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EQUIVALENT A POSTERIORI ERROR ESTIMATES FOR A CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY PARABOLIC EQUATIONS

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Abstract. In this paper, we study adaptive finite element approximation in the backward Euler scheme for a constrained optimal control problem by parabolic equations on multi-meshes. The control constraint is given in an integral sense: $K = \{u(t) \in L^2(\Omega) : a \le \int_\Omega u(t) \le b\}$. We derive equivalent a posteriori error estimates with lower and upper bounds for both the state and the control approximation, which are used as indicators in adaptive multi-meshes finite element scheme. The error estimates are then implemented and tested with promising numerical experiments.

Key Words. constrained optimal control problem, adaptive finite element approximation, equivalent a posteriori error estimates, parabolic equations, multi-meshes

1. Introduction

For the optimal control problems governed by linear elliptic or parabolic state equations, a priori error estimates of the finite element approximation were established long ago, see [1, 2, 3, 4, 5]. In order to obtain a numerical solution of acceptable accuracy for the optimal control problem, the finite element meshes have to be refined according to a mesh refinement scheme. Adaptive finite element approximation uses a posteriori indicators to guide the mesh refinement procedure. Only the area where the error indicator is larger will be refined so that a higher density of nodes is distributed over the area where the solution is difficult to be approximated. In this sense adaptive finite element approximation relies very much on the error indicator used.

It has been recently found that suitable adaptive meshes can greatly reduce the control approximation errors, see [6, 7, 8, 9, 10]. If the computational meshes are not properly generated, then there may be large error around the singularities of the control, which cannot be removed later on. Furthermore in a constrained control problem, the optimal control and the state usually have very different regularities and their locations. This indicates that the all-in-one mesh strategy may be inefficient. Adaptive multi-meshes, that is, separate adaptive meshes which are adjusted according to different error indicators, are often necessary, see [11]. Using different adaptive meshes for the control and the state allows very coarse meshes to be used in solving the state and co-state equations. Thus much computational work can be saved because one of the major computational loads is to solve the state and co-state equations repeatedly. This can be clearly seen from numerical experiments in [11] and our numerical tests in Section 4.

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Up to now, a posteriori error estimates have mainly been developed for elliptic control problems particularly with point-wise type control constraints. The details can be found in the book [12]. In a recent work [13], a posteriori error estimates were derived for the constrained optimal control governed by an elliptic equation, where the control constraint is given in an average sense: $K = \{\int u \geq 0\}$. These estimates were held on different multi-meshes for the control and the state.

Although there are so much progress for elliptic control problems, it is much more complicated to study and implement adaptive multi-meshes computational schemes for evolutional control problems. There are some papers on a posteriori error estimates for optimal control problems governed by parabolic equations, e.g. [10, 11]. They mainly used the well-known stability results of the dual equations [14] to derive a posteriori error upper bounds, which were presented mainly in $C(0,T;L^2(\Omega))$-norm. However, a posteriori error lower bounds were not provided in these papers.

The purpose of this work is to derive equivalent a posteriori error estimates for the following constrained parabolic optimal control problem:

$$\min_{u\in X, u(t)\in K} \frac{1}{2}\int_0^T \left\{\|y-y_d\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right\} dt, \tag{1.1}$$

subject to

$$\begin{cases} \dfrac{\partial y}{\partial t} - div(\nabla y) = f+u, & (x,t)\in\Omega\times(0,T], \\ y|_{\partial\Omega} = 0, & t\in[0,T], \\ y(x,0) = y_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where: Ω is a bounded open set in R^n $(n\geq 2)$ with the Lipschitz boundary $\partial\Omega$, $y_0 \in H_0^1(\Omega)$, $f\in L^2(0,T;L^2(\Omega))$, $U = L^2(\Omega)$, $X = L^2(0,T;U)$. Let $K = \{u(t)\in L^2(\Omega) : a \leq \int_\Omega u(t) \leq b\}$ be a closed convex set, where a and b are known constants. We obtain a posteriori error estimates with lower and upper bounds and present numerical experiments to confirm the effectiveness of the error estimates.

The plan of the paper is as follows. In Section 2, we will construct the multi-meshes finite element approximation in the backward Euler scheme for (1.1)-(1.2). In Section 3, equivalent a posteriori error estimates are derived for both the state and the control approximation. Our methods are very different from that of [10, 11]. For the reason to derive the lower bounds, we do not use the stability results of the dual equations and derive the error estimates mainly in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$-norm (see Theorem 3.1). Finally numerical experiments are presented in Section 4. To our best knowledge, this paper appears to be the first trial to consider this case in the literature.

In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $W_0^{m,q}(\Omega)\equiv\{w\in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$. In addition, c or C denotes a general positive constant independent of h.

2. Finite element approximation

In the rest of the paper, we will take the state space $W = L^2(0,T;V)$ with $V = H_0^1(\Omega)$, the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega)$.

