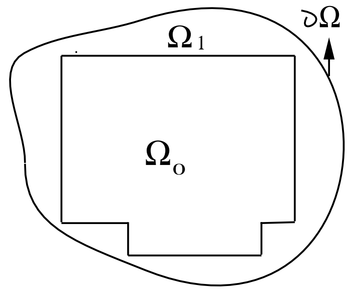


FIGURE 2. The boundary layer Ω_1

We define the boundary layer solution $u^{\varepsilon,b}_{s,k}(x)$, $k \geq 1$ given by

$$\begin{cases} \mathcal{L}_\varepsilon u^{\varepsilon,b}_{s,k} = 0, & \text{in} \quad \Omega_1, \\ u^{\varepsilon,b}_{s,k} = 0, & \text{on} \quad \Gamma_0, \\ u^{\varepsilon,b}_{s,k} = u^{\varepsilon}_{s,k}, & \text{on} \quad \Gamma^*, \\ \sigma_\varepsilon(u^{\varepsilon,b}_{s,k}) = \lambda^{(0)}_k u^{\varepsilon,b}_{s,k}(x), & \text{on} \quad \Gamma_1, \end{cases} \tag{7}$$

where $\lambda^{(0)}_k$, $u^\varepsilon_{s,k}(x)$ are given in (6) and (3).

We define the multiscale asymptotic solution by

$$\tilde{u}^\varepsilon_{s,k}(x) = \begin{cases} u^0_k(x) + \sum\limits_{l=1}^{s} \varepsilon^l \sum\limits_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi) \dfrac{\partial^l u^0_k(x)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}, \ x \in \overline{\Omega}_0 \\ u^{\varepsilon,b}_{s,k}(x), \quad x \in \Omega_1, \quad s = 1, 2. \end{cases} \tag{8}$$

Lemma 2.2 (see [13], Theorem 2.2) Suppose that Ω is a bounded Lipschitz polygonal domain or a smooth domain in R^n, $n \geq 2$ with boundary $\partial\Omega$, $\Omega_0 \subset\subset \Omega$, $\Omega_1 = \Omega \setminus \overline{\Omega}_0$. Let $(\lambda^\varepsilon_k, u^\varepsilon_k)$ be the k-th eigenpair of problem (1), and let $\tilde{u}^\varepsilon_{s,k}(x)$ be the multiscale solutions as defined in (8) associated with u^ε_k. Then we have the following estimates:

$$|\lambda^\varepsilon_k - \lambda^{(0)}_k| \leq C(k)\varepsilon^{1/2}, \quad k \geq 1. \tag{9}$$

If the multiplicity of the eigenvalues $\lambda^{(0)}_k$ is equal to t, then

$$\|\bar{u}^\varepsilon_k - \tilde{u}^\varepsilon_{s,k}\|_{1,\Omega} \leq C_s(k)\varepsilon^{1/2}, \quad s = 1, 2, \quad k \geq 1, \tag{10}$$

where $\bar{u}^\varepsilon_k$ is a linear combination of the eigenfunctions of problem (1) corresponding to $\lambda^\varepsilon_k, \cdots, \lambda^\varepsilon_{k+t-1}$. In particular, if the eigenvalue $\lambda^{(0)}_k$ is simple, then

$$\|u^\varepsilon_k - \tilde{u}^\varepsilon_{s,k}\|_{1,\Omega} \leq C_s(k)\varepsilon^{1/2}, \quad s = 1, 2, \quad k \geq 1, \tag{11}$$

where $C_s(k)$ is a constant independent of ε.

3. Finite Element Computations

In this section, we present the finite element numerical algorithms for the related problems. In particular, we obtain a superapproximation estimate for solving the homogenized Steklov eigenvalue problem.

3.1. Adaptive finite elements for calculating cell functions $N_{\alpha_1}(\xi)$ and $N_{\alpha_1\alpha_2}(\xi)$, α_1,
$\alpha_2 = 1, 2, \cdots, n$. Since the elements $a_{ij}(\xi)$ of the coefficients matrix $A(\xi)$ of (4) and (5) are discontinuous, we employ an adaptive finite element method (see [14, 43]). For convenience, we present the *a posteriori* error estimates for solving the cell problems (4). In solving (5), we use the same mesh as in solving (4). We first introduce the following notation: Let $\mathcal{T}_p$ be a sequence of tetrahedrons of the reference cell Q, and let $\mathcal{F}_p$ be the set of faces not lying on ∂Q , $p \geq 0$, i.e. $\mathcal{F}_p \cap \partial Q = \emptyset$. Note that the tetrahedrons must be aligned with the boundary of Q to employ the periodic boundary conditions on the boundary ∂Q. The finite element space U_p over $\mathcal{T}_p$ is defined by

$$U_p(Q) = \{v \in C(\overline{Q})| \quad v|_T \in P_1(T), \quad v \quad \text{takes the same value on the opposite faces of} \quad Q, \quad \forall T \in \mathcal{T}_p\}. \tag{12}$$

For any $T \in \mathcal{T}_p$ and $F \in \mathcal{F}_p$, we denote the diameters of T and F by h_T and h_F, respectively.

Let $N^{h_0}_{\alpha_1,p}(\xi)$ denote the approximate solution of $N_{\alpha_1}(\xi)$ in the finite element space $U_p(Q)$, respectively, where h_0 is the final mesh parameter of Q for the adaptive finite element method.

Following the lines of Theorems 5.2.1 of [43], also see [14], *a posteriori* error estimates for $N_{\alpha_1}(\xi)$ are given by

$$\|N_{\alpha_1} - N^{h_0}_{\alpha_1,p}\|^2_{1,Q} \leq C\Big(\sum_{T\in\mathcal{T}_p} \zeta_T^2 + \sum_{F\in\mathcal{F}_p} \zeta_F^2\Big), \tag{13}$$

where

$$\begin{aligned} \zeta_T^2 &= h_T^2 \int_T \Big(\frac{\partial a_{i\alpha_1}(\xi)}{\partial \xi_i} + \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)\frac{\partial N^{h_0}_{\alpha_1,p}(\xi)}{\partial \xi_j})\Big)^2 d\xi, \\ \zeta_F^2 &= h_F \int_F \Big(\nu_i(a_{i\alpha_1}(\xi) + a_{ij}(\xi)\frac{\partial N^{h_0}_{\alpha_1,p}(\xi)}{\partial \xi_j})\Big)^2 d\Gamma_\xi, \end{aligned}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is the outward unit normal to the element edge $F \in \mathcal{F}_p$.

We then obtain the following proposition.

Proposition 3.1 Let $N_{\alpha_1}(\xi)$ and $N_{\alpha_1\alpha_2}(\xi)$ be the weak solutions of problems (4) and (5), respectively, and let $N^{h_0}_{\alpha_1}(\xi)$, $N^{h_0}_{\alpha_1\alpha_2}(\xi)$ be the corresponding finite element solutions, where h_0 is the final mesh parameter of Q for the adaptive finite element method. If N_{α_1}, $N_{\alpha_1\alpha_2} \in H^2(Q)$, then it holds

$$\|N_{\alpha_1} - N^{h_0}_{\alpha_1}\|_{\sigma,Q} \leq Ch_0^{2-\sigma}\|N_{\alpha_1}\|_{2,Q}, \quad \sigma = 0, 1, \tag{14}$$

and

$$\|N_{\alpha_1\alpha_2} - N^{h_0}_{\alpha_1\alpha_2}\|_{\sigma,Q} \leq Ch_0^{2-\sigma}\Big(\|N_{\alpha_1}\|_{2,Q} + \|N_{\alpha_1\alpha_2}\|_{2,Q}\Big), \quad \sigma = 0, 1, \tag{15}$$

where C is a constant independent of ε, h_0.

3.2. FEM for computing the eigenvalues and the eigenfunctions of the homogenized Steklov eigenvalue problem. In the numerical computation, we actually solve the modified homogenized Steklov eigenvalue problem

$$\begin{cases} \mathcal{L}_{h_0}\tilde{u}^0_k \equiv -\dfrac{\partial}{\partial x_i}(\hat{a}^{h_0}_{ij}\dfrac{\partial \tilde{u}^0_k(x)}{\partial x_j}) + \langle a_0\rangle \tilde{u}^0_k(x) = 0 \quad \text{in} \quad \Omega \\ \tilde{u}^0_k(x) = 0, \quad \text{on} \quad \Gamma_0 \\ \nu_i\hat{a}^{h_0}_{ij}\dfrac{\partial \tilde{u}^0_k(x)}{\partial x_j} = \tilde{\lambda}^{(0)}_k \tilde{u}^0_k(x) \quad \text{on} \quad \Gamma_1, \end{cases} \tag{16}$$

where

$$\hat{a}_{ij}^{h_0} = \int_Q \Big(a_{ij}(\xi) + a_{ip}(\xi) \frac{\partial N_j^{h_0}(\xi)}{\partial \xi_p} \Big) d\xi,$$

and $N_j^{h_0}(\xi)$ is the finite element solution of $N_j(\xi)$.

Remark 3.1 It follows from $(A_1)-(A_3)$ and Proposition 3.1 that the coefficients matrix $(\hat{a}_{ij}^{h_0})$ is symmetric and positive-definite, and

$$\bar{\mu}_1 |\eta|^2 \leq \hat{a}_{ij}^{h_0} \eta_i \eta_j \leq \bar{\mu}_2 |\eta|^2, \tag{17}$$

where $\bar{\mu}_1$, $\bar{\mu}_2$ are constants independent of h_0, and $\eta = (\eta_1, \cdots, \eta_n) \in R^n$, $|\eta|^2 = \sum_{i=1}^{n} \eta_i^2$.

Next we compare (6) with (16), and estimate the difference between the eigenpair $(\lambda_k^{(0)}, u_k^0)$ of problem (6) and $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0)$ of problem (16).

Proposition 3.2 Let $(\lambda_k^{(0)}, u_k^0)$ and $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0)$, $k = 1, 2, \cdots$ be the eigenvalues and the eigenfunctions of the Steklov eigenvalue problems (6) and (16), respectively. Under assumptions $(A_1) - (A_4)$, then it holds

$$|\tilde{\lambda}_k^{(0)} - \lambda_k^{(0)}| \leq C(k) h_0^2. \tag{18}$$

Moreover, if the multiplicity of the eigenvalue $\lambda_k^{(0)}$ is equal to t, i.e.

$$\lambda_{k-1}^{(0)} < \lambda_k^{(0)} = \cdots = \lambda_{k+t-1}^{(0)} < \lambda_{k+t}^{(0)},$$

then

$$\|u_k^0 - \bar{\tilde{u}}_k^0\|_{1,\Omega} \leq C(k) h_0^2, \tag{19}$$

where $\bar{\tilde{u}}_k^0$ is a linear combination of the eigenfunctions of problem (16) corresponding to the eigenvalues $\tilde{\lambda}_k^{(0)}, \cdots, \tilde{\lambda}_{k+t-1}^{(0)}$. In particular, if $\lambda_k^{(0)}$ is simple, then

$$\|u_k^0 - \tilde{u}_k^0\|_{1,\Omega} \leq C(k) h_0^2, \tag{20}$$

where $C(k)$ is a constant independent of ε, h_0.

Proof. The proof of Proposition 3.2 follows along the lines of the proof of Theorem 4.1 of [11].

Since equations (6) and (16) are Steklov eigenvalue problems with constant coefficients, we can take higher-order derivatives on both sides of (6) and (16) in the interior subdomain Ω_0, respectively. Subtracting Eq.(16) from Eq.(6), using the interior regularity estimates for elliptic equations (see, e.g. [20]) and Proposition 3.2, we obtain

Proposition 3.3 Suppose that $\Omega_0 \subset\subset \Omega' \subset\subset \Omega$. Under assumptions $(A_1) - (A_4)$, if $u_k^0 \in H^{r+2}(\Omega')$, $0 \leq r \leq s$, then the following estimate holds:

$$\|u_k^0 - \tilde{u}_k^0\|_{r,\Omega_0} \leq C(k) h_0^2 \|u_k^0\|_{r+2,\Omega'}, \tag{21}$$

where $r = 0, \cdots, s$, $s = 1, 2$, $k \geq 1$, and $C(k)$ is a constant independent of ε, h_0.

For simplicity, we assume that $\Omega \subset R^2$ is a bounded Lipschitz convex domain, the higher dimensional cases can be discussed similarly. Let $\mathcal{J}^h = \{e\}$ be a regular family of subdivision of Ω, and satisfy the following properties:

$(\mathbf{F_1})$. The elements are uniform triangles (or rectangles) in the interior domain $\Omega_0 \subset\subset \Omega$.

$(\mathbf{F_2})$. The elements are regular triangles in region $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, and the elements are (curved) triangles near the boundary $\partial\Omega$.

$(\mathbf{F_3})$. Any face of any element e_1 is either a subset of the boundary $\partial\Omega$, or a face of another element e_2 in the subdivision.

We define a finite element space as follows: For $m \geq 1$, let

$$V_h(\Omega) = \{v \in C(\overline{\Omega}) : \quad v|_e \in \overline{P}_m(e), \quad v|_{\Gamma_0} = 0\} \subset H^1(\Omega, \Gamma_0), \tag{22}$$

where $H^1(\Omega,\Gamma_0) = \{v \in H^1(\Omega),\, v|_{\Gamma_0} = 0\}$, and $\overline{P}_m = \begin{cases} Q_m, & e \quad \text{is a rectangle} \\ P_m, & e \quad \text{is a triangle} \end{cases}$.
We follow Ciarlet's notation of finite element spaces (see [15]).

The discrete variational formulation of the modified homogenized Steklov eigenvalue problem (16) is then given by

$$A(\tilde{u}^0_{k,h}, v_h) = \tilde{\lambda}^{(0)}_{k,h} \langle \tilde{u}^0_{k,h}, v_h \rangle, \quad \forall v_h \in V_h(\Omega), \quad k = 1, 2, \cdots, \tag{23}$$

where

$$A(u,v) = \int_\Omega \Big(\hat{a}^{h_0}_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \langle a_0 \rangle uv \Big) dx, \quad \langle u, v \rangle = \int_{\Gamma_1} uv d\sigma. \tag{24}$$

For the error estimates of the finite element solutions for problem (23), we have the following known convergence results.

Lemma 3.1 ([2], Theorem 1) If $\tilde{u}^0_k \in H^{m+1}(\Omega)$, $k \geq 1$, then

$$0 \leq \tilde{\lambda}^{(0)}_{k,h} - \tilde{\lambda}^{(0)}_k \leq C(k) h^{2m}, \quad \|\tilde{u}^0_k - \tilde{u}^0_{k,h}\|_{0,\Omega} \leq C(k) h^{m+\frac{1}{2}}, \tag{25}$$

where $C(k)$ is a constant independent of ε.

Lemma 3.2 ([4]) If $\tilde{u}^0_k \in H^{m+1}(\Omega)$, $k \geq 1$, then it follows that

$$\|\tilde{u}^0_k - \tilde{u}^0_{k,h}\|_{1,\Omega} \leq C(k) h^m, \tag{26}$$

where $C(k)$ is a constant independent of ε.

Armentano [4] gave a priori estimates for the finite element method applied to the Steklov eigenvalue problem. Andreev and Todorov [2] obtained error estimates for the isoparametric finite element approximation of a Steklov eigenvalue problem. Armentano and Padra [3] presented a posteriori error estimates for this problem. As mentioned earlier, a goal of this paper is to obtain a superapproximation estimate for solving the Steklov eigenvalue problem (16). To the best of our knowledge, there are no other results of this type in the literature. We remark that our technique for superapproximation is valid only in two dimensional cases. However, superapproximation error estimates for other types eigenvalue problems have been the subject of a considerable number of theoretical and numerical results. We refer the interested reader to some classical books (see, e.g. [32], [44]).

To begin, we introduce the notation: Set $\|w\|^2_A = A(w,w)$ and define a Ritz-Galerkin projection operator $R^{(k)}_h : V_k \subset H^1(\Omega, \Gamma_0) \to V_h(\Omega)$ such that

$$A(u - R^{(k)}_h u, v_h) = 0, \quad u \in V_k, \quad \forall v_h \in V_h(\Omega). \tag{27}$$

where $V_k \subset V$ denotes the orthogonal complement of the first $k-1$ eigenspaces, $k \geq 1$ with respect to the inner product $A(u,v)$ defined in (24), i.e.
$V_k = \{v \in V,\, A(v, u_i) = 0,\, i = 1, \cdots, k-1\}$. In particular, for $k = 1$, we have $V_k = V_1 = V = H^1(\Omega, \Gamma_0)$. For simplicity, we write $R^{(1)}_h = R_h$ in the sequel.

For convenience, we set $\lambda = \tilde{\lambda}^{(0)}_k$, $\lambda_h = \tilde{\lambda}^{(0)}_{k,h}$, and assume that $H_\lambda(\Gamma_1)$ is the restriction of the eigenspace of the operator $\mathcal{L}_{h_0}$ with respect to the eigenvalue $\lambda = \tilde{\lambda}^{(0)}_k$, $k \geq 1$ on the boundary Γ_1. Define a projection operator $P_\lambda : L^2(\Gamma_1) \to H_\lambda(\Gamma_1)$ such that

$$P_\lambda v = \sum_{\lambda_i = \lambda} \langle v, u_i \rangle u_i, \tag{28}$$

where $\langle \varphi, g\rangle = \int_{\Gamma_1} \varphi g d\sigma$, and $\{u_i\}$ form a set of orthonormal basis of $H_\lambda(\Gamma_1)$ with respect to the inner product $\langle \varphi, g\rangle$.

To derive the superapproximation estimate for the finite element method of problem (23), see Theorem 3.1, we have to present Propositions 3.4, 3.5 and 3.6. We would like to emphasize that these propositions are proved only for the first eigenpair $(\tilde{\lambda}_1^{(0)}, \tilde{u}_1^{(0)})$ of the modified homogenized Steklov eigenvalue problem (16), and they are actually valid for other cases $k > 1$.

Proposition 3.4 Suppose that $\Omega \subset R^2$ is a bounded Lipschitz convex domain or a smooth domain. Let $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0)$ and $(\tilde{\lambda}_{k,h}^{(0)}, \tilde{u}_{k,h}^0)$, $k \geq 1$ be k-th eigenvalues and eigenfunctions of problems (16) and (23), respectively. Then the following relations hold

$$0 \leq \frac{A(w,w)}{\langle w,w\rangle} - \tilde{\lambda}_k^{(0)} \leq \frac{\|w-\tilde{u}_k^0\|_A^2}{\langle w,w\rangle}, \quad \forall w \in V_k, \tag{29}$$

$$0 < \tilde{\lambda}_k^{(0)} \leq \frac{A(\tilde{u}_{k,h}^0, \tilde{u}_{k,h}^0)}{\langle \tilde{u}_{k,h}^0, \tilde{u}_{k,h}^0\rangle} = \tilde{\lambda}_{k,h}^{(0)} \leq \frac{A(v,v)}{\langle v,v\rangle}, \quad \forall v \in V_h(\Omega), \tag{30}$$

and

$$0 \leq \tilde{\lambda}_{k,h}^{(0)} - \tilde{\lambda}_k^{(0)} \leq \frac{A(R_h^{(k)}\tilde{u}_k^0, R_h^{(k)}\tilde{u}_k^0)}{\langle R_h^{(k)}\tilde{u}_k^0, R_h^{(k)}\tilde{u}_k^0\rangle} - \tilde{\lambda}_k^{(0)} \leq \frac{\|R_h^{(k)}\tilde{u}_k^0 - \tilde{u}_k^0\|_A^2}{\langle R_h^{(k)}\tilde{u}_k^0, R_h^{(k)}\tilde{u}_k^0\rangle}, \quad k \geq 1, \tag{31}$$

where $R_h^{(k)}$ is defined in (27), in particular, $R_h^{(1)} = R_h$. $A(w,w)$, $\langle w,w\rangle$ are given in (24).

Proof. We prove Proposition 3.4 only for the first eigenvalue and the first eigenfunction, i.e. $k = 1$.

Estimate (30) is a straightforward consequence of the relation $V_h(\Omega) \subset V = H^1(\Omega, \Gamma_0)$. It remains to give the proofs of (29) and (31).

For any fixed $w \in V$, $w \neq 0$, since $\|w\|_A^2 = \tilde{\lambda}_1^{(0)}\langle w, \tilde{u}_1^0\rangle^2 + \|w - \langle w, \tilde{u}_1^0\rangle \tilde{u}_1^0\|_A^2$, we have

$$\tilde{\lambda}_1^{(0)} = \min_{v\in V,\ v\neq 0} \frac{\|v\|_A^2}{\|v\|_{0,\Gamma_1}^2} \leq \frac{\|w\|_A^2}{\|w\|_{0,\Gamma_1}^2} \leq \tilde{\lambda}_1^{(0)} + \frac{\|w-(w,\tilde{u}_1^0)\tilde{u}_1^0\|_A^2}{\|w\|_{0,\Gamma_1}^2}. \tag{32}$$

On the other hand, since $\|\tilde{u}_1^0\|_{0,\Gamma_1} = 1$ (see Remark 2.1), for any fixed $w \in V$, $w \neq 0$, one can directly verify that

$$A(w - \langle w, \tilde{u}_1^0\rangle \tilde{u}_1^0, \alpha \tilde{u}_1^0) = 0, \quad \forall \alpha \in R, \tag{33}$$

and consequently

$$\|w-\tilde{u}_1^0\|_A^2 = \|w - \langle w,\tilde{u}_1^0\rangle\tilde{u}_1^0\|_A^2 + \|\langle w,\tilde{u}_1^0\rangle\tilde{u}_1^0 - \tilde{u}_1^0\|_A^2 \geq \|w - \langle w,\tilde{u}_1^0\rangle\tilde{u}_1^0\|_A^2. \tag{34}$$

Combining (32) and (34) gives

$$0 \leq \frac{A(w,w)}{\langle w,w\rangle} - \tilde{\lambda}_1^{(0)} \leq \frac{\|w - \langle w,\tilde{u}_1^0\rangle\tilde{u}_1^0\|_A^2}{\|w\|_{0,\Gamma_1}^2} \leq \frac{\|w-\tilde{u}_1^0\|_A^2}{\|w\|_{0,\Gamma_1}^2}. \tag{35}$$

(35) actually proves (29). Putting $w = R_h\tilde{u}_1^0$ into (29), and recalling (30) , we get

$$0 \leq \tilde{\lambda}_{1,h}^{(0)} - \tilde{\lambda}_1^{(0)} \leq \frac{\|R_h\tilde{u}_1^0 - \tilde{u}_1^0\|_A^2}{\langle R_h\tilde{u}_1^0, R_h\tilde{u}_1^0\rangle}. \tag{36}$$

The proof of Proposition 3.4 is complete.

In order to obtain Propositions 3.5 and 3.6, we first consider the mixed boundary value problem given by

$$
\left\{
\begin{array}{l}
\mathcal{L}_{h_0} w(x) = 0, \quad x \in \Omega, \\
w(x) = 0, \quad x \in \Gamma_0, \\
\nu_i \hat{a}_{ij}^{h_0} \dfrac{\partial w(x)}{\partial x_j} = g(x), \quad x \in \Gamma_1.
\end{array}
\right. \tag{37}
$$

where the operator $\mathcal{L}_{h_0}$ is defined in (16).

The variational form of (37) is to find $w \in V = H^1(\Omega, \Gamma_0)$ such that

$$
A(w, v) = \langle g, v \rangle, \quad g \in L^2(\Gamma_1), \quad \forall v \in V. \tag{38}
$$

It follows from (17) that problem (38) is uniquely solvable. For $g \in L^2(\Gamma_1)$ the solution w is in V. We define the operator $\mathcal{B} : L^2(\Gamma_1) \to V$ by $\mathcal{B}g = w$. Now let us consider the operator $\mathcal{T} : L^2(\Gamma_1) \to L^2(\Gamma_1)$ as the restriction of $\mathcal{B}$ on Γ_1, i.e. $\mathcal{T}g = (\mathcal{B}g)|_{\Gamma_1}$. We can check that $\mathcal{T} : L^2(\Gamma_1) \to L^2(\Gamma_1)$ is a linear self-adjoint compact operator in a Hilbert space $L^2(\Gamma_1)$, see [7] and [2].

Proposition 3.5 Suppose that $\Omega \subset R^2$ is a bounded Lipschitz convex domain or a smooth domain. Let $R_h^{(k)} : V_k \to V_h(\Omega)$, $k \geq 1$ be the Ritz-Galerkin projection operator defined in (27), in particular, $R_h^{(1)} = R_h$. Under assumptions $(A_1) - (A_4)$, $(F_2) - (F_3)$, we have

$$
\|\lambda_h R_h^{(k)} \mathcal{B} - \lambda \mathcal{B}\|_b \leq C(k) h^{1/2}, \tag{39}
$$

and

$$
\|R_h^{(k)} \mathcal{B}\|_b \leq C(k), \tag{40}
$$

where $\|K\|_b = \|K\|_{L^2(\Gamma_1) \to L^2(\Gamma_1)}$ is the norm of an operator K: $L^2(\Gamma_1) \to L^2(\Gamma_1)$, and $C(k)$ is a constant independent of ε, h, h_0.

Proof. We only prove Proposition 3.5 for the case $k = 1$. If Ω is a bounded Lipschitz convex domain or a smooth domain, for $g \in L^2(\Gamma_1)$, then we can infer that $\mathcal{B}g \in H^{3/2}(\Omega)$ and $\|\mathcal{B}g\|_{3/2,\Omega} \leq C\|g\|_{0,\Gamma_1}$ (see [2, 7]). On the other hand, it follows from Theorem 4.4.4 of ([8], p.104) and the interpolation error estimates of Sobolev spaces that

$$
\|(R_h \mathcal{B} - \mathcal{B})g\|_{1,\Omega} \leq C h^{1/2} \|\mathcal{B}g\|_{3/2,\Omega}.
$$

By using the trace theorem, we get

$$
\begin{aligned}
\|(R_h \mathcal{B} - \mathcal{B})g\|_{0,\Gamma_1} &\leq C\|(R_h \mathcal{B} - \mathcal{B})g\|_{1,\Omega} \\
&\leq C h^{1/2} \|\mathcal{B}g\|_{3/2,\Omega} \leq C h^{1/2} \|g\|_{0,\Gamma_1},
\end{aligned}
$$

and consequently

$$
\|(R_h \mathcal{B} - \mathcal{B})\|_b \leq C h^{1/2}. \tag{41}
$$

It follows from Lemma 3.1 and (41) that

$$
\begin{aligned}
\|\lambda_h R_h \mathcal{B} - \lambda \mathcal{B}\|_b &\leq \|\lambda_h R_h \mathcal{B} - \lambda_h \mathcal{B} + \lambda_h \mathcal{B} - \lambda \mathcal{B}\|_b \\
&\leq \lambda_h \|(R_h \mathcal{B} - \mathcal{B})\|_b + |\lambda_h - \lambda| \|\mathcal{B}\|_b \\
&\leq C h^{1/2} + C h^2 \leq C h^{1/2}.
\end{aligned}
$$

Since $\|\lambda_h R_h \mathcal{B}\|_b \leq \|\lambda_h R_h \mathcal{B} - \lambda \mathcal{B}\|_b + \|\lambda \mathcal{B}\|_b$, using (39) and $\|\mathcal{B}\|_b \leq C$, we get $\|R_h \mathcal{B}\|_b \leq C$.

Therefore the proof of Proposition 3.5 is complete.

Proposition 3.6 Suppose that $\Omega \subset R^2$ is a bounded Lipschitz convex domain or a smooth domain. Let $R_h^{(k)} : V_k \to V_h(\Omega)$, $k \geq 1$ be the Ritz-Galerkin projection operator defined in (27), in particular, $R_h^{(1)} = R_h$. If $u \in W^{m+1,q}(\Omega)$, $q > 2$, then we have the following estimate:

$$\|R_h^{(k)}\mathcal{B}(I - R_h^{(k)})u\|_{0,\Gamma_1} \leq C(k)h^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}, \quad m \geq 1, \tag{42}$$

where I is an identity operator, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $\frac{4}{3} < p < 2$, $q > 2$, and $C(k)$ is a constant independent of ε, h, h_0.

Proof. We only prove Proposition 3.6 for the case $k = 1$. We define a solution $\tilde{w} \in V = H^1(\Omega, \Gamma_0)$ such that

$$A(\tilde{w}, v) = \langle R_h\mathcal{B}(I - R_h)u, v\rangle, \quad \forall v \in V. \tag{43}$$

Using a priori estimates for elliptic equations (cf.[20]), it is obvious that $\|\tilde{w}\|_{1,\Omega} \leq C\|R_h\mathcal{B}(I - R_h)u\|_{0,\Gamma_1}$.

Let $\mathcal{B}^*$ be the adjoint operator of $\mathcal{B}$ with respect to the inner product $\langle \varphi, g\rangle$, in such a way that for all $\varphi \in H^1(\Omega)$, $A(\varphi, \mathcal{B}^*) = \langle \varphi, g\rangle$. Using (3.3) of [2], we have

$$\|\mathcal{B}^*g\|_{3/2,\Omega} \leq C^*\|g\|_{0,\Gamma_1}, \quad \forall g \in L^2(\Gamma_1), \tag{44}$$

where C^* is a constant independent of ε, h_0, h.

Let us introduce an interpolation operator $I_m : V \to V_h(\Omega)$. It follows from Theorem 4.4.4 of ([8], p.104) and the interpolation error estimates of Sobolev spaces that

$$\|\mathcal{B}^*R_h\tilde{w} - I_m\mathcal{B}^*R_h\tilde{w}\|_{1,p,\Omega} \leq Ch^{1/2}\|\mathcal{B}^*R_h\tilde{w}\|_{3/2,p,\Omega}, \quad m \geq 1, \tag{45}$$

where C is a constant independent of ε, h_0, h.