Let

$$a(v,w) = \int_\Omega (\nabla v)\cdot\nabla w, \ \forall\, v,w\in V; \qquad (u,v) = \int_\Omega uv, \ \forall\, u,\, v\in U.$$

It follows that

$$c\|v\|_1^2 \le a(v,v), \qquad |a(v,w)| \le C\|v\|_1\|w\|_1, \qquad \forall\ v,\, w \in V. \tag{2.1}$$

Then a weak formula of the convex optimal control problem (1.1)-(1.2) reads: find $y \in H^1(0,T;L^2(\Omega)) \cap W$ such that

$$(OCP): \quad \begin{cases} \min\limits_{u\in X, u(t)\in K} \dfrac{1}{2}\displaystyle\int_0^T \Big\{ \|y-y_d\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \Big\}\, dt, \\ (\dfrac{\partial y}{\partial t}, w) + a(y,w) = (f+u, w), \quad \forall\ w \in V,\ \ t \in\ (0,T], \\ y(x,0) = y_0(x), \qquad x \in\ \Omega. \end{cases} \tag{2.2}$$

It follows from [15] that the control problem (OCP) has a unique solution (y,u), and that a pair (y,u) is the solution of (OCP) iff there is a co-state $p \in H^1(0,T;L^2(\Omega)) \cap W$ such that the triplet (y,p,u) satisfies the following optimality conditions (OCP-OPT):

$$(OCP-OPT): \quad \begin{cases} (\dfrac{\partial y}{\partial t}, w) + a(y,w) = (f+u,w), \qquad \forall\ w \in V, \\ \qquad y(x,0) = y_0(x), \\ -(\dfrac{\partial p}{\partial t}, w) + a(q,p) = (y-y_d, q), \qquad \forall\ q \in V, \\ \qquad p(x,T) = 0, \\ \displaystyle\int_0^T (u+p, v-u)\, dt \ge 0, \qquad \forall\ v(t) \in K,\ \ v \in X. \end{cases} \tag{2.3}$$

Let us consider the finite element approximation of the control problem (OCP). Let Ω^h be a polygonal approximation to Ω with a boundary $\partial\Omega^h$. For simplicity, we assume that $\Omega^h = \Omega$ in this paper. Let T^h be a partitioning of Ω^h into disjoint regular n-simplices τ, so that $\bar{\Omega}^h = \bigcup_{\tau\in T^h} \bar{\tau}$. Each element τ has at most one face on $\partial\Omega^h$, and the adjoining elements $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if τ and $\tau' \in T^h$. Let h_τ denote the maximum diameter of the element τ in T^h.

Associated with T^h is a finite dimensional subspace S^h of $C(\bar{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of m-order ($m \ge 1$) for all $\chi \in S^h$ and $\tau \in T^h$. Let $V^h = \{v_h \in S^h : v_h|_{\partial\Omega} = 0\}$, $W^h = L^2(0,T;V^h)$. It is easy to see that $V^h \subset V,\ W^h \subset W$.

Let T^h_U be a partitioning of Ω^h into disjoint regular n-simplices τ_U, so that $\bar{\Omega}^h = \bigcup_{\tau_U\in T^h_U} \bar{\tau}_U$. Each element τ_U has at most one face on $\partial\Omega^h$, and the adjoining elements $\bar{\tau}_U$ and $\bar{\tau}'_U$ have either only one common vertex or a whole edge or face if τ_U and $\tau'_U \in T^h_U$. Let h_{τ_U} denote the maximum diameter of the element τ_U in T^h_U.

Associated with T^h_U is another finite dimensional subspace U^h of $L^2(\Omega^h)$, such that $\chi|_{\tau_U}$ are polynomials of m-order ($m = 0$ or 1) for all $\chi \in U^h$ and $\tau_U \in T^h_U$. An optimal control of a constrained problem normally has lower regularity so that we shall use discontinuous base functions to approximate the control. Hence there is no requirement for continuity of the functions in U^h. Let $X^h = L^2(0,T;U^h)$. It is easy to see that $X^h \subset X$.

Let

$$K^h = \{u_h(t) \in U^h :\ a \le \int_\Omega u_h(t) \le b\}. \tag{2.4}$$

Then a possible semi-discrete finite element approximation of (OCP) is as follows:
(2.5)

$$(OCP)^h: \quad \begin{cases} \min\limits_{u_h\in X^h,\, u_h(t)\in K^h} \dfrac{1}{2}\displaystyle\int_0^T \left\{ \|y_h-y_d\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2(\Omega)} \right\} dt, \\ (\dfrac{\partial y_h}{\partial t}, w_h) + a(y_h, w_h) = (f+u_h, w_h), \qquad \forall\, w_h \in V^h,\ t\in(0,T], \\ y_h(x,0) = y_0^h(x), \quad x\in\Omega, \end{cases}$$

where $y_h \in H^1(0,T;V^h)$, K^h is a closed convex set in U^h, and $y_0^h(x) \in V^h$ is an approximation of $y_0(x)$.

Let y and y_h be the solutions of (2.2) and (2.5) respectively. Let

$$J(u) = \frac{1}{2}\int_\Omega (y-y_d)^2 + \frac{1}{2}\int_\Omega u^2, \quad J_h(u_h) = \frac{1}{2}\int_{\Omega^h}(y_h-y_d)^2 + \frac{1}{2}\int_{\Omega^h} u_h^2.$$

Then the reduced problems of (2.2) and (2.5) read

$$\min_{u\in X,\, u(t)\in K}\{\int_0^T J(u)dt\} \qquad \text{and} \qquad \min_{u_h\in X^h,\, u_h(t)\in K^h}\{\int_0^T J_h(u_h)dt\},$$

respectively.

Since (2.2) is a linear control problem, the reduced objective functions are convex. Furthermore, we assume that $J(\cdot)$ is uniformly convex in the sense that there is a $c>0$, independent of h, such that

$$\int_0^T (J'(u)-J'(v), u-v)\, dt \geq c\|u-v\|^2_{L^2(0,T;L^2(\Omega))}, \tag{2.6}$$

where $u, v\in X$.