Using (27) and the interpolation error estimates, it is not difficult to check that

$$\|u - R_hu\|_{1,q,\Omega} \leq Ch^m\|u\|_{m+1,q,\Omega}, \quad m \geq 1, \tag{46}$$

where C is a constant independent of ε, h_0, h.

Using (43)-(46), Hölder inequality and the trace theorem, we obtain

$$\begin{aligned}
\|R_h\mathcal{B}(I - R_h)u\|^2_{0,\Gamma_1} &= A(\tilde{w}, R_h\mathcal{B}(I - R_h)u) \\
&= A(\mathcal{B}^*R_h\tilde{w}, (I - R_h)u) \\
&= A(\mathcal{B}^*R_h\tilde{w} - I_m\mathcal{B}^*R_h\tilde{w}, (I - R_h)u) \text{ (see (3.14), } k = 1) \\
&\leq C\|\mathcal{B}^*R_h\tilde{w} - I_m\mathcal{B}^*R_h\tilde{w}\|_{1,p,\Omega}\|u - R_hu\|_{1,q,\Omega} \\
&\leq Ch^{m+\frac{1}{2}}\|\mathcal{B}^*R_h\tilde{w}\|_{3/2,p,\Omega}\|u\|_{m+1,q,\Omega} \\
&\leq Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|R_h\tilde{w}\|_{0,p,\Gamma_1} \\
&\leq Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|R_h\tilde{w}\|_{0,\Gamma_1} \text{ (since } \quad 4/3 < p < 2) \\
&\leq Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|R_h\tilde{w}\|_{1,\Omega} \\
&\leq Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|\tilde{w}\|_{1,\Omega} \text{ (see (40))} \\
&\leq Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|R_h\mathcal{B}(I - R_h)u\|_{0,\Gamma_1},
\end{aligned}$$

and consequently

$$\|R_h\mathcal{B}(I - R_h)u\|_{0,\Gamma_1} \leq Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}.$$

The proof of Proposition 3.6 is complete.

Theorem 3.1 Suppose that $\Omega \subset R^2$ is a bounded Lipschitz convex domain or a smooth domain. Let H_λ be the eigenspace of the k-th eigenvalue $\lambda = \tilde{\lambda}_k^{(0)}$ for problem (16), and $V_{\lambda_h} \subset V_h$ be the eigenspace of $\lambda_h = \tilde{\lambda}_{k,h}^{(0)}$ for problem (23). Assume that $H_\lambda \subset W^{m+1,q}(\Omega)$, $q > 2$, $m \geq 1$. For $\forall u_h \in V_{\lambda_h}$, then there exists $u \in H_\lambda$ such that

$$\|R_h^{(k)}u - u_h\|_{1,\Omega} \leq C(k)h^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}, \tag{47}$$

where $R_h^{(k)} : V_k \to V_h(\Omega)$, $k \geq 1$ are the Ritz-Galerkin projection operators defined in (27), in particular, $R_h^{(1)} = R_h$ and $C(k)$ is a constant independent of ε, h_0, h.

Proof. We prove Theorem 3.1 only for the case $k = 1$. Other cases can be discussed similarly. Let $\lambda = \tilde{\lambda}_1^{(0)}$, $\lambda_h = \tilde{\lambda}_{1,h}^{(0)}$ and have $\lambda > 0$, $\lambda_h > 0$. Set

$$\mathcal{M} = \{\varphi \in L^2(\Gamma_1), \quad \langle \varphi, v\rangle = 0, \forall v \in H_\lambda\}.$$

We recall $\mathcal{T} : L^2(\Gamma_1) \to L^2(\Gamma_1)$ is a self-adjoint compact operator. For $\forall \varphi \in \mathcal{M}$, since $\langle \mathcal{T}\varphi, v\rangle = \langle \varphi, \mathcal{T}v\rangle = \frac{1}{\lambda}\langle \varphi, v\rangle = 0$, we infer that $\mathcal{T}\mathcal{M} \subset \mathcal{M}$. Denote by $\mathcal{T}|_{\mathcal{M}}$ the restriction of $\mathcal{T}$ on $\mathcal{M}$. It follows from Fredholm's alternative theorem that the operator $(I - \lambda\mathcal{T}|_{\mathcal{M}})$ has a bounded inverse operator. Hence there exists a constant $\gamma > 0$ such that

$$\gamma\|v\|_{0,\Gamma_1} \leq \|(I - \lambda\mathcal{T}|_{\mathcal{M}})v\|_{0,\Gamma_1}, \quad \forall v \in \mathcal{M}. \tag{48}$$

We know $P_\lambda : L^2(\Gamma_1) \to H_\lambda$ is a projection operator and $P_\lambda v = \sum\limits_{\lambda_i=\lambda} \langle v, u_i\rangle u_i$, $v \in L^2(\Gamma_1)$. For $\forall u_h \in V_{\lambda_h} \subset V_h$, we set $u = P_\lambda u_h \in H_\lambda$. It remains to prove that u satisfies (47).

Setting $\bar{u} = R_h u - u_h - P_\lambda(R_h u - u_h)$, where $R_h = R_h^{(1)} : V_1 = V \to V_h(\Omega)$ is the Ritz-Galerkin projection operator defined in (27), it is easy to check that $\bar{u} \in \mathcal{M}$. On the other hand, we have

$$\begin{aligned}\lambda\mathcal{B}u &= \lambda\mathcal{B}P_\lambda u_h = \lambda\mathcal{B}\sum_{\lambda_i=\lambda}\langle u_h, u_i\rangle u_i\\ &= \lambda\sum_{\lambda_i=\lambda}\langle u_h, u_i\rangle\mathcal{B}u_i = \lambda\sum_{\lambda_i=\lambda}\langle u_h, u_i\rangle\frac{1}{\lambda}u_i\\ &= P_\lambda u_h = u.\end{aligned}$$

Since

$$\begin{aligned}A(\lambda_h R_h\mathcal{B}u_h, v_h) &= \lambda_h A(R_h\mathcal{B}u_h, v_h) = \lambda_h A(\mathcal{B}u_h, v_h)\\ &= \lambda_h A(\frac{1}{\lambda_h}u_h, v_h) = A(u_h, v_h), \quad \forall v_h \in V_h(\Omega),\end{aligned}$$

we thus get $A(\lambda_h R_h\mathcal{B}u_h - u_h, v_h) = 0$, $\forall v_h \in V_h$ and $\lambda_h R_h\mathcal{B}u_h = u_h$.

Given $\bar{u} \in \mathcal{M}$, we have $(I - \lambda\mathcal{T}|_{\mathcal{M}})\bar{u} = (I - \lambda\mathcal{T})\bar{u}$. We recall the operator $\mathcal{T} : L^2(\Gamma_1) \to L^2(\Gamma_1)$ as the restriction of $\mathcal{B}$ on Γ_1 and obtain

$$(I - \lambda\mathcal{T})P_\lambda(R_h u - u_h) = 0,$$

$$\begin{aligned}\|(I - \lambda\mathcal{T})\bar{u}\|_{0,\Gamma_1} &= \|(I - \lambda\mathcal{T})(R_h u - u_h)\|_{0,\Gamma_1}\\ &= \|(I - \lambda\mathcal{B})(R_h u - u_h)\|_{0,\Gamma_1}.\end{aligned}$$

Hence we derive

$$\begin{array}{rcl}\gamma\|\bar{u}\|_{0,\Gamma_1} & \leq & \|(I - \lambda\mathcal{T}|_{\mathcal{M}})\bar{u}\|_{0,\Gamma_1} \quad \text{(since } (48))\\ & = & \|(I - \lambda\mathcal{T})\bar{u}\|_{0,\Gamma_1} = \|(I - \lambda\mathcal{B})(R_h u - u_h)\|_{0,\Gamma_1}\\ & = & \|\lambda R_h\mathcal{B}(I - R_h)u + (\lambda_h R_h\mathcal{B} - \lambda\mathcal{B})(R_h u - u_h)\\ & + & (\lambda - \lambda_h)R_h\mathcal{B}R_h u\|_{0,\Gamma_1} \leq \lambda\|R_h\mathcal{B}(I - R_h)u\|_{0,\Gamma_1}\\ & + & \|(\lambda_h R_h\mathcal{B} - \lambda\mathcal{B})(R_h u - u_h)\|_{0,\Gamma_1} + \|(\lambda - \lambda_h)R_h\mathcal{B}R_h u\|_{0,\Gamma_1}\\ & \leq & \lambda\|R_h\mathcal{B}(I - R_h)u\|_{0,\Gamma_1} + \|\lambda_h R_h\mathcal{B} - \lambda\mathcal{B}\|_b\|R_h u - u_h\|_{0,\Gamma_1}\\ & + & Ch^{2m}\|R_h\mathcal{B}\|_b\|R_h u\|_{0,\Gamma_1}.\end{array} \tag{49}$$

Given

$$\begin{aligned}P_\lambda u &= P_\lambda(P_\lambda u_h) = P_\lambda\sum_{\lambda_i=\lambda}\langle u_h, u_i\rangle u_i = \sum_{\lambda_i=\lambda}\langle u_h, u_i\rangle P_\lambda u_i\\ &= \sum_{\lambda_i=\lambda}\langle u_h, u_i\rangle u_i = P_\lambda u_h,\end{aligned}$$

here we have used $\langle u_j, u_i\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. By using standard error interpolation, we thus obtain

$$\begin{aligned}
\|P_\lambda(R_hu-u_h)\|_{0,\Gamma_1} &= \|P_\lambda(R_hu-u)\|_{0,\Gamma_1}\\
&= \|\sum_{\lambda_i=\lambda}\langle R_hu-u,u_i\rangle u_i\|_{0,\Gamma_1}\\
&\le \sum_{\lambda_i=\lambda}|\langle R_hu-u,u_i\rangle|\|u_i\|_{0,\Gamma_1}\\
&= \sum_{\lambda_i=\lambda}\frac{1}{\lambda}|A(R_hu-u,u_i)|\\
&= \sum_{\lambda_i=\lambda}\frac{1}{\lambda}|A(R_hu-u,u_i-I_mu_i)|\ (\text{since}\quad(27))\\
&\le Ch^{2m}.
\end{aligned}\tag{50}$$

We recall $\bar{u} = R_hu - u_h - P_\lambda(R_hu-u_h)$, and derive

$$\|\bar{u}\|_{0,\Gamma_1} \ge \|R_hu-u_h\|_{0,\Gamma_1} - \|P_\lambda(R_hu-u_h)\|_{0,\Gamma_1}.\tag{51}$$

Substituting (51) into (49), we obtain

$$\begin{aligned}
&\Big(1-\frac{1}{\gamma}\|\lambda_hR_h\mathcal{B}-\lambda\mathcal{B}\|_b\Big)\|R_hu-u_h\|_{0,\Gamma_1}\\
&\le \frac{\lambda}{\gamma}\|R_h\mathcal{B}(I-R_h)u\|_{0,\Gamma_1} + Ch^{2m}(C+\|R_h\mathcal{B}\|_b\|R_hu\|_{0,\Gamma_1}).
\end{aligned}\tag{52}$$

Combining (39), (40) and (42) yields

$$\|R_hu-u_h\|_{0,\Gamma_1} \le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}.\tag{53}$$

Here we have used the estimate: $\|R_hu\|_{0,\Gamma_1} \le C\|u\|_{1,\Omega}$.

On the other hand, since $A(u-R_hu, v_h) = 0, \quad \forall v_h \in V_h$, we have

$$\begin{aligned}
A(u_h-R_hu,v_h) &= A(u_h-u,v_h)\\
&= \lambda_h\langle u_h,v_h\rangle - \lambda\langle u,v_h\rangle\\
&= (\lambda_h-\lambda)\langle u_h,v_h\rangle + \lambda\langle u_h-u,v_h\rangle\\
&= (\lambda_h-\lambda)\langle u_h,v_h\rangle + \lambda\langle u_h-R_hu,v_h\rangle\\
&+ \lambda\langle R_hu-u,v_h\rangle,\quad \forall v_h\in V_h.
\end{aligned}\tag{54}$$

We recall (38) and the operator $\mathcal{B} : L^2(\Gamma_1) \to V$ by $\mathcal{B}g = w$. For any fixed $v_h \in V_h$, if we set $g = v_h$, then we have $w = \mathcal{B}v_h$ and

$$A(w, R_hu-u) = \langle v_h, R_hu-u\rangle.$$

Using (27), Hölder inequality, the interpolation error estimates and the trace theorem, we obtain

$$\begin{aligned}
\langle R_hu-u,v_h\rangle &= A(R_hu-u,w-I_mw)\\
&\le \|R_hu-u\|_{1,q,\Omega}\|w-I_mw\|_{1,p,\Omega}\\
&\le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|w\|_{3/2,p,\Omega}\\
&\le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|v_h\|_{0,p,\Gamma_1}\\
&\le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|v_h\|_{1,p,\Omega}\\
&\le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|v_h\|_{1,\Omega},
\end{aligned}\tag{55}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $q > 2$, $4/3 < p < 2$. Substituting (55) into (54), we get

$$|A(u_h-R_hu,v_h)| \le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}\|v_h\|_{1,\Omega},\quad \forall v_h\in V_h.$$

Furthermore, we have

$$\|u_h-R_hu\|_{1,\Omega} \le Ch^{m+\frac{1}{2}}\|u\|_{m+1,q,\Omega}.$$

where C is a constant independent of ε, h_0, h. Therefore, the proof of Theorem 3.1 is complete.

Remark 3.2 The basic idea of the proof of Theorem 3.1 originates from the proof of Proposition 4.5 of [11]. However, there is the essential difference between them. In (38), we defined the operator $\mathcal{B} : L^2(\Gamma_1) \to H^1(\Omega, \Gamma_0)$ by $\mathcal{B}g = w$. In [11], the corresponding operator $\mathcal{K}$ was defined as $L^2(\Omega) \to H_0^1(\Omega)$. Since we cannot use the embedding theorems in the former case, the proof of Theorem 3.1 is much more complicated than that of Proposition 4.5 of [11].

Now we use the superconvergence results of Theorem 3.1 to implement the post-processing technique for calculating the higher-order derivatives $\dfrac{\partial^l \tilde{u}_k^0}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}}$, $l = 1, 2$, $\alpha_1, \alpha_2 = 1, 2$, where $\tilde{u}_k^0$ is the k-th eigenfunction of problem (16).

Following the terminology of [32], we define the novel bi-2m-th (2m-th) order interpolation operator and denote by $\mathcal{I}_{2h}^{(2m)}$ the operator. The crucial idea of the interpolated finite element method is the following: If we know the nodal values of the bi-m-th (or m-th) finite element solution in a fine mesh, then we use these nodal values to define a bi-2m-th (or 2m-th) interpolation function at a new larger element with respect to a coarse mesh as shown as in Figs.3 and 4. It should be emphasized that the mesh must be uniform, i.e. the condition (F_1). We refer the interested reader to Lin's book [32].

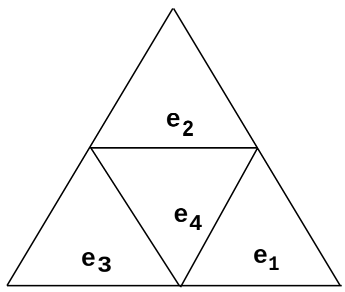


FIGURE 3. Triangular mesh.

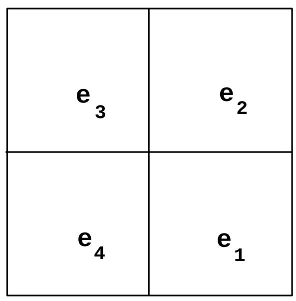


FIGURE 4. Rectangular mesh.

In order to obtain the global superapproximation estimate for the postprocessing operator (see Theorem 3.2), we need to employ the following lemmas.

Lemma 3.3 [32] Let $I_m : V \to V_h(\Omega)$ be a Lagrange's interpolation operator. Then the interpolation operators I_m and $\mathcal{I}_{2h}^{(2m)}$ satisfy the following properties:

$$\|\mathcal{I}_{2h}^{(2m)} v_h\|_{\sigma,p} \le C\|v_h\|_{\sigma,p}, \quad 1 \le p \le \infty,\, \sigma = 0, 1,\ \forall\, v_h \in V_h(\Omega), \tag{56}$$

$$(\mathcal{I}_{2h}^{(2m)})^2 = \mathcal{I}_{2h}^{(2m)}, \quad \mathcal{I}_{2h}^{(2m)} I_m = \mathcal{I}_{2h}^{(2m)}, \quad I_m \mathcal{I}_{2h}^{(2m)} = I_m, \tag{57}$$

$$\forall\, z_i \in \mathcal{N}_h,\, \mathcal{I}_{2h}^{(2m)} v(z_i) = I_m v(z_i) = v(z_i), \quad v \in C(\overline{\Omega}),$$

where $\mathcal{N}_h$ is the set of nodal points of $\mathcal{J}^h = \{e\}$.

$$\|v - \mathcal{I}_{2h}^{(2m)} v\|_{\sigma,p,E} \le Ch^{2m+1-\sigma}\|v\|_{2m+1,p,E}, \tag{58}$$

where

$$\forall v \in W^{2m+1,p}(E), \quad \sigma = 0, 1,\, 1 \le p \le +\infty,\, \forall E \in \mathcal{J}^{2h}.$$

To prove Theorem 3.1, we introduce the following lemma:

Lemma 3.4 [45] Suppose that $\Omega \subset R^2$ is a bounded Lipschitz convex domain or a smooth domain. Under the assumptions of $(F_1) - (F_3)$, we have the following estimate:

(59)
$$|A(u - I_m u, v_h)| \le \begin{cases} Ch^{m+1}\|u\|_{m+2,q,\Omega}\|v_h\|_{1,p,\Omega},\ m = 1,2, & e \text{ is a triangle;} \\ Ch^{m+1}\|u\|_{m+2,q,\Omega}\|v_h\|_{1,p,\Omega},\ m \ge 1, & e \text{ is a rectangle,} \end{cases}$$

for $\forall v_h \in V_h(\Omega)$, $I_m : C(\overline{\Omega}) \to V_h(\Omega)$ is a Lagrange's interpolation operator, and $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q < \infty$.

Next we give the global superapproximation estimate for the postprocessing operator.

Theorem 3.2 Suppose that $\Omega \subset R^2$ is a bounded Lipschitz convex domain or a smooth domain. Let $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ be the k-th eigenpair of problem (16), and let $(\tilde{\lambda}_{k,h}^{(0)}, \tilde{u}_{k,h}^0(x))$ be the corresponding finite element solution of $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ in $V_h(\Omega)$. Under assumptions $(A_1) - (A_4)$, $(F_1) - (F_3)$, if $\tilde{u}_k^0 \in H^{m+2}(\Omega)$, then we obtain the following superapproximation estimate:

(60)
$$\|\tilde{u}_k^0(x) - \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0(x)\|_{1,\Omega} \le \begin{cases} C(k)h^{m+\frac{1}{2}}\|\tilde{u}_k^0\|_{m+2,\Omega},\ m = 1,2, & e \text{ is a triangle;} \\ C(k)h^{m+\frac{1}{2}}\|\tilde{u}_k^0\|_{m+2,\Omega},\ m \ge 1, & e \text{ is a rectangle,} \end{cases}$$

where $C(k) > 0$ is a constant independent of ε, h_0, h; $k \ge 1$.

Proof. As mentioned above, we prove Theorem 3.2 only for the first eigenpair of problem (16) and for the rectangular mesh. Other cases can be discussed similarly.

It follows from Lemma 3.4 that

$$\begin{aligned} \bar{\mu}_1\|I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0\|_{1,\Omega}^2 &\le A(I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0, I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0) \\ &= A(I_m\tilde{u}_1^0 - \tilde{u}_1^0, I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0) \\ &\le Ch^{m+1}\|\tilde{u}_1^0\|_{m+2,\Omega}\|I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0\|_{1,\Omega}, \end{aligned}$$

and

$$\|I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0\|_{1,\Omega} \le Ch^{m+1}\|\tilde{u}_1^0\|_{m+2,\Omega}, \tag{61}$$

where C is a constant independent of ε, h_0, h.

It follows from Lemma 3.3, Theorem 3.1, and (61) that

$$\begin{aligned} \|\mathcal{I}_{2h}^{(2m)}\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2m)}\tilde{u}_{1,h}^0\|_{1,\Omega} &= \|\mathcal{I}_{2h}^{(2m)}(I_m\tilde{u}_1^0 - \tilde{u}_{1,h}^0)\|_{1,\Omega} \\ &\le C\|I_m\tilde{u}_1^0 - \tilde{u}_{1,h}^0\|_{1,\Omega} \le C\|I_m\tilde{u}_1^0 - R_h\tilde{u}_1^0\|_{1,\Omega} \\ &+ C\|R_h\tilde{u}_1^0 - \tilde{u}_{1,h}^0\|_{1,\Omega} \le Ch^{m+1}\|\tilde{u}_1^0\|_{m+2,\Omega} \\ &+ C\|R_h\tilde{u}_1^0 - \tilde{u}_{1,h}^0\|_{1,\Omega} \le Ch^{m+\frac{1}{2}}\|\tilde{u}_1^0\|_{m+2,\Omega}, \end{aligned} \tag{62}$$

and consequently

$$\begin{aligned} \|\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2m)}\tilde{u}_{1,h}^0\|_{1,\Omega} &\le \|\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2m)}\tilde{u}_1^0\|_{1,\Omega} \\ &+ \|\mathcal{I}_{2h}^{(2m)}\tilde{u}_1^0 - \mathcal{I}_{2h}^{(2m)}\tilde{u}_{1,h}^0\|_{1,\Omega} \\ &\le Ch^{m+\frac{1}{2}}\|\tilde{u}_1^0\|_{m+2,\Omega}. \end{aligned} \tag{63}$$

Therefore the proof of Theorem 3.2 is complete.

We define

$$\tilde{\lambda}_{k,2h}^{(2m)} = \frac{A(\mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0, \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0)}{\langle \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0, \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0\rangle}, \tag{64}$$

where the operator $\mathcal{I}_{2h}^{(2m)}$ is given in Lemma 3.3, and $\tilde{u}_{k,h}^0(x)$ is the finite element solution of the k-th eigenfunction $\tilde{u}_k^0(x)$ of problem (16). $A(u,v)$, $\langle u,v\rangle$ are defined in (24).

To get Corollary 3.1, we need to introduce the following lemma:

Lemma 3.5 (see [44]) Let (λ, u) be the eigenpair of the Steklov eigenvalue problem (16). Then, for $\forall w \in V$, $\|w\|_{0,\Gamma_1} \neq 0$, we have

$$\frac{A(w,w)}{\langle w,w\rangle} - \lambda = \frac{\|w-u\|_A^2}{\|w\|_{0,\Gamma_1}^2} - \lambda\frac{\|w-u\|_{0,\Gamma_1}^2}{\|w\|_{0,\Gamma_1}^2}, \tag{65}$$

and consequently

$$\left|\frac{A(w,w)}{\langle w,w\rangle} - \lambda\right| \leq C\frac{\|w-u\|_A^2}{\|w\|_{0,\Gamma_1}^2}, \tag{66}$$

where C is a constant, $\|w\|_A = A(w,w)$, $\|w\|_{0,\Gamma_1}^2 = \langle w,w\rangle$, $A(u,v)$, $\langle u,v\rangle$ are defined in (24).

Proof. We directly check that

$$\|w-u\|_A^2 - \lambda\|w-u\|_{0,\Gamma_1}^2 = A(w,w) - \lambda\langle w,w\rangle.$$

Since $\|w\|_{0,\Gamma_1}^2 \neq 0$, we thus get

$$\frac{A(w,w)}{\langle w,w\rangle} - \lambda = \frac{\|w-u\|_A^2}{\|w\|_{0,\Gamma_1}^2} - \lambda\frac{\|w-u\|_{0,\Gamma_1}^2}{\|w\|_{0,\Gamma_1}^2}.$$

Therefore, (65) holds. From Lemma 2.1, we know $\lambda \geq 0$. Furthermore, using the trace theorem, we complete the proof of (66).

Corollary 3.1 Under the assumptions of Theorem 3.2, we have

$$|\tilde{\lambda}_k^0 - \tilde{\lambda}_{k,2h}^{(2m)}| \leq \begin{cases} C(k)h^{2m+1}, & m = 1,2, \quad \text{triangles;} \\ C(k)h^{2m+1}, & m \geq 1, \quad \text{rectangles,} \end{cases} \tag{67}$$

where $C(k)$ is a constant independent of h, h_0.

Proof. When the mesh parameter h is sufficiently small, using Theorem 3.2, we have

$$\begin{aligned} \|\mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0\|_{0,\Gamma_1}^2 &= \|\mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0 - \tilde{u}_k^0 + \tilde{u}_k^0\|_{0,\Gamma_1}^2 \\ &\geq \|\tilde{u}_k^0\|_{0,\Gamma_1}^2 - \|\mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0 - \tilde{u}_k^0\|_{0,\Gamma_1}^2 \\ &\geq 1 - Ch^{m+1/2} \geq 1/2. \end{aligned}$$

In (66), if we set $w = \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0$, $u = \tilde{u}_k^0$, then we obtain

$$|\tilde{\lambda}_k^0 - \tilde{\lambda}_{k,2h}^{(2m)}| \leq C\|\mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0 - \tilde{u}_k^0\|_{1,\Omega}^2.$$

Theorem 3.2 implies that (67) holds.

Remark 3.3 It should be emphasized that we can obtain the superapproximation estimate of the k-th eigenvalue and the k-th eigenfunction, $k \geq 1$, for the Steklov eigenvalue problem with constant coefficients only in two dimensional cases (see, Theorem 3.2 and Corollary 3.1). An interesting question is: The numerical results presented in Section 5 (e.g. Example 5.2) clearly show that there are not the usual superapproximation estimates for the eigenvalues and the eigenfunctions of the Steklov eigenvalue problems with constant coefficients in three dimensional cases.

3.3. Adaptive finite elements for solving the boundary layer equation. We recall the boundary layer equation (7). In practice, we solve the modified boundary layer equation given by

$$\left\{\begin{array}{lll} \mathcal{L}_\varepsilon \tilde{u}_{s,k}^{\varepsilon,b} = 0, & \text{in} & \Omega_1, \\ \tilde{u}_{s,k}^{\varepsilon,b} = 0, & \text{on} & \Gamma_0, \\ \tilde{u}_{s,k}^{\varepsilon,b} = u_{s,k}^{\varepsilon,h_0,h}(x), & \text{on} & \Gamma^*, \\ \sigma_\varepsilon(\tilde{u}_{s,k}^{\varepsilon,b}) = \tilde{\lambda}_{k,h}^{(0)} \tilde{u}_{s,k}^{\varepsilon,b}(x), & \text{on} & \Gamma_1, \end{array}\right. \tag{68}$$

where $\tilde{\lambda}_{k,h}^{(0)}$ is the finite element solution of the $k-$th eigenfunction $\tilde{\lambda}_k^{(0)}$ for the modified homogenized Steklov eigenvalue problem (16). The function $u_{s,k}^{\varepsilon,h_0,h}$ denotes the multiscale approximate solutions as given in (75).

Similarly to the computations of cell functions $N_{\alpha_1}(\xi)$, we employ an adaptive finite element method to solve the boundary layer equation (68). The details of the procedure are omitted. Let $\mathcal{F}^{h_1} = \{\tau\}$ be a family of regular triangulations of subdomain $\Omega_1 = \Omega \setminus \overline{\Omega}_0$ as shown in Figure 2 . Let $h_1 = \max_{\tau \in \mathcal{F}^{h_1}} \{h_\tau\}$, $\frac{h_1}{\varepsilon^2} << 1$. Define a piecewise linear finite element space

$$W_{h_1}^\varepsilon(\Omega_1) = \Big\{ v \in C(\overline{\Omega}_1) : \quad v|_\tau \in P_1(\tau),\, v|_{\Gamma_0} = 0,\, v|_{\Gamma^*} = u_{s,k}^{\varepsilon,h_0,h} \Big\}. \tag{69}$$

Denote by $\tilde{u}_{s,k,h_1}^{\varepsilon,b}(x)$ the finite element solution of $\tilde{u}_{s,k}^{\varepsilon,b}(x)$, $k \geq 1$ in $W_{h_1}^\varepsilon(\Omega_1)$. Note that $\tilde{u}_{s,k,h_1}^{\varepsilon,b}(x)$ does depend on h_0, h.

3.4. The numerical algorithm for solving the algebraic eigenvalue problem. In this section, we introduce the numerical algorithm for solving the algebraic eigenvalue problem (23). We follow Andreev's idea, see [2]. Let $\mathcal{N}_h$ denote the set of the nodes of the subdivision $\mathcal{J}^h$, and $\mathcal{N}_{Bh}$ be the set of the nodes on the boundary Γ_h. Let $\mathcal{N}_{Ih} = \mathcal{N}_h \setminus \mathcal{N}_{Bh}$ and $\{\psi_i\}$ be the nodal basis in a finite element space $V_h(\Omega)$. We define the spaces

$$V_{Bh} = Span\{\psi_i\}_{i:z_i \in \mathcal{N}_{Bh}}, \quad V_{Ih} = Span\{\psi_i\}_{i:z_i \in \mathcal{N}_{Ih}}.$$

It is also convenient to introduce some vectors and matrices

$$\begin{array}{l} U_{Ih} = (u_h(z_i))_{i:z_i \in \mathcal{N}_{Ih}}, \quad U_{Bh} = (u_h(z_i))_{i:z_i \in \mathcal{N}_{Bh}} \\ K_{II} = (A(\psi_i, \psi_j))_{i,j:z_i,z_j \in \mathcal{N}_{Ih}}, \quad K_{BB} = (A(\psi_i, \psi_j))_{i,j:z_i,z_j \in \mathcal{N}_{Bh}} \\ K_{IB} = (A(\psi_i, \psi_j))_{i,j:z_i \in \mathcal{N}_{Ih}, z_j \in \mathcal{N}_{Bh}}, M_{BB} = (\langle \psi_i, \psi_j \rangle)_{i,j:z_i,z_j \in \mathcal{N}_{Bh}}. \end{array}$$

We rewrite (16) in the following algebraic form:

$$\begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^t & K_{BB} \end{pmatrix} \begin{pmatrix} U_{Ih} \\ U_{Bh} \end{pmatrix} = \lambda_h \begin{pmatrix} 0 & 0 \\ 0 & M_{BB} \end{pmatrix} \begin{pmatrix} U_{Ih} \\ U_{Bh} \end{pmatrix}. \tag{70}$$

It is obvious that the matrix K_{II} is symmetric and positive definite. We write the complete Cholesky factorization of the matrix $K_{II} = LL^t$, where the matrix L is a lower triangular matrix. Then for the corresponding Schur complement we have

$$S = K_{BB} - K_{IB}^t L^{-t} L^{-1} K_{IB}.$$

Eliminating U_{Ih} from (70), we get

$$SU_{Bh} = \lambda_h M_{BB} U_{Bh}. \tag{71}$$

We use the subspace iterative algorithm (see, e.g. [22]) to solve the general algebraic eigenvalue problem (71).