It follows that the control problem $(OCP)^h$ has a unique solution (y_h, u_h) and that a pair $(y_h, u_h) \in V^h\times U^h$ is the solution of $(OCP)^h$ iff there is a co-state $p_h\in V^h$ such that the triplet (y_h, p_h, u_h) satisfies the following optimality conditions:

$$(OCP-OPT)^h: \quad \begin{cases} (\dfrac{\partial y_h}{\partial t}, w_h) + a(y_h, w_h) = (f+u_h, w_h), \quad \forall\, w_h\in V^h, \\ \qquad y_h(x,0) = y_0^h(x), \quad x\in\Omega, \\ -(\dfrac{\partial p_h}{\partial t}, w_h) + a(q_h, p_h) = (y_h - y_d, q_h), \quad \forall\, q_h \in V^h, \\ \qquad p_h(x,T) = 0, \quad x\in\Omega, \\ \int_0^T (u_h+p_h, v_h-u_h)dt \geq 0, \quad \forall\, v_h \in K^h. \end{cases} \tag{2.7}$$

We now consider the fully discrete approximation for above semi-discrete problem by using the backward Euler scheme in time ([10, 11]).

Let $0=t_0<t_1<\cdots<t_N=T, k_i=t_i-t_{i-1}, i=1,2,\cdots,N, k=max_{i\in[1,N]}\{k_i\}$. For $i=1,2,\cdots,N$, construct the finite element spaces $V_i^h\subset H_0^1(\Omega)$ (similar as V^h) with the mesh T_i^h. Similarly, construct the finite element spaces $U_i^h\subset L^2(\Omega)$ (similar as U^h) with the mesh $(T_U^h)_i$. Let $h_{\tau^i}(h_{\tau_U^i})$ denote the maximum diameter of the element $\tau^i(\tau_U^i)$ in $T_i^h((T_U^h)_i)$. Define mesh functions $\tau(\cdot)$, $\tau_U(\cdot)$ and mesh size functions $h_\tau(\cdot)$, $h_{\tau_U}(\cdot)$ such that $\tau(t)|_{t\in(t_{i-1},t_i]}=\tau^i, \tau_U(t)|_{t\in(t_{i-1},t_i]}=\tau_U^i$, $h_\tau(t)|_{t\in(t_{i-1},t_i]}=h_{\tau^i}, h_{\tau_U}(t)|_{t\in(t_{i-1},t_i]}=h_{\tau_U^i}$. For ease of exposition, we shall denote $\tau(t), \tau_U(t), h_\tau(t)$ and $h_{\tau_U}(t)$ by τ, τ_U, h_τ and h_{τ_U}, respectively. Let $K_i^h\subset U_i^h\cap K$. The fully discrete approximation scheme $(OCP)^{hk}$

is to find $(y_h^i, u_h^i) \in V_i^h \times K_i^h$, $i = 1, 2, \cdots, N$, such that

(2.8)

$$(OCP)^{hk}: \quad \begin{cases} \min\limits_{u_h^i \in K_i^h} \dfrac{1}{2} \displaystyle\sum_{i=1}^{N} k_i \Big\{ \|y_h^i - y_d\|^2_{L^2(\Omega)} + \|u_h^i\|^2_{L^2(\Omega)} \Big\}, \\ (\dfrac{y_h^i - y_h^{i-1}}{k_i}, w_h) + a(y_h^i, w_h) = (f(x, t_i) + u_h^i, w_h), \quad \forall\, w_h \in V_i^h, \\ y_h^0 = y_0^h(x), \quad x \in \Omega. \end{cases}$$

It follows that the control problem $(OCP)^{hk}$ has a unique solution $(Y_h^i, U_h^i), i = 1, 2, \cdots, N$, and that a pair $(Y_h^i, U_h^i) \in V_i^h \times K_i^h, i = 1, 2, \cdots, N$, is the solution of $(OCP)^{hk}$ iff there is a co-state $P_h^{i-1} \in V_{i-1}^h, i = 1, 2, \cdots, N$, such that the triplet $(Y_h^i, P_h^{i-1}, U_h^i) \in V_i^h \times V_{i-1}^h \times K_i^h, i = 1, 2, \cdots, N$, satisfies the following optimality conditions: $(OCP - OPT)^{hk}$

(2.9)

$$\begin{cases} (\dfrac{Y_h^i - Y_h^{i-1}}{k_i}, w_h) + a(Y_h^i, w_h) = (f(x, t_i) + U_h^i, w_h), \quad \forall\, w_h \in V_i^h,\ i = 1, \cdots, N, \\ \quad Y_h^0 = y_0^h(x), \quad x \in \Omega, \\ (\dfrac{P_h^{i-1} - P_h^i}{k_i}, w_h) + a(q_h, P_h^{i-1}) = (Y_h^i - y_d, q_h), \quad \forall\, q_h \in V_{i-1}^h,\ i = N, \cdots, 1, \\ \quad P_h^N = 0, \quad x \in \Omega, \\ (U_h^i + P_h^{i-1}, v_h - U_h^i) \geq 0, \quad \forall\, v_h \in K_i^h,\ i = 1, 2, \cdots, N. \end{cases}$$

For $i = 1, 2, \cdots, N$, let

$$\begin{aligned} Y_h|_{(t_{i-1}, t_i]} &= \Big((t_i - t) Y_h^{i-1} + (t - t_{i-1}) Y_h^i \Big) / k_i, \\ P_h|_{(t_{i-1}, t_i]} &= \Big((t_i - t) P_h^{i-1} + (t - t_{i-1}) P_h^i \Big) / k_i, \\ U_h|_{(t_{i-1}, t_i]} &= U_h^i. \end{aligned}$$