4. The Multiscale Numerical Algorithm

We recall (8), and summarize the multiscale finite element method for solving the Steklov eigenvalue problem (1) by means of the following parts:

Part I. Compute the cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\alpha_1, \alpha_2 = 1, 2, \cdots, n$ in a reference cell $Q = (0, 1)^n$.

Part II. Solve the modified homogenized Steklov eigenvalue problem (16) in a whole domain Ω in a coarse mesh.

Part III. Calculate the higher-order derivatives $\dfrac{\partial^l \tilde{u}_k^0(x)}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}}$, $l = 1, 2$, $\alpha_1, \alpha_2 = 1, 2, \cdots,$
n; $k \geq 1$ by using the finite difference method, where $\tilde{u}_k^0(x)$ is the k-th eigenfunction of the modified homogenized Steklov eigenvalue problem (16). The key step of our method is to replace the derivatives $\dfrac{\partial \tilde{u}_k^0(x)}{\partial x_i}$, $\dfrac{\partial^2 \tilde{u}_k^0(x)}{\partial x_i \partial x_q}$ at the nodal point N_p by the first-order difference quotients $\delta_{x_i} \tilde{u}_{k,h}^0(N_p)$ and the second-order difference quotients $\delta^2_{x_i x_q} \tilde{u}_{k,h}^0(N_p)$, respectively. We remark that one cannot directly compute the higher-order derivatives from the finite element solutions.

Part IV. Solve the modified boundary layer equation (68) in a fine mesh.

We first introduce the first-order difference quotients given by

$$\delta_{x_i} \tilde{u}_{k,h}^0(N_p) = \frac{1}{\tau(N_p)} \sum_{e \in \sigma(N_p)} \left(\frac{\partial \tilde{u}_{k,h}^0}{\partial x_i} \right)_e (N_p), \tag{72}$$

where $\sigma(N_p)$ is the set of elements with node N_p, $\tau(N_p)$ is the number of elements of $\sigma(N_p)$, $\tilde{u}_{k,h}^0(x)$ is the finite element solution of $\tilde{u}_k^0(x)$ in $V_h(\Omega)$, and $(\dfrac{\partial \tilde{u}_{k,h}^0}{\partial x_i})_e(N_p)$ is the value of the derivative $\dfrac{\partial \tilde{u}_{k,h}^0}{\partial x_i}$ at node N_p relative to element e.

Analogously, the second-order difference quotients are then given by

$$\delta^2_{x_i x_q} \tilde{u}_{k,h}^0(N_p) = \frac{1}{\tau(N_p)} \sum_{e \in \sigma(N_p)} \sum_{j=1}^{d} \delta_{x_q} \tilde{u}_{k,h}^0(P_j) \left(\frac{\partial \psi_j}{\partial x_i} \right)_e (N_p), \tag{73}$$

where d is the number of nodes in e, P_j are the nodes of e, $\psi_j(x)$ are Lagrange's shape functions, $j = 1, 2, \ldots, d$.

In summary, we define the multiscale finite element scheme as follows:

$$U_{s,k,h_1}^{\varepsilon,h_0,h}(N_p) = \begin{cases} u_{s,k}^{\varepsilon,h_0,h}(N_p), & N_p \in \overline{\Omega}_0 \\ \tilde{u}_{s,k,h_1}^{\varepsilon,b}(N_p), & N_p \in \Omega_1, \end{cases} \tag{74}$$

where

$$u_{s,k}^{\varepsilon,h_0,h}(N_p) = \tilde{u}_{k,h}^0(N_p) + \sum_{l=1}^{s} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi(N_p)) \delta^l_{x_{\alpha_1}\cdots x_{\alpha_l}} \tilde{u}_{k,h}^0(N_p), \tag{75}$$

and $\tilde{u}_{s,k,h_1}^{\varepsilon,b}$ denotes the finite element solution of $\tilde{u}_{s,k}^{\varepsilon,b}$ in $W_{h_1}^{\varepsilon}(\Omega_1)$, $k \geq 1$, $s = 1, 2$, $N_p \in \overline{\Omega}$ is a nodal point, h_0, h, h_1 are the mesh parameters of Q, Ω, Ω_1, respectively.

In order to improve the numerical accuracy, we employ the postprocessing technique given by

$$\begin{aligned} &\mathcal{P}u_{s,k}^{\varepsilon,h_0,h}(x) = \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0(x) \\ &+\sum_{l=1}^{s}\varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi)\delta_{x_{\alpha_1}\cdots x_{\alpha_l}}^l \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0(x),\ x\in\overline{\Omega}_0, \quad s=1,2 \end{aligned} \tag{76}$$

where the interpolation operator $\mathcal{I}_{2h}^{(2m)}$ is as given in Lemma 3.3.

Finally, we state the error estimates for the multiscale finite element method.

Theorem 4.1 Suppose that $\Omega\subset R^n$, $n\geq 2$, is a bounded Lipschitz polygonal convex domain or a smooth domain with the boundary $\partial\Omega$, $\Omega_0\subset\subset\Omega$, $\Omega_1=\Omega\setminus\overline{\Omega}_0$. Let $(\lambda_k^\varepsilon, u_k^\varepsilon(x))$ be the k-th eigenpair of the Steklov eigenvalue problem (1). Assume that $(\lambda_k^{(0)}, u_k^0(x))$, $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ are the k-th eigenpairs of the homogenized Steklov eigenvalue problems (6) and (16), respectively, $(\tilde{\lambda}_{k,h}^{(0)}, \tilde{u}_{k,h}^0(x))$ are the corresponding finite element solutions of $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ in $V_h(\Omega)$. $u_{s,k}^{\varepsilon,h_0,h}(x)$, $u_{s,k,h_1}^{\varepsilon,b}(x)$ are defined in (75). Under assumptions $(A_1)-(A_4)$ and $(F_2)-(F_3)$, we have the following error estimates:

$$|\lambda_k^\varepsilon-\tilde{\lambda}_{k,h}^{(0)}|\leq C(k)\Big\{\varepsilon^{1/2}+h_0^2+h^{2m}\Big\}, \tag{77}$$

$$\|u_k^0-\tilde{u}_{k,h}^0\|_{1,\Omega}\leq C(k)\Big\{h_0^2+h^m\Big\}, \tag{78}$$

$$\|u_k^\varepsilon-u_{s,k}^{\varepsilon,h_0,h}\|_{1,\Omega_0}\leq C(k)\Big\{\varepsilon^{1/2}+h_0+h^m+\varepsilon^2h^{m-2}\Big\},\quad m\geq 1,\ s=1,2,\ k\geq 1, \tag{79}$$

and

$$\begin{aligned} &\|u_k^\varepsilon-u_{s,k,h_1}^{\varepsilon,b}\|_{1,p,\Omega_1}\leq C(k)\Big\{\varepsilon^{1/2}+h_0+h^m+\varepsilon^2h^{m-2}+(\frac{h_1}{\varepsilon^2})\Big\}, \\ &m\geq 1,\quad k\geq 1,\quad 1<p\leq p_0<2 \end{aligned} \tag{80}$$

where $C(k)$ is a constant independent of ε, h_0, h, h_1; h_0, h, h_1 are mesh parameters of Q, Ω and Ω_1, respectively.

Proof. Given

$$\lambda_k^\varepsilon-\tilde{\lambda}_{k,h}^{(0)}=\lambda_k^\varepsilon-\lambda_k^{(0)}+\lambda_k^{(0)}-\tilde{\lambda}_k^{(0)}+\tilde{\lambda}_k^{(0)}-\tilde{\lambda}_{k,h}^{(0)}.$$

Using Lemma 2.2, Proposition 3.2 and Lemma 3.1, we complete the proof of (77).

Since $u_k^0-\tilde{u}_{k,h}^0=u_k^0-\tilde{u}_k^0+\tilde{u}_k^0-\tilde{u}_{k,h}^0$, it follows from Proposition 3.2 and Lemma 3.2 that the proof of (78) is complete.

We recall that

$$u_k^\varepsilon-u_{s,k}^{\varepsilon,h_0,h}=u_k^\varepsilon-u_{s,k}^\varepsilon+u_{s,k}^\varepsilon-u_{s,k}^{\varepsilon,h_0,h}. \tag{81}$$

On the other hand, we have

$$\begin{aligned} &u_{s,k}^\varepsilon(x)-u_{s,k}^{\varepsilon,h_0,h}(x)=u_k^0(x)-\tilde{u}_k^0(x)+\tilde{u}_k^0(x)-\tilde{u}_{k,h}^0(x) \\ &+\sum_{l=1}^{s}\varepsilon^l\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}\Big[N_{\alpha_1\cdots\alpha_l}(\xi)-N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi)\Big]\frac{\partial^l u_k^0(x)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}} \\ &+\sum_{l=1}^{s}\varepsilon^l\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi)\frac{\partial^l(u_k^0(x)-\tilde{u}_k^0(x))}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}} \\ &+\sum_{l=1}^{s}\varepsilon^l\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}N_{\alpha_1\cdots\alpha_l}^{h_0}(\xi)\Big[\frac{\partial^l\tilde{u}_k^0(x)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}-\delta_{x_{\alpha_1}\cdots x_{\alpha_l}}^l\tilde{u}_{k,h}^0(x)\Big]. \end{aligned} \tag{82}$$

It follows from Proposition 3.2 that

$$\|u_k^0(x)-\tilde{u}_k^0(x)\|_{1,\Omega_0}\leq Ch_0^2. \tag{83}$$

By using Lemma 3.2, we have

$$\|\tilde{u}_k^0(x) - \tilde{u}_{k,h}^0(x)\|_{1,\Omega_0} \le Ch^m. \tag{84}$$

Thanks to the interior regularity estimates for elliptic equations (see [20]), taking into account $\dfrac{d}{dx_i} \to \dfrac{\partial}{\partial x_i} + \varepsilon^{-1}\dfrac{\partial}{\partial \xi_i}$ and using Proposition 3.1, we get

(85)
$$\|\sum_{l=1}^{s} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} \Big[N_{\alpha_1\cdots\alpha_l}(\xi) - N^{h_0}_{\alpha_1\cdots\alpha_l}(\xi)\Big] \frac{\partial^l u_k^0(x)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\|_{1,\Omega_0} \le Ch_0, \quad s = 1,2.$$

Recalling $\dfrac{d}{dx_i} \to \dfrac{\partial}{\partial x_i} + \varepsilon^{-1}\dfrac{\partial}{\partial \xi_i}$, it follows from Propositions 3.1 and 3.3 that

$$\|\sum_{l=1}^{s} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N^{h_0}_{\alpha_1\cdots\alpha_l}(\xi) \frac{\partial^l (u_k^0(x) - \tilde{u}_k^0(x))}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\|_{1,\Omega_0} \le Ch_0^2, \quad s = 1,2. \tag{86}$$

We observe the final terms of (82) and estimate their errors. First, taking into account $\dfrac{d}{dx_i} \to \dfrac{\partial}{\partial x_i} + \varepsilon^{-1}\dfrac{\partial}{\partial \xi_i}$, we have

$$\begin{aligned}
&\|\varepsilon N^{h_0}_{\alpha_1}(\xi)\Big(\frac{\partial \tilde{u}_k^0(x)}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0(x)\Big)\|_{1,\Omega_0} \le C\Big\{\|\frac{\partial \tilde{u}_k^0(x)}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0\|_{0,\Omega_0} \\
&\quad + \varepsilon\|\frac{\partial \tilde{u}_k^0(x)}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0\|_{1,\Omega_0}\Big\}.
\end{aligned} \tag{87}$$

Since

$$\Big(\frac{\partial \tilde{u}_k^0}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0\Big)(N_p) = \frac{1}{\tau(N_p)} \sum_{e\in\sigma(N_p)} \Big(\frac{\partial(\tilde{u}_k^0 - \tilde{u}_{k,h}^0)}{\partial x_{\alpha_1}}\Big)_e (N_p),$$

using Lemma 3.2, we have

$$\|\frac{\partial \tilde{u}_k^0(x)}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0\|_{0,\Omega_0} \le Ch^m, \quad m \ge 1. \tag{88}$$

On the other hand, using (72) and (73), it is obvious that

$$\Big(\frac{\partial^2 \tilde{u}_k^0}{\partial x_i \partial x_{\alpha_1}} - \frac{\partial}{\partial x_i}\delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0\Big)(N_p) = \Big(\frac{\partial^2 \tilde{u}_k^0}{\partial x_i \partial x_{\alpha_1}} - \delta^2_{x_i x_{\alpha_1}} \tilde{u}_{k,h}^0\Big)(N_p).$$

Recalling (73) and using $\|\psi_j\|_{1,e} \le Ch^{-1}$, we get

$$\|\frac{\partial \tilde{u}_k^0(x)}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0(x)\|_{1,\Omega_0} \le Ch^{m-1}, \quad m \ge 1. \tag{89}$$

Combining (87)-(89) gives

$$\|\varepsilon N^{h_0}_{\alpha_1}(\xi)\Big(\frac{\partial \tilde{u}_k^0(x)}{\partial x_{\alpha_1}} - \delta_{x_{\alpha_1}} \tilde{u}_{k,h}^0(x)\Big)\|_{1,\Omega_0} \le C\Big\{h^m + \varepsilon h^{m-1}\Big\} \le C\Big\{h^m + \varepsilon\Big\}. \tag{90}$$

Similarly, we can prove

$$\begin{aligned}
&\|\varepsilon^2 N^{h_0}_{\alpha_1\alpha_2}(\xi)\Big[\frac{\partial^2 \tilde{u}_k^0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \delta^2_{x_{\alpha_1}x_{\alpha_2}} \tilde{u}_{k,h}^0(x)\Big]\|_{1,\Omega_0} \\
&\le C\Big\{\varepsilon\|\frac{\partial^2 \tilde{u}_k^0(x)}{\partial x_i \partial x_{\alpha_1}} - \delta^2_{x_i x_{\alpha_1}} \tilde{u}_{k,h}^0(x)\|_{0,\Omega_0} \\
&+ \varepsilon^2\|\frac{\partial^2 \tilde{u}_k^0(x)}{\partial x_i \partial x_{\alpha_1}} - \delta^2_{x_i x_{\alpha_1}} \tilde{u}_{k,h}^0(x)\|_{1,\Omega_0}\Big\} \\
&\le C\Big\{\varepsilon h^{m-1} + \varepsilon^2 h^{m-2}\Big\} \le C\Big\{\varepsilon + \varepsilon^2 h^{m-2}\Big\}, \quad m \ge 1.
\end{aligned} \tag{91}$$

Combining (81)-(91), and using Lemma 2.2, we obtain

$$\|u_k^\varepsilon - u_{s,k}^{\varepsilon,h_0,h}\|_{1,\Omega_0} \le C\Big\{\varepsilon^{1/2} + h_0 + h^m + \varepsilon^2 h^{m-2}\Big\}, \quad m \ge 1,\ s = 1,2.$$

Therefore, (79) holds.