For any function $\omega \in C(0, T; L^2(\Omega))$, let $\hat{\omega}(x, t)|_{t \in (t_{i-1}, t_i]} = \omega(x, t_i)$, $\tilde{\omega}(x, t)|_{t \in (t_{i-1}, t_i]} = \omega(x, t_{i-1})$. Then the optimality conditions (2.9) can be restated as

$$\begin{cases} (\dfrac{\partial Y_h}{\partial t}, w_h) + a(\hat{Y}_h, w_h) = (\hat{f} + U_h, w_h), \quad \forall\, w_h \in V_i^h,\ i = 1, \cdots, N, \\ \quad Y_h(x, 0) = y_0^h(x), \quad x \in \Omega, \\ -(\dfrac{\partial P_h}{\partial t}, w_h) + a(q_h, \tilde{P}_h) = (\hat{Y}_h - y_d, q_h), \quad \forall\, q_h \in V_{i-1}^h,\ i = N, \cdots, 1, \\ \quad P_h(x, T) = 0, \quad x \in \Omega, \\ (U_h + \tilde{P}_h, v_h - U_h) \geq 0, \quad U_h \in K_i^h,\ \forall\, v_h \in K_i^h,\ i = 1, 2, \cdots, N. \end{cases} \tag{2.10}$$

The variational inequality in (2.10) can be easily solved numerically as follows:

Lemma 2.1. *Let (Y_h, P_h, U_h) be the solutions of (2.10). Then the solution of the variational inequality in (2.10) is*

$$U_h = \Pi_h(-P_h + \max\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\}), \tag{2.11}$$

where $\bar{v}$ is the integral average value of v on the element τ such that $\bar{v}|_\tau = \frac{\int_\tau v}{\int_\tau 1}$ and Π_h is the L^2-projection from $L^2(\Omega)$ to U^h.

Proof. (I) Since Π_h is the $L^2-projection$ from $L^2(\Omega)$ to U^h, we have

$$\int_\Omega (\Pi_h v - v)\phi = 0, \quad \forall\ \phi \in U^h,\ v \in K. \tag{2.12}$$

It follows from (2.12) that $\int_\Omega (\Pi_h v - v) = 0$, which infers $a \le \int_\Omega \Pi_h v \le b$. Thus, we see $\Pi_h v \in K^h$.

We know that

$$\int_\Omega \Big(-P_h + \max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\}\Big) = \begin{cases} a, & \text{if} \quad -\overline{P_h} \le \bar{a}, \\ -\int_\Omega P_h, & \text{if} \quad \bar{a} < -\overline{P_h} < \bar{b}, \\ b, & \text{if} \quad \bar{b} \le -\overline{P_h}, \end{cases} \tag{2.13}$$

which shows that $-P_h + \max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\} \in K \subset L^2(\Omega)$. Thus, U_h defined by (2.11) is in K^h.

(II) Since $\forall\ \phi \in U^h$

$$\int_\Omega \Big(\Pi_h(-P_h + \max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\}) - (-P_h + \max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\})\Big)\phi = 0,$$

we have $\forall\ v_h \in K^h$,

$$\int_\Omega (U_h + P_h)(v_h - U_h) = \int_\Omega (\max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\})(v_h - U_h). \tag{2.14}$$

Further, we can discuss (2.14) as follows:

Case (i): If $-\overline{P_h} \le \bar{a}$, then since

$$\int_\Omega U_h = \int_\Omega (-P_h + \max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\}) = a,$$

and $\int_\Omega v_h \ge a$, we have

$$\int_\Omega (U_h + P_h)(v_h - U_h) = \int_\Omega (\overline{P_h} + \bar{a})(v_h - U_h) \ge 0. \tag{2.15}$$

Case (ii): If $\bar{a} < -\overline{P_h} < \bar{b}$, then

$$\int_\Omega (U_h + P_h)(v_h - U_h) = 0. \tag{2.16}$$

Case (iii): If $\bar{b} \le -\overline{P_h}$, then since

$$\int_\Omega U_h = \int_\Omega (-P_h + \max\big\{\overline{P_h} + \bar{a}, \min\{\overline{P_h} + \bar{b}, 0\}\big\}) = b,$$

and $\int_\Omega v_h \le b$, we have

$$\int_\Omega (U_h + P_h)(v_h - U_h) = \int_\Omega (\overline{P_h} + \bar{b})(v_h - U_h) \ge 0. \tag{2.17}$$

Thus, U_h defined by (2.11) is the solution of the variational inequality in (2.10). The proof of Lemma 2.1 is completed. $\square$

3. Equivalent a posteriori error estimates

Now, we turn to deriving equivalent a posteriori error estimates for finite element approximation (2.10) allowing different meshes to be used for the state and the control. In order to derive a posteriori error estimates of residual type, we need the following important lemmas:

Lemma 3.1. *Let π_h be the average interpolation operator defined in* [16]. *For $m = 0$ or 1, $1 \le q \le \infty$ and $\forall\, v \in W^{1,q}(\Omega)$,*

$$|v - \pi_h v|_{W^{m,q}(\tau)} \le \sum_{\bar\tau' \cap \bar\tau \neq \emptyset} C h_\tau^{1-m} |v|_{W^{1,q}(\tau')}. \tag{3.1}$$

Lemma 3.2. [17] *For all $v \in W^{1,q}(\Omega)$, $1 \le q < \infty$,*

$$\|v\|_{W^{0,q}(\partial\tau)} \le C\Big(h_\tau^{-\frac{1}{q}} \|v\|_{W^{0,q}(\tau)} + h_\tau^{1-\frac{1}{q}} |v|_{W^{1,q}(\tau)}\Big). \tag{3.2}$$

3.1. Upper bound estimates.

Define $J(\cdot)$ and $J_h(\cdot)$ as before. It can be shown that

$$\int_0^T (J'(u), v)\, dt = \int_0^T (u + p, v)\, dt,$$

$$\int_0^T (J_h'(U_h), v)\, dt = \int_0^T (U_h + P_h, v)\, dt,$$

$$\int_0^T (J'(U_h), v)\, dt = \int_0^T (U_h + p(U_h), v)\, dt,$$

where $p(U_h) \in V$ is the solution of the following auxiliary equation:

$$\begin{cases} (\dfrac{\partial y(U_h)}{\partial t}, w) + a(y(U_h), w) = (f + U_h, w), & \forall\, w \in V = H_0^1(\Omega), \\ \quad y(U_h)(x, 0) = y_0(x), & x \in \Omega, \\ -(\dfrac{\partial p(U_h)}{\partial t}, w) + a(q, p(U_h)) = (y(U_h) - y_d, q), & \forall\, q \in V = H_0^1(\Omega), \\ \quad p(U_h)(x, T) = 0, & x \in \Omega. \end{cases} \tag{3.3}$$

The following lemma is the first step to derive our a posteriori error estimates.

Lemma 3.3. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (2.3) and (2.10), respectively. Then,*

$$\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} \le C\big\{\eta_1^2 + \|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\big\}, \tag{3.4}$$

where $p(U_h)$ is the solution of equation (3.3), and

$$\eta_1^2 = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \sum_{\tau_U} \int_{\tau_U} (P_h - \Pi_h P_h)^2\, dt.$$

Proof. (I). It follows from (2.6) that

$$
\begin{aligned}
c\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))} &\le \int_0^T (J'(u),u-U_h)dt - \int_0^T (J'(U_h),u-U_h)dt \\
&\le -\int_0^T (J'(U_h),u-U_h)dt \\
&= \int_0^T (J_h'(U_h),U_h-u)dt + \int_0^T (J_h'(U_h)-J'(U_h),u-U_h)dt \\
&= \inf_{v_h(t)\in K^h}\int_0^T (J_h'(U_h),v_h-u)dt + \int_0^T (J_h'(U_h)-J'(U_h),u-U_h)dt.
\end{aligned}
\tag{3.5}
$$

(II). Note that

$$
\int_0^T (J_h'(U_h),v_h-u)dt = \int_0^T (U_h+P_h,v_h-u)dt. \tag{3.6}
$$

By Lemma 2.1 and taking $v_h=\Pi_h u\in U^h$ in (3.6), we know

$$
\begin{aligned}
&\int_0^T (U_h+P_h,\Pi_h u-u)dt \\
&= \int_0^T \sum_{\tau_U}\int_{\tau_U}(-\Pi_h P_h+P_h)(\Pi_h(u-U_h)-(u-U_h))\,dt \\
&\le C(\delta)\int_0^T \sum_{\tau_U}\int_{\tau_U}(-\Pi_h P_h+P_h)^2\,dt + \delta\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
\tag{3.7}
$$

From (3.5), (3.6) and (3.7), we obtain

$$
\begin{aligned}
c\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))} \le{}& C(\delta)\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h-\Pi_h P_h)^2\,dt + \delta\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))} \\
&+\int_0^T (J_h'(U_h)-J'(U_h),u-U_h)\,dt.
\end{aligned}
\tag{3.8}
$$

By the formulas of J' and J_h', it follows that

$$
\begin{aligned}
&\int_0^T (J_h'(U_h)-J'(U_h),u-U_h)\,dt = \int_0^T (P_h-p(U_h),u-U_h)\,dt \\
&\le \frac{1}{2\delta}\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\delta}{2}\|U_h-u\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
\tag{3.9}
$$

Therefore, (3.4) follows from (3.8) and (3.9) by setting $\delta=\dfrac{c}{3}$ in the above inequalities. □

Using Lemma 3.3, we then can derive upper error bounds as stated in the following lemma. The detailed proof is rather long, so we only state the main parts that are different from those in [10, 11].

Lemma 3.4. *Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of (2.3) and* (2.10)*, respectively. Let $y(U_h)$ and $p(U_h)$ be the solutions of the auxiliary equation* (3.3)*. Suppose all* *the conditions in Lemma 3.3 are valid. Then,*

(3.10)

$$\|Y_h-y(U_h)\|^2_{L^\infty(0,T;L^2(\Omega))}+\|Y_h-y(U_h)\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(Y_h-y(U_h))\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

$$+\|P_h-p(U_h)\|^2_{L^\infty(0,T;L^2(\Omega))}+\|P_h-p(U_h)\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(P_h-p(U_h))\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

$$\leq C\sum_{i=2}^{7}\eta_i^2,$$

where

$$\eta_2^2=\sum_{i=1}^N k_i\Big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat{Y}_h-y_d+\partial_tP_h+\Delta\tilde{P}_h)^2+\sum_l\int_l h_l[\nabla\tilde{P}_h\cdot n]^2\Big),$$
$$\eta_3^2=\sum_{i=1}^N\frac{k_i}{3}|P_h^{i-1}-P_h^i|_1^2,$$
$$\eta_4^2=\sum_{i=1}^N k_i\Big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat{f}+U_h-\partial_tY_h+\Delta\hat{Y}_h)^2+\sum_l\int_l h_l[\nabla\hat{Y}_h\cdot n]^2\Big),$$
$$\eta_5^2=\|f-\hat{f}\|^2_{L^2(0,T;L^2(\Omega))},$$
$$\eta_6^2=\sum_{i=1}^N\frac{k_i}{3}|Y_h^i-Y_h^{i-1}|_1^2,$$
$$\eta_7^2=\|Y_h(x,0)-y_0(x)\|^2_{L^2(\Omega)},$$