Following the lines of the proof of Theorem 3.3 of [11], we obtain

$$\|\tilde{u}_{s,k}^{\varepsilon,b}\|_{2,p,\Omega_1} \le C(k)\varepsilon^{-2}\|\tilde{u}_{s,k}^{0}\|_{2,p,\Omega} \le C(k)\varepsilon^{-2}, \tag{92}$$

where $C(k)$ is a constant independent of ε; $1 < p < 2$.

It is obvious that

$$\begin{aligned} u_k^\varepsilon - \tilde{u}_{s,k,h_1}^{\varepsilon,b} &= u_k^\varepsilon - u_{s,k}^{\varepsilon,b} + u_{s,k}^{\varepsilon,b} - \tilde{u}_{s,k,h_1}^{\varepsilon,b} \\ &= u_k^\varepsilon - u_{s,k}^{\varepsilon,b} + u_{s,k}^{\varepsilon,b} - \tilde{u}_{s,k}^{\varepsilon,b} + \tilde{u}_{s,k}^{\varepsilon,b} - \tilde{u}_{s,k,h_1}^{\varepsilon,b}. \end{aligned} \tag{93}$$

We recall (7) and (68), and get

$$\begin{aligned} \|u_{s,k}^{\varepsilon,b} - \tilde{u}_{s,k}^{\varepsilon,b}\|_{1,p,\Omega_1} &\le C(k)\Big\{\|u_{s,k}^\varepsilon - u_{s,k}^{\varepsilon,h_0,h}\|_{1,\Omega_0} \\ &\quad + |\lambda_k^{(0)} - \tilde{\lambda}_{k,h}^{(0)}|\Big\}. \end{aligned} \tag{94}$$

It follows from (94),(79) and (77) that

$$\|u_{s,k}^{\varepsilon,b} - \tilde{u}_{s,k}^{\varepsilon,b}\|_{1,p,\Omega_1} \le C(k)\Big\{\varepsilon^{1/2} + h_0 + h^m + \varepsilon^2 h^{m-2}\Big\}. \tag{95}$$

Using the error estimates of the finite element method and (92), we have

$$\|\tilde{u}_{s,k}^{\varepsilon,b} - \tilde{u}_{s,k,h_1}^{\varepsilon,b}\|_{1,p,\Omega_1} \le C(k)h_1\|\tilde{u}_{s,k}^{\varepsilon,b}\|_{2,p,\Omega_1} \le C(k)\Big(\frac{h_1}{\varepsilon^2}\Big). \tag{96}$$

By using Lemma 2.2, (93)-(96) and the triangle inequality, we complete the proof of (80). Therefore the proof of Theorem 4.1 is complete.

In conclusion, we obtain a superapproximation estimate for the multiscale finite element method:

Theorem 4.2 Suppose that $\Omega \subset R^2$ is a bounded Lipschitz polygonal convex domain or a smooth domain with the boundary $\partial\Omega$, $\Omega_0 \subset\subset \Omega$, $\Omega_1 = \Omega \setminus \overline{\Omega}_0$. Let $(\lambda_k^\varepsilon, u_k^\varepsilon(x))$ be the k-th eigenpair of the Steklov eigenvalue problem (1). Assume that $(\lambda_k^{(0)}, u_k^0(x))$, $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ are the k-th eigenpairs of the homogenized Steklov eigenvalue problems (6) and (16), respectively, $(\tilde{\lambda}_{k,h}^{(0)}, \tilde{u}_{k,h}^0(x))$ are the corresponding finite element solutions of $(\tilde{\lambda}_k^{(0)}, \tilde{u}_k^0(x))$ in $V_h(\Omega)$. $\mathcal{P}u_{s,k}^{\varepsilon,h_0,h}$, $\mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0$ are given in (76) and (60), respectively. Under assumptions $(A_1) - (A_4)$, $(F_1) - (F_3)$, if $u_k^0, \tilde{u}_k^0 \in H^{s+2}(\Omega_0) \cap H^{m+1}(\Omega_0)$, then we have the following superapproximation estimates:

$$\|u_k^0 - \mathcal{I}_{2h}^{(2m)}\tilde{u}_{k,h}^0\|_{1,\Omega} \le C(k)\Big\{h_0^2 + h^{m+\frac{1}{2}}\Big\}, \tag{97}$$

and

$$\begin{aligned} \|u_k^\varepsilon - \mathcal{P}u_{s,k}^{\varepsilon,h_0,h}\|_{1,\Omega_0} &\le C(k)\Big\{\varepsilon^{1/2} + h_0 + h^{m+\frac{1}{2}} + \varepsilon^2 h^{m-\frac{3}{2}}\Big\}, \\ & m \ge 1,\ s = 1,2,\ k \ge 1, \end{aligned} \tag{98}$$

where $C(k)$ is a constant independent of ε, h_0, h; h_0, h are the mesh parameters of Q and Ω, respectively.

Following the lines of the proof of Theorem 4.1, and using Theorem 3.2, we complete the proof of Theorem 4.2.

Remark 4.1 We would like to state that Theorem 4.1 is valid in any higher-dimensional cases, but the superapproximation estimates in Theorem 4.2 are true only in two dimensional cases.

TABLE 1. The computational results in Example 5.1: the first eigenvalue

h	1/4	1/6	1/8	1/10	1/12
$\lambda_{1,h}$	1.030841	1.013773	1.007768	1.004979	1.003461
$\lambda_{1,2h}^{(2m)}$	1.000344	1.000090	1.000032	1.000013	1.000005

5. Numerical Case Studies

To validate the developed multiscale algorithm and to confirm the theoretical analysis reported in this paper, we present numerical simulations for the following case studies.

Example 5.1 We first present the numerical example, which supports the superapproximation results of Theorem 3.2. To this end, we consider the following Steklov eigenvalue problem with constant coefficients

$$\begin{cases} -\Delta u = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = \lambda u, \quad x \in \Gamma_1, \end{cases} \tag{99}$$

where $\Omega = (0,1)^2$, $\Gamma_0 = \{(x_1,x_2)| \quad x_1x_2 = 0\}$, $\Gamma_1 = \{(x_1,x_2)| \quad 0 < x_1 \leq 1,\, x_2 = 1\} \cup \{(x_1,x_2)| \quad x_1 = 1,\, 0 < x_2 < 1\}$.

It is obvious that the exact first eigenpair of problem (99) is (λ_1, u_1), where $\lambda_1 = 1$, $u_1(x_1,x_2) = x_1x_2$. In the standard approach, we first apply linear triangular elements to solve problem (99). The numerical results for the first eigenvalue of (99) are illustrated in Table 1, where $\lambda_{1,h}$ denotes the finite element solution of the first eigenvalue λ_1. It can be verified that $\lambda_1 \leq \lambda_{1,h} \leq \lambda_1 + C_1h^{\beta}$, $\beta = 1.9909$, which is consistent with Lemma 3.1. In addition, we use the following formula to compute the first eigenvalue of (99):

$$\lambda_{1,2h}^{(2m)} = \frac{a(\mathcal{I}_{2h}^{(2m)}u_{1,h}, \mathcal{I}_{2h}^{(2m)}u_{1,h})}{\langle \mathcal{I}_{2h}^{(2m)}u_{1,h}, \mathcal{I}_{2h}^{(2m)}u_{1,h}\rangle}, \quad m = 1, \tag{100}$$

where the operator $\mathcal{I}_{2h}^{(2m)}$ is given in Lemma 3.3, and $u_{1,h}$ is the finite element solution of the first eigenfunction u_1 of problem (99). The numerical results are listed in Table 1. It can be verified that $\lambda_1 \leq \lambda_{1,2h}^{(2m)} \leq \lambda_1 + C_1h^{\beta}$, $\beta = 3.7887$

Remark 5.1 We observe the computational results in Table 1, and conclude that the formula (100) improves the numerical accuracy for computing the eigenvalues of problem (99) in two dimensional cases.

For the first eigenfunction $u_1(x_1,x_2)$, we have the following numerical errors:

$$\|u_1 - u_{1,h}\|_{0,\Omega} \leq C_1h^{1.67}, \quad \|u_1 - \mathcal{I}_{2h}^{(2m)}u_{1,h}\|_{0,\Omega} \leq C_2h^{3.18}, \tag{101}$$

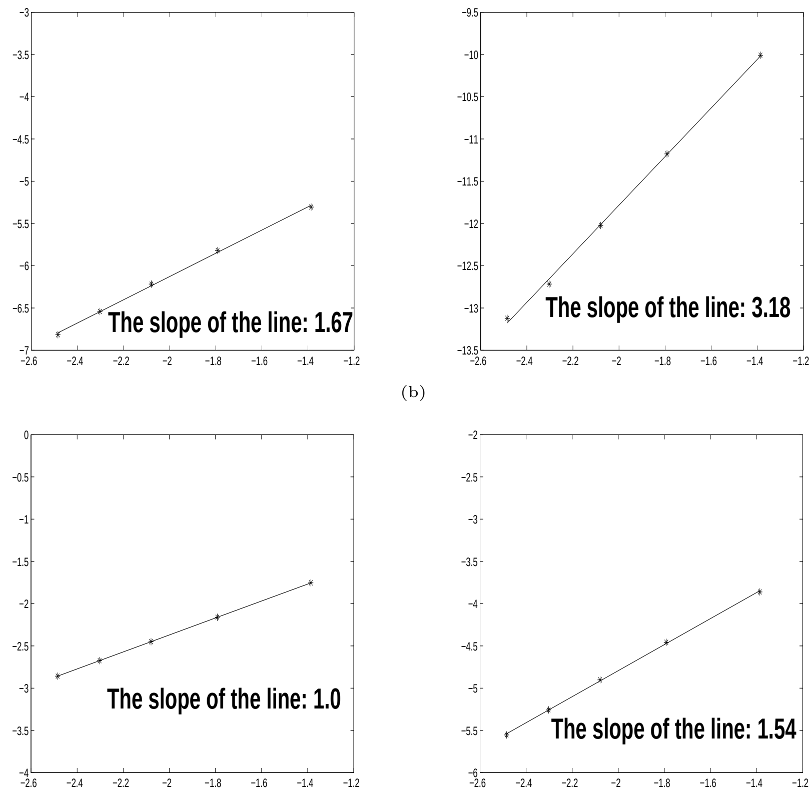
and

$$\|u_1 - u_{1,h}\|_{1,\Omega} \leq C_3h, \quad \|u_1 - \mathcal{I}_{2h}^{(2m)}u_{1,h}\|_{1,\Omega} \leq C_4h^{1.54}, \tag{102}$$

where C_i, $i = 1,2,3,4$ are constants independent of h, and the higher-order interpolation operator $\mathcal{I}_{2h}^{(2m)}$, $m = 1$ has been defined in Lemma 3.3. The further computational results are illustrated in Fig.5:(a)-(d).

Remark 5.2 The error estimates both (101) in the $L^2(\Omega)$ norm and (102) in the $H^1(\Omega)$ norm demonstrate the superapproximation result of Theorem 3.2.

Example 5.2 We consider the similar Steklov eigenvalue problem to Example 5.1, where a domain $\Omega = (0,1)^3$, $\Gamma_0 = \{(x_1,x_2,x_3)| \quad x_1x_2x_3 = 0\}$, $\Gamma_1 =$



(a) (b) (c) (d)

FIGURE 5. In Example 5.1: (a) The relationship between $ln(\|u_1 - u_{1,h}\|_{0,\Omega})$ and $ln(h)$; (b) the relationship between $ln(\|u_1 - \mathcal{I}_{2h}^{(2m)} u_{1,h}\|_{0,\Omega})$ and $ln(h)$; (c) the relationship between $ln(\|u_1 - u_{1,h}\|_{1,\Omega})$ and $ln(h)$; (d) the relationship between $ln(\|u_1 - \mathcal{I}_{2h}^{(2m)} u_{1,h}\|_{1,\Omega})$ and $ln(h)$.

$\{(x_1, x_2, x_3)| \quad x_3 = 1, 0 < x_1 \leq 1, 0 < x_2 \leq 1\} \cup \{(x_1, x_2, x_3)| \quad x_2 = 1, 0 < x_1 \leq 1, 0 < x_3 < 1\} \cup \{(x_1, x_2, x_3)| \quad x_1 = 1, 0 < x_2 < 1, 0 < x_3 < 1\}$.

The exact first eigenpair (λ_1, u_1) of the problem is $\lambda_1 = 1$, $u_1(x_1, x_2, x_3) = x_1 x_2 x_3$. To verify that whether Theorem 3.2 is valid or not in three dimensional cases, we do some numerical simulations. We employ linear tetrahedral elements to solve the problem, where $\lambda_{1,h}$ denotes the finite element solution of the first eigenvalue λ_1. It can be verified that $\lambda_1 \leq \lambda_{1,h} \leq \lambda_1 + C_1 h^{\beta}$, $\beta = 1.9905$, which is consistent with Lemma 3.1. Also we use the formula (100) to compute the first eigenvalue λ_1 and obtain the following numerical error: $\lambda_1 \leq \lambda_{1,2h}^{(2m)} \leq \lambda_1 + C_1 h^{\beta}$, $\beta = 2.1533$. The further computational results are illustrated in Table 2.

For the first eigenfunction $u_1(x_1, x_2, x_3)$, we have the following numerical errors:

$$\|u_1 - u_{1,h}\|_{0,\Omega} \leq C_1 h^{1.99}, \quad \|u_1 - \mathcal{I}_{2h}^{(2m)} u_{1,h}\|_{0,\Omega} \leq C_2 h^{2.12}, \tag{103}$$

and

$$\|u_1 - u_{1,h}\|_{1,\Omega} \leq C_3 h, \quad \|u_1 - \mathcal{I}_{2h}^{(2m)} u_{1,h}\|_{1,\Omega} \leq C_4 h^{1.14}, \tag{104}$$

TABLE 2. The computational results in Example 5.2: the first eigenvalue

h	1/4	1/8	1/10	1/20
$\lambda_{1,h}$	1.031391	1.007933	1.005085	1.001275
$\lambda_{1,2h}^{(2m)}$	1.010156	1.002103	1.001374	1.000315

where C_i, $i = 1, 2, 3, 4$ are constants independent of h. The further computational results are shown in Fig.6:(a)-(d).