where l is a face of an element τ, h_l is the maximum diameter of l, $[\nabla\tilde{P}_h\cdot n]$ and $[\nabla\hat{Y}_h\cdot n]$ are the normal derivative jumps over the interior face l, defined by

$$[\nabla\tilde{P}_h\cdot n]_l=(\nabla\tilde{P}_hh|_{\tau_l^1}-\nabla\tilde{P}_h|_{\tau_l^2})\cdot n,$$

$$[\nabla\hat{Y}_h\cdot n]_l=(\nabla\hat{Y}_h|_{\tau_l^1}-\nabla\hat{Y}_h|_{\tau_l^2})\cdot n,$$

where n is the unit normal vector on $l=\bar{\tau}_l^1\cap\bar{\tau}_l^2$ outwards τ_l^1. For later convenience, we define $[\nabla\tilde{P}_h\cdot n]_l=0$ and $[\nabla\hat{Y}_h\cdot n]_l=0$ when $l\subset\partial\Omega$. Let ∂_t denote $\frac{\partial}{\partial t}$.

Proof. Let π_h be the average interpolation operator defined as in Lemma 3.1 and

$$<R(U_h),v>=-(\partial_t(p(U_h)-P_h),v)+a(p(U_h)-P_h,v).$$

Then it follows from (2.10) and (3.3) that

$$-(\partial_t(p(U_h)-P_h),v)+(\nabla(p(U_h)-P_h),\nabla v)$$

$$=-(\partial_t(p(U_h)-P_h),v-\pi_hv)+(\nabla(p(U_h)-P_h),\nabla(v-\pi_hv))$$

$$-(\partial_t(p(U_h)-P_h),\pi_hv)+(\nabla(p(U_h)-P_h),\nabla\pi_hv)$$

$$=\sum_{\tau\in T^h}\int_\tau(\hat{Y}_h-y_d+\partial_tP_h)(v-\pi_hv)-\sum_{\tau\in T^h}\int_\tau\nabla\tilde{P}_h\nabla(v-\pi_hv)$$
$$+(y(U_h)-\hat{Y}_h,v)+(\nabla(\tilde{P}_h-P_h),\nabla v) \tag{3.11}$$

$$\leq\Big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat{Y}_h-y_d+\partial_tP_h+\Delta\tilde{P}_h)^2+\sum_l h_l\int_l[\nabla\tilde{P}_h\cdot n]^2\Big)^{\frac{1}{2}}\|v\|_1$$
$$+(y(U_h)-\hat{Y}_h,v)+(\nabla(\tilde{P}_h-P_h),\nabla v).$$

Taking $v = p(U_h) - P_h$ in (3.11), we obtain

$$-\frac{1}{2}\frac{d}{dt}\|p(U_h) - P_h\|^2_{L^2(\Omega)} + |p(U_h) - P_h|^2_{H^1(\Omega)}$$

$$\leq \Big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h)^2 + \sum_l h_l\int_l[\nabla\tilde{P}_h\cdot n]^2\Big)^{\frac{1}{2}}\|p(U_h) - P_h\|_1$$
$$+\|y(U_h) - \hat{Y}_h\|_{L^2(\Omega)}\|p(U_h) - P_h\|_{L^2(\Omega)} + \|\nabla(\tilde{P}_h - P_h)\|_{L^2(\Omega)}\|\nabla(p(U_h) - P_h)\|_{L^2(\Omega)}.$$

Then integrating time from 0 to T and by Schwarz's inequality, we have

$$\|p(U_h) - P_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|p(U_h) - P_h\|^2_{L^2(0,T;H^1(\Omega))}$$

$$\leq \sum_{i=1}^N k_i\Big(\int_\tau h_\tau^2(\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h)^2 + \sum_l h_l\int_l[\nabla\tilde{P}_h\cdot n]^2\Big) \tag{3.12}$$
$$+\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^T|\tilde{P}_h - P_h|_1^2dt.$$

Since we know that

$$\tilde{P}_h - P_h = \frac{t - t_{i-1}}{k_i}(P_h^{i-1} - P_h^i),$$

then we have

$$\int_0^T|\tilde{P}_h - P_h|_1^2dt = \sum_{i=1}^N\frac{k_i}{3}|P_h^{i-1} - P_h^i|_1^2. \tag{3.13}$$

Further, from (3.11) and (3.13) we know

$$\|\partial_t(p(U_h) - P_h)\|_{L^2(0,T;H^{-1}(\Omega))}$$

$$= \sup_{v\in L^2(0,T;H_0^1(\Omega))}\frac{\int_0^T(-< R(U_h), v > +a(p(U_h) - P_h, v))dt}{\|v\|_{L^2(0,T;H_0^1(\Omega))}}$$

$$\leq \sum_{i=1}^N k_i\Big(\int_\tau h_\tau^2(\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h)^2 + \sum_l h_l\int_l[\nabla\tilde{P}_h\cdot n]^2\Big) \tag{3.14}$$

$$+\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;L^2(\Omega))} + \sum_{i=1}^N\frac{k_i}{3}|P_h^{i-1} - P_h^i|_1^2.$$

Combining (3.12) with (3.14), we derive

(3.15)

$$\|p(U_h) - P_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|p(U_h) - P_h\|^2_{L^2(0,T;H^1(\Omega))} + \|\partial_t(p(U_h) - P_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

$$\leq \sum_{i=1}^N k_i\Big(\int_\tau h_\tau^2(\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h)^2 + \sum_l h_l\int_l[\nabla\tilde{P}_h\cdot n]^2\Big)$$
$$+\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;L^2(\Omega))} + \sum_{i=1}^N\frac{k_i}{3}|P_h^{i-1} - P_h^i|_1^2.$$

Similarly analyzing for $\|y(U_h) - Y_h\|_{L^2(0,T;L^2(\Omega))}$, we let

$$< Q(U_h), v >= -(\partial_t(y(U_h) - Y_h), v) + a(y(U_h) - Y_h, v).$$

We see that

$$
\begin{aligned}
&-(\partial_t(y(U_h)-Y_h),v)+(\nabla(y(U_h)-Y_h),\nabla v)\\
&\le\Big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat f+U_h-\partial_tY_h+\Delta\hat Y_h)^2+\sum_l\int_l h_l[\nabla\tilde Y_h\cdot n]^2\Big)\|v\|_1\\
&+(f-\tilde f,v)-(\nabla(Y_h-\hat Y_h),\nabla v)
\end{aligned}
\tag{3.16}
$$

Then taking $v=y(U_h)-Y_h$ and deducing similarly as (3.15), we have

$$
\begin{aligned}
&\|y(U_h)-Y_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y(U_h)-Y_h\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(y(U_h)-Y_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}\\
&\le\sum_{i=1}^N k_i\Big(\int_\tau h_\tau^2(\hat f+U_h-\partial_tY_h+\Delta\hat Y_h)^2+\sum_l h_l\int_l[\nabla\hat Y_h\cdot n]^2\Big)\\
&\quad+\|f-\hat f\|^2_{L^2(0,T;L^2(\Omega))}+\sum_{i=1}^N\frac{k_i}{3}|Y_h^i-Y_h^{i-1}|_1^2+\|Y_h(x,0)-y_0(x)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.17}
$$

Now by (3.15) and (3.17), we can prove Lemma 3.4. □

Using Lemmas 3.3 and 3.4, we can obtain a posteriori upper bounds as follows.

Theorem 3.1. *Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of (2.3) and (2.10), respectively. Suppose all the conditions in Lemmas 3.3 and 3.4 are valid. Then,*

$$
\begin{aligned}
&\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}+\|y-Y_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y-Y_h\|^2_{L^2(0,T;H^1(\Omega))}\\
&+\|\partial_t(y-Y_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|p-P_h\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&+\|p-P_h\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(p-P_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}\\
&\le C\sum_{i=1}^7\eta_i^2,
\end{aligned}
\tag{3.18}
$$

where η_1 is defined in Lemma 3.3, $\eta_i,\ i=2,\cdots,7$ are defined in Lemma 3.4.

Proof. It follows from Lemmas 3.3 and 3.4 that

$$
\begin{aligned}
\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}&\le C\{\eta_1^2+\|\tilde P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\}\\
&\le C\eta_1^2+C\|\tilde P_h-P_h\|^2_{L^2(0,T;L^2(\Omega))}+C\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\\
&\le C\eta_1^2+C\|\tilde P_h-P_h\|^2_{L^2(0,T;L^2(\Omega))}+C\sum_{i=1}^7\eta_i^2.
\end{aligned}
\tag{3.19}
$$

By Poincaré's inequality and (3.13), it follows

$$
\|\tilde P_h-P_h\|^2_{L^2(0,T;L^2(\Omega))}\le C\|\nabla(\tilde P_h-P_h)\|^2_{L^2(0,T;L^2(\Omega))}\le C\eta_3^2.
\tag{3.20}
$$

Combining (3.19) with (3.20), we see that

$$
\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}\le C\sum_{i=1}^7\eta_i^2.
\tag{3.21}
$$

Note that

(3.22)
$$\|y-Y_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y-Y_h\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leq \|y-y(U_h)\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y-y(U_h)\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(y-y(U_h))\|^2_{L^2(0,T;H^{-1}(\Omega))}$$
$$+\|Y_h-y(U_h)\|^2_{L^\infty(0,T;L^2(\Omega))}+\|Y_h-y(U_h)\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(Y_h-y(U_h))\|^2_{L^2(0,T;H^{-1}(\Omega))},$$

and

(3.23)
$$\|y-y(U_h)\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y-y(U_h)\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(y-y(U_h))\|^2_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leq C\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}.$$

The similar results can be derived for $p-P_h$ and $p-P(U_h)$ as (3.22) and (3.23).

Then by (3.21), (3.22)-(3.23) and Lemma 3.4, we get

(3.24)
$$\|y-Y_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y-Y_h\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}$$
$$+\|p-P_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|p-P_h\|^2_{L^2(0,T;H^1(\Omega))}+\|\partial_t(p-P_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leq C\sum_{i=1}^{7}\eta_i^2+C\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}\leq C\sum_{i=1}^{7}\eta_i^2.$$

Therefore, (3.18) follows from (3.21) and (3.24). □

3.1. Lower bound estimates. Now we are in the position of deriving a posteriori lower bounds. We will use some lemmas. First, we give the following lemma using the standard bubble function technique in [18, 19, 20].