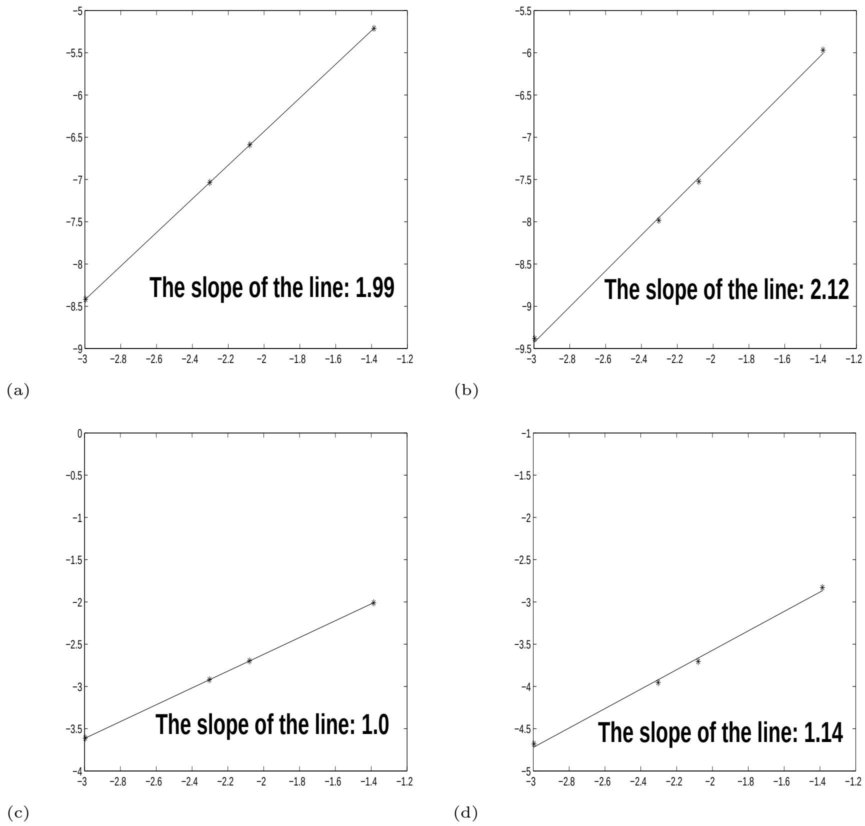


FIGURE 6. In Example 5.2: (a) The relationship between $ln(\|u_1 - u_{1,h}\|_{0,\Omega})$ and $ln(h)$; (b) the relationship between $ln(\|u_1 - \mathcal{I}_{2h}^{(2m)} u_{1,h}\|_{0,\Omega})$ and $ln(h)$; (c) the relationship between $ln(\|u_1 - u_{1,h}\|_{1,\Omega})$ and $ln(h)$; (d) the relationship between $ln(\|u_1 - \mathcal{I}_{2h}^{(2m)} u_{1,h}\|_{1,\Omega})$ and $ln(h)$.

Remark 5.3 Observing the computational results listed in Table 2, (103)-(104) and Fig.6:(a)-(d), we conclude that we can not obtain the similar superapproximation estimates for the eigenvalues and the eigenfunctions to Theorem 3.2 in three dimensional cases.

Example 5.3 We consider the Steklov eigenvalue problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u_k^{\varepsilon}(x)}{\partial x_j}\Big) + a_0(\frac{x}{\varepsilon})u_k^{\varepsilon}(x) = 0, \quad x \in \Omega, \\ \nu_i a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u_k^{\varepsilon}(x)}{\partial x_j} = \lambda_k^{\varepsilon} u_k^{\varepsilon}(x), \quad x \in \partial\Omega. \end{cases} \tag{105}$$

where $\Omega = (0,1)^2$ is a periodic structure as illustrated in Figure 7, and the reference cell Q is shown as in Figure 8, $\Gamma_0 = \{(x_1,x_2)| \quad 0 < x_1 < 0.2, x_2 = 0.2 - x_1\} \cup \{(x_1,x_2)| \quad 0 < x_1 < 0.2, x_2 = 0.8 + x_1\} \cup \{(x_1,x_2)| \quad 0.8 < x_1 < 1, x_2 = 1.8 - x_1\} \cup \{(x_1,x_2)| \quad 0.8 < x_1 < 1, x_2 = x_1 - 0.8\}$, $\Gamma_1 = \{(x_1,x_2)| \quad 0.2 \le x_1 \le 0.8,\ x_2 = 0\} \cup \{(x_1,x_2)| \quad x_1 = 1,\ 0.2 \le x_2 \le 0.8\} \cup \{(x_1,x_2)| \quad 0.2 \le x_1 \le 0.8,\ x_2 = 1\} \cup \{(x_1,x_2)| \quad x_1 = 0,\ 0.2 \le x_2 \le 0.8\}$, $\nu = (\nu_1, \cdots, \nu_n)$ is the outward unit normal to Γ_1. We take $\varepsilon = \frac{1}{5}$.

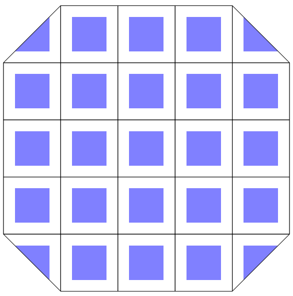


FIGURE 7. Domain Ω.

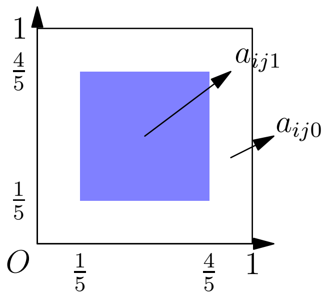


FIGURE 8. The reference cell Q.

In (105), let $a_0(\frac{x}{\varepsilon}) = 0$ and δ_{ij} be a Kronecker symbol.
Case 2.1: $a_{ij0} = \delta_{ij}$, $a_{ij1} = 0.1\delta_{ij}$;
Case 2.2: $a_{ij0} = \delta_{ij}$, $a_{ij1} = 0.01\delta_{ij}$.
Case 2.3: $a_{ij0} = \delta_{ij}$, $a_{ij1} = 0.001\delta_{ij}$.

In order to show the numerical accuracy of the method presented in this paper, we need to know the exact solution of the original problem (105). Since this is extremely difficult, we replace the exact solution with the finite element solution in a fine mesh. We employ linear triangular elements to solve the original problem (105). In real applications, this step is not necessary, and we can apply our method to solve numerically problem (105) in more complicated structures.

Here we use the linear triangular elements to compute the cell functions $N_{\alpha_1}(\xi)$, $N_{\alpha_1\alpha_2}(\xi)$, $\alpha_1, \alpha_2 = 1, 2$ defined in (4) and (5), the modified homogenized Steklov problem (16) and the boundary layer solution (7), respectively. The numbers of elements and nodes are listed in Table 3.

The numerical results of several eigenvalues and eigenfunctions of the related problems in Example 5.3 are illustrated as in Tables 4-9, respectively. Here λ_k^{ε}, $k = 1,2,3,4$ are the finite element solutions of the four minimal eigenvalues of the original problem (105) in a fine mesh, and $\lambda_k^{(0)}$, $k = 1,2,3,4$ are the finite element solutions of the corresponding eigenvalues of the modified homogenized Steklov eigenvalue problem (16) in a coarse mesh. The functions $u_k^{\varepsilon}(x)$, $k = 1,2,3,4$ are the finite element solutions of the eigenfunctions associated with four minimal eigenvalues of problem (105) in a fine mesh, while $u_k^0(x)$, $k = 1,2,3,4$ denote the finite element solutions of the corresponding eigenfunctions for the modified homogenized Steklov eigenvalue problem (16) in a coarse mesh. Finally, it should be noted that the functions $U_{1,k}^{\varepsilon}(x)$, $U_{2,k}^{\varepsilon}(x)$, $k=1,2,3,4$ are respectively the first-order and the second-order multiscale finite element solutions based on the expansion (74). We set $e_{0,k}=u_k^{\varepsilon}-u_k^0$, $e_{1,k}=u_k^{\varepsilon}-U_{1,k}^{\varepsilon}$, $e_{2,k}=u_k^{\varepsilon}-U_{2,k}^{\varepsilon}$.

Table 3. Comparison of computational cost

	original problem	cell problem	homogenized equation	boundary layer
elements	18400	800	1150	11200
nodes	9361	441	616	5880

The numerical results for the second eigenfunction in Case 2.1, the second eigenfunction in Case 2.2 and the second eigenfunction in Case 2.3 are illustrated in Figs.9-14, respectively.

Remark 5.4 The numerical results as shown as in Tables 4, 6 and 8, show that the eigenvalues of the modified homogenized Steklov eigenvalue problem (16) in a coarse mesh are close to those of the original Steklov eigenvalue problem (105) in a fine mesh. This implies that, in order to calculate the eigenvalues for the Steklov eigenvalue problem (105) with rapidly oscillating coefficients, we only need to compute the associated eigenvalues for the homogenized Steklov eigenvalue problem (16) in a coarse mesh.

Remark 5.5 The computational results that are illustrated in Tables 5, 7 and 9, show that the error estimates of Theorem 4.1 are correct. Figs.9-14 support

Table 4. Comparison of computational results in Case 2.1: four minimal eigenvalues

	original problem (λ_k^{ε})	homogenized equation ($\lambda_k^{(0)}$)	relative error
k=1	1.989091	2.016775	0.013726
k=2	2.726728	2.777857	0.018405
k=3	2.726728	2.777857	0.018405
k=4	3.018254	3.077715	0.019319

Table 5. Comparison of computational results in Case 2.1: eigenfunctions

	$\frac{\|e_{0,k}\|_{L^2}}{\|u_k^0\|_{L^2}}$	$\frac{\|e_{1,k}\|_{L^2}}{\|U_{1,k}^{\varepsilon}\|_{L^2}}$	$\frac{\|e_{2,k}\|_{L^2}}{\|U_{2,k}^{\varepsilon}\|_{L^2}}$	$\frac{\|e_{0,k}\|_{H^1}}{\|u_k^0\|_{H^1}}$	$\frac{\|e_{1,k}\|_{H^1}}{\|U_{1,k}^{\varepsilon}\|_{H^1}}$	$\frac{\|e_{2,k}\|_{H^1}}{\|U_{2,k}^{\varepsilon}\|_{H^1}}$
k=1	0.046642	0.010421	0.010569	0.503885	0.096686	0.096983
k=2	0.077027	0.017251	0.017247	0.447108	0.058885	0.056487
k=3	0.077011	0.017289	0.017283	0.447112	0.058889	0.056483
k=4	0.093801	0.028623	0.028950	0.490832	0.092353	0.095392

Table 6. Comparison of computational results in Case 2.2: four minimal eigenvalues

	original problem	homogenized solutions	relative error
1	1.778715	1.740666	0.021858
2	2.410815	2.397551	0.005532
3	2.410815	2.397551	0.005532
4	2.649853	2.656357	0.002448

Table 7. Comparison of computational results in Case 2.2: eigenfunctions

	$\frac{\|e_{0,k}\|_{L^2}}{\|u_k^0\|_{L^2}}$	$\frac{\|e_{1,k}\|_{L^2}}{\|U_{1,k}^\varepsilon\|_{L^2}}$	$\frac{\|e_{2,k}\|_{L^2}}{\|U_{2,k}^\varepsilon\|_{L^2}}$	$\frac{\|e_{0,k}\|_{H^1}}{\|u_k^0\|_{H^1}}$	$\frac{\|e_{1,k}\|_{H^1}}{\|U_{1,k}^\varepsilon\|_{H^1}}$	$\frac{\|e_{2,k}\|_{H^1}}{\|U_{2,k}^\varepsilon\|_{H^1}}$
k=1	0.053723	0.017170	0.017380	0.547313	0.110316	0.111782
k=2	0.077907	0.017476	0.017273	0.478182	0.058563	0.058872
k=3	0.078021	0.017343	0.017435	0.471127	0.059898	0.059838
k=4	0.107685	0.023138	0.024143	0.543290	0.100363	0.107822

Table 8. Comparison of computational results in Case 2.3: four minimal eigenvalues

	original problem	homogenized solutions	relative error
1	1.755958	1.711058	0.026241
2	2.376704	2.356769	0.008458
3	2.376704	2.356769	0.008458
4	2.610057	2.611173	0.000427

Table 9. Comparison of computational results in Case 2.3: eigenfunctions

	$\frac{\|e_{0,k}\|_{L^2}}{\|u_k^0\|_{L^2}}$	$\frac{\|e_{1,k}\|_{L^2}}{\|U_{1,k}^\varepsilon\|_{L^2}}$	$\frac{\|e_{2,k}\|_{L^2}}{\|U_{2,k}^\varepsilon\|_{L^2}}$	$\frac{\|e_{0,k}\|_{H^1}}{\|u_k^0\|_{H^1}}$	$\frac{\|e_{1,k}\|_{H^1}}{\|U_{1,k}^\varepsilon\|_{H^1}}$	$\frac{\|e_{2,k}\|_{H^1}}{\|U_{2,k}^\varepsilon\|_{H^1}}$
k=1	0.054601	0.018881	0.019093	0.552150	0.113700	0.115508
k=2	0.078702	0.018135	0.018724	0.480898	0.058798	0.058327
k=3	0.078931	0.018219	0.018385	0.481236	0.058901	0.058433
k=4	0.109257	0.024977	0.023893	0.548967	0.110832	0.102708

the results of Theorem 4.1. They show that the multiscale finite element method has better numerical accuracy compared with the homogenization method. Finally, we observe that the multiscale correctors presented this paper are essential for the improvement of the numerical accuracy.

Remark 5.6 We observe the numerical results presented in Tables 5, 7 and 9, and conclude that the first-order multiscale method should be a better choice compared with the homogenization method and is sufficient to describe the detail of solutions compared with the second-order multiscale method for the Steklov eigenvalue problem (1), which is different from other eigenvalue problems, see [11, 12, 13].

To support the convergence results of Lemma 2.2, we present the following numerical example:

Example 5.4 We consider the following Steklov eigenvalue problem

$$
\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u_k^\varepsilon(x)}{\partial x_j}\Big) = 0, \quad x \in \Omega, \\ u_k^\varepsilon(x) = 0, \quad x \in \Gamma_0 \\ \nu_i a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u_k^\varepsilon(x)}{\partial x_j} = \lambda_k^\varepsilon u_k^\varepsilon(x), \quad x \in \Gamma_1. \end{cases} \tag{106}
$$

where $\Omega = (0,1)^2$, $\Gamma_0 = \{(x_1,x_2)| \quad x_1x_2 = 0\}$, $\Gamma_1 = \{(x_1,x_2)| \quad 0 < x_1 \le 1,\ x_2 = 1\} \cup \{(x_1,x_2)| \quad x_1 = 1,\ 0 < x_2 < 1\}$.

We respectively use the homogenization method, the first-order and the second-order multiscale methods to compute the first eigenfunctions of problem (106) with respect to different small periodic parameters ε. Without confusion $u_1^\varepsilon(x)$ denotes the finite element solution of the first eigenfunction in a fine mesh and replace

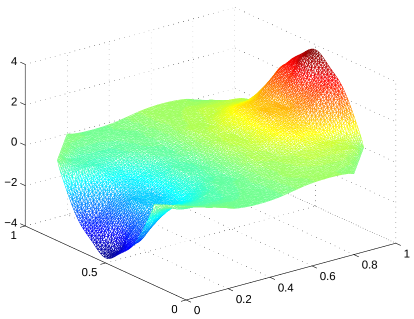


FIGURE 9. Case 2.1: u_2^ε.

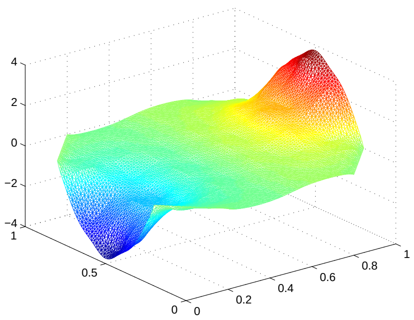


FIGURE 10. Case 2.1: $U_{1,2}^\varepsilon$.

TABLE 10. Comparison of the computational results for different ε

ε	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\|e_{0,1}\|_0/\|u_1^\varepsilon\|_0$	0.029457	0.015057	0.007578	0.003791	0.001856
$\|e_{1,1}\|_0/\|u_1^\varepsilon\|_0$	0.005226	0.002192	0.001091	0.000542	0.000272
$\|e_{2,1}\|_0/\|u_1^\varepsilon\|_0$	0.004835	0.002171	0.001091	0.000542	0.000272
$\|e_{0,1}\|_1/\|u_1^\varepsilon\|_1$	0.174892	0.176840	0.175498	0.175572	0.170115
$\|e_{1,1}\|_1/\|u_1^\varepsilon\|_1$	0.035144	0.029460	0.027802	0.026155	0.024534
$\|e_{2,1}\|_1/\|u_1^\varepsilon\|_1$	0.030461	0.025159	0.023484	0.023074	0.022518

the exact solution of problem (106) by the approximate solution. $u_1^0(x)$ is the finite element solution of the first eigenfunction for the corresponding homogenized Steklov problem (16) in a coarse mesh. $u_{1,1}^\varepsilon(x)$, $u_{2,1}^\varepsilon(x)$ denote the first-order and the second-order multiscale solutions for the first eigenfunction of problem (106), respectively. The numbers of elements and nodes are listed in Table 3. We set $e_{0,1} = u_1^\varepsilon - u_1^0$, $e_{1,1} = u_1^\varepsilon - u_{1,1}^\varepsilon$, $e_{2,1} = u_1^\varepsilon - u_{2,1}^\varepsilon$. The numerical results for different ε are illustrated in Table 10 and Fig.15, where $\|v\|_0 = \|v\|_{L^2(\Omega)}$, $\|v\|_1 = \|v\|_{H^1(\Omega)}$.

Observing the numerical results for the first eigenfunction u_1^ε, we can obtain the following error estimates:

(107)
$$\|u_1^\varepsilon - u_1^0\|_{0,\Omega} \le C_0\varepsilon^{0.9966}, \quad \|u_1^\varepsilon - u_{1,1}^\varepsilon\|_{0,\Omega} \le C_1\varepsilon^{1.0544}, \quad \|u_1^\varepsilon - u_{2,1}^\varepsilon\|_{0,\Omega} \le C_2\varepsilon^{1.0306},$$

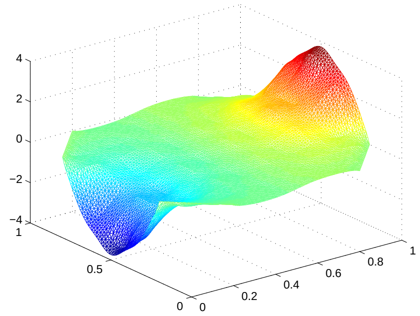


FIGURE 11. Case 2.1: $U^{\varepsilon}_{2,2}$.

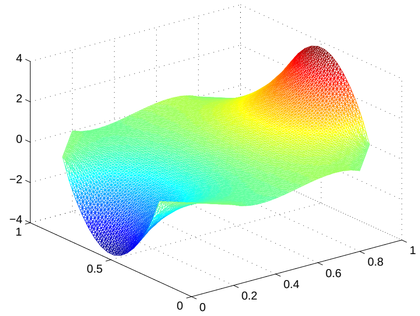


FIGURE 12. Case 2.1: u^0_2.

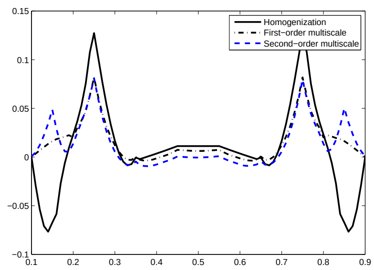


FIGURE 13. In Case 2.2, comparison of the numerical errors $e_{s,2}$, $s = 0, 1, 2$ of the second eigenfunction along the diagonal line $x_2 = x_1$.

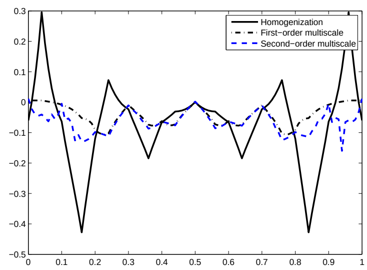


FIGURE 14. In Case 2.3, comparison of the numerical errors $e_{s,2}$, $s = 0, 1, 2$ of the second eigenfunction along the line $x_2 = 0.5$.

where C_i, $i = 0, 1, 2$ are constants independent of ε.

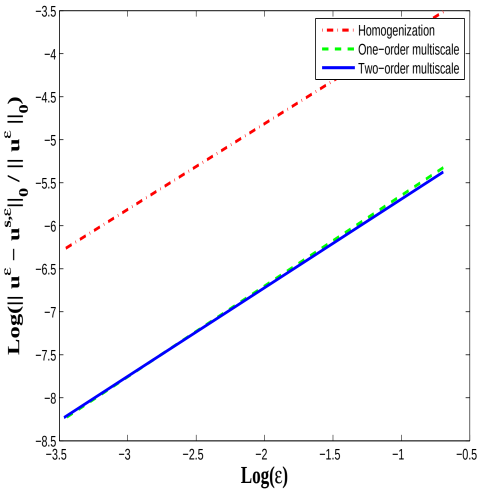


FIGURE 15. Homogenization: The relationship between $ln(\|u_1^\varepsilon - u_1^0\|_{0,\Omega})$ and $ln(\varepsilon)$. The first-order multiscale method: The relationship between $ln(\|u_1^\varepsilon - u_{1,1}^\varepsilon\|_{0,\Omega})$ and $ln(\varepsilon)$. The second-order multiscale method: The relationship between $ln(\|u_1^\varepsilon - u_{2,1}^\varepsilon\|_{0,\Omega})$ and $ln(\varepsilon)$.

Conclusions.

This paper discussed the multiscale finite element computation of a Steklov eigenvalue problem with rapidly oscillating coefficients. The new contributions obtained in this paper were to present the multiscale finite element method and to derive the convergence result (see Theorems 4.1 and 4.2). In particular, a superapproximation estimate for solving the homogenized Steklov eigenvalue problem was obtained. To our knowledge, there are no other results in the literature on this problem. The numerical results given in Section 5 validated the theoretical results presented earlier in the paper.

References

[1] H.J. Ahn, Vibration of a pendulum consisting of a bob suspended from a wire, *Quart. Appl. Math.,* **39**: 109-117, 1981.

[2] A.B. Andreev and T.D. Todorov, Isoparametric finite-element approximation of a Steklov eigenvalue problem, *IMA J. Numer. Anal.*(2004), **24**: 309-322.

[3] M.G. Armentano and C. Padra, A posteriori error estimates for the Steklov eigenvalue problem, *Appl. Numer. Math.*(2008), **58**: 593-601.

[4] M.G. Armentano, The effect of reduced integration in the Steklov eigenvalue problem, *Math. Mod. Numer. Anal., (M^2AN)*(2004), 38(1): 27-36.

[5] I. Babuska and J.E. Osborn, *Eigenvalue problem, Handbook of Numerical Analysis, Vol.II,* North-Holland, Amsterdam, 1991.

[6] A. Bensoussan, J.L. Lions and G. Papanicolaou, *Asymptotic Analysis of Periodic Structures,* North-Holland, Amsterdam, 1978.

[7] J.H. Bramble and J.E. Osborn, Approximation of Steklov eigenvalues of non-selfadjoint second order elliptic operators. *Math. Foundations of the Finite Element Method with Applications to PDE.* (A.K.Aziz, ed.) Academic Press, New York, 387-408, 1972.

[8] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, Berlin, 1994.

[9] D. Bucur and I. Ionescu, Asymptotic analysis and scaling of friction parameters,*Z.Angew. Math. Phys.(ZAMP)*(2006), **57**: 1-15.

[10] J. Canavati and A. Minsoni, A discontinuous Steklov problem with an application to water waves,*J. Math. Anal. Appl.*(1979), **69**: 540-558.

[11] L.Q. Cao and J.Z. Cui, Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problem for second order elliptic equations in perforated domains, *Numer. Math.*(2004), **96**:525-581.

[12] L.Q. Cao and J.L. Luo, Multiscale Numerical Algorithm for the Elliptic Eigenvalue Problem with the Mixed Boundary in Perforated Domains, *Appl. Numer. Math.*(2008), **58**: 1349-1374.

[13] L.Q. Cao, L. Zhang, W. Allegretto and Y.P. Lin, Multiscale asymptotic method of Steklov eigenvalue equations with rapidly oscillating coefficients, Submitted, 2009.

[14] Z. Chen, L. Wang and W. Zheng, An adaptive multilevel method for time-harmonic Maxwell equations with sigularities, *SIAM J.Sci. Comput.*(2007), 29: 118-138.

[15] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[16] R. Courant and D. Hilbert,*Methods of Mathematical Physics, Vol.I,* Interscience Publishers Inc., New York, 1953.

[17] W. E and B. Engquist, The heterogeneous multi-scale methods, *Comm. in Math. Sci.*(2003), 1(1): 87-132.

[18] W. E, B. Engquist and X.T. Li, W.Q. Ren, E. Vanden-Eijnden, Heterogeneous multiscale methods: a review, *Commun. Comput. Phys.*(2007), 2(3): 367-450.

[19] Y. Efendiev, T.Y. Hou, Multiscale finite element methods. Theory and applications. Surveys and Tutorials in the Applied Mathematical Sciences, 4, Springer, New York, 2009.

[20] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*(2nd ed)., Springer-Verlag, Berlin, 1983.

[21] Y. Golovaty, D. Gómez, M. Lobo and E. Pérez, On vibrating membranes with very heavy thin inlcusions, *Math. Models and Methods in Appl. Sci.*(2004), **14**: 987-1034.

[22] G.H. Golub and C.F. Van Loan, *Matrix Computations*, London: John Hopkins, 1996.

[23] D. Gómez, M. Lobo, S.A. Nazarov and E. Pérez, Spectral stiff problems in domains surrounded by thin bands: asymptotic and uniform estimates for eigenvalues, *J. Math.Pures Appl.*(2006), **85**: 598-632.

[24] H. Han and Z. Guan, An analysis of the boundary element approximation of Steklov eigenvalue problems, *Numerical Methods for Partial Differential Equations,* World Scientific, 35-51, 1994.

[25] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators,* Birkhäuser Verlag, Basel, 2006.

[26] D.B. Hinton and J.K. Shaw, Differential operators with spectral parameter incompletely in the boundary conditions, *Funkcialaj Ekvacioj*(Serio Internacia)(1990), **33**: 363-385.

[27] T.Y. Hou, X.H. Wu, A multiscale finite element method for elliptic problems in composite materials and porous media, *J. Comp. Phys.*(1997), 134:169-189.

[28] T.Y. Hou, X.H. Wu, and Z. Cai, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, *Math. Comp.*(1999), 68:913-943.

[29] J. Huang and T. Lü, The mechanical quadrature methods and their extropolation for solving BIE of Steklov eigenvalue problems, *J.Comp. Math.*(2004), **22**(5): 719-726.

[30] I. Ionescu, D. Onofrei and B. Vernescu, $\Gamma-$ convergence for a fault model with slip-weakening friction on periodic barriers, *Quart. Appl. Math.*(2005), **63**: 747-778.

[31] V.V. Jikov , S.M. Kozlov, and O.A. Oleinik , *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.

[32] Q. Lin and Q.D. Zhu, *The Preprocessing and Postprocessing for the Finite Element Method*, Shanghai Scientific & Technical Publishers, 1994. (in Chinese)

[33] M. Lobo and E. Pérez, On the vibrations of a body with many concentrated masses near the boundary, *Math. Models and Methods in Appl. Sci.*(1993), **3**: 249-273.

[34] S. Moskow, M. Vogelius, First-order corrections to the homogenised eigenvalues of a periodic composite medium , A convergence proof , *Proc. of the Royal Society of Edinburgh*(1997) **127 A**: 1263-1299.

[35] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian, *Mathematical Problems in Elasticity and Homogenization,* North-Holland, Amsterdam, 1992.

[36] J.E. Osborn, Spectral approximation for compact operators, *Math. Comp.*(1975), **29**: 712-725.

[37] E. Pérez, On periodic Steklov type eigenvalue problems on half-bands and the spectral homogenization problem, *Discrete and Continuous Dynamical Systems-Ser.B*(2007), **7**(4): 859-883.
[38] F. Santosa and M. Vogelius, First-order corrections to homogenized eigenvalues of a periodic composite medium, *SIAM J. Appl. Math.*(1993), **53**:1636-1668.
[39] M. Sarkis and H. Versieux, Convergence analysis for the numerical boundary corrector for elliptic equations with rapidly oscillating coefficients, *SIAM J. Numer. Anal.* (2008), 46(2): 545-576.
[40] M. Vanninathan, Homogenization of eigenvalue problems in perforated domains, *Proc. Indian Acad. Sci. Math. Sci.*(1981), **90**:239-271.
[41] H. Versieux and M. Sarkis, Numerical boundary corrector for elliptic equations with rapidly oscillating coefficients, *Comm. Numer. Methods Engrg.* (2006), 22(6): 577-589.
[42] H.F. Weinberger, *Variational Methods for Eigenvalue Approximation*, SIAM, Philadelphia, 1974.
[43] N.N.Yan, *Superconvergence Analysis and a Posteriori Error Estimation in Finite Element Methods*, Science Press, Beijing, 2008.
[44] Y.D. Yang, *An Analysis of the Finite Element Method for Eigenvalue Problems*, Guizhou People Publishers, 2004. (in Chinese)
[45] Q.D. Zhu and Q. Lin, *The Superconvergence Theory on Finite Element Methods*, Hunan Scientific and Technical Publishers, Changsha, 1989.(in Chinese)

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