Lemma 3.5. *Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of (2.3) and (2.10), respectively. Then,*

$$k_i\eta_{4i}^2+\frac{k_i}{3}|Y_h^i-Y_h^{i-1}|_1^2$$
$$\leq C_{10}\Big\{\|f-\hat f\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}$$
$$+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))} \tag{3.25}$$
$$+k_i\sum_{\tau\in T^h}\int_\tau h_\tau^2\Big(\hat f+U_h-\partial_tY_h+\Delta\hat Y_h-\overline{\hat f+U_h-\partial_tY_h+\Delta\hat Y_h}\Big)^2\Big\},$$

where

$$\eta_{4i}^2=\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,\hat f+U_h-\partial_tY_h+\Delta\hat Y_h\,)^2+\sum_l\int_l h_l[\nabla\hat Y_h\cdot n]^2.$$

Proof. (I). Since we know that

$$\frac{k_i}{3}|Y_h^i-Y_h^{i-1}|_1^2=\int_{t_{i-1}}^{t_i}|\hat Y_h-Y_h|_1^2dt,$$

and

$$(\partial_t(y-Y_h),v)+(\nabla(y-Y_h),\nabla v)$$
$$=(f+u,v)-(\partial_t Y_h,v)-(\nabla(Y_h-\hat{Y}_h),\nabla v)-(\nabla\hat{Y}_h,\nabla v),$$

we obtain

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i}(\nabla(Y_h-\hat{Y}_h),\nabla v)dt\\
=&\int_{t_{i-1}}^{t_i}\Big((f+u,v)-(\partial_t Y_h,v)-(\nabla\hat{Y}_h,\nabla v)-(\partial_t(y-Y_h),v)-(\nabla(y-Y_h),\nabla v)\Big)dt\\
=&\int_{t_{i-1}}^{t_i}\Big(\big(f+u-(\hat{f}+U_h),v\big)+\sum_{\tau\in T^h}\int_\tau(\hat{f}+U_h-\partial_t Y_h+\Delta\hat{Y}_h)(v-\pi_h v)\\
&-\sum_l\int_l[\nabla\hat{Y}_h\cdot n](v-\pi_h v)-(\partial_t(y-Y_h),v)-(\nabla(y-Y_h),\nabla v)\Big)dt\\
\le&\Big(\|f-\hat{f}\|_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|_{L^2(t_{i-1},t_i;L^2(\Omega))}\\
&+k_i^{\frac{1}{2}}(\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat{f}+U_h-\partial_t Y_h+\Delta\hat{Y}_h)^2+\sum_l\int_l h_l[\nabla\hat{Y}_h\cdot n]^2)^{\frac{1}{2}}\\
&+\|y-Y_h\|_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\Big)\|\nabla v\|_{L^2(t_{i-1},t_i;L^2(\Omega))}.
\end{aligned}$$

Taking $v=\hat{Y}_h-Y_h$ in the above inequality, we have

(3.26)
$$\begin{aligned}
\frac{k_i}{3}|Y_h^i-Y_h^{i-1}|_1^2\quad&\le C\Big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}\\
&+k_i\eta_{4i}^2+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\Big).
\end{aligned}$$

(II). Then, we take three steps to analyze the term $k_i\eta_{4i}^2$.

Step 1: the bubble functions. Let

$$R_\tau=\hat{f}+U_h-\partial_t Y_h+\Delta\hat{Y}_h,\qquad R_l=[\nabla\hat{Y}_h\cdot n].$$

And denote by ϕ_τ and ϕ_l the corresponding bubble functions which are the scaled product of the barycentric co-ordinates of the vertices of τ and l, respectively. We denote for any edge l the union of the elements sharing l by ω_l. We then have

$$\sup\phi_\tau\subset\tau,\qquad\sup\phi_l\subset\omega_l.$$

Letting $\bar{R}_\tau=\frac{\int_\tau R_\tau}{\int_\tau 1}$, then using the standard bubble function technique ([18, 19, 20]), it can be proved that there exist polynomials $w_\tau\in H_0^1(\tau)$ and $w_l\in H_0^1(\omega_l)$ such that

$$\int_\tau h_\tau^2\overline{(\hat{f}+U_h-\partial_t Y_h+\Delta\hat{Y}_h)^2}=\int_\tau h_\tau^2\overline{(\hat{f}+U_h-\partial_t Y_h+\Delta\hat{Y}_h)}w_\tau,\tag{3.27}$$

$$\int_l h_l[\nabla\hat{Y}_h\cdot n]^2=\int_l h_l[\nabla\hat{Y}_h\cdot n]w_l,\tag{3.28}$$

where $w_\tau = C_1\bar{R}_\tau\phi_\tau$ and $w_l = C_2 R_l\phi_l$. And from the standard scaling technique, we have:

$$\left\{\begin{array}{l} \|\nabla(\bar{R}_\tau\phi_\tau)\|_{0,\tau} \leq C_3 h_\tau^{-1}\|\bar{R}_\tau\|_{0,\tau}, \\ \|\nabla(R_l\phi_l)\|_{0,\omega_l} \leq C_4 h_l^{-\frac{1}{2}}\|R_l\|_{0,l}, \\ \|R_l\phi_l\|_{0,\omega_l} \leq C_5 h_l^{\frac{1}{2}}\|R_l\|_{0,l}. \end{array}\right. \tag{3.29}$$

The constants $C_1,\cdots,C_5$ depend on the maximal polynomial degree of the finite element functions; the constants C_3, C_4, C_5 in addition depend on the maximal ratio of the diameter of any element to the diameter of its largest inscribed ball.

We now set

$$w_n = \beta\sum_{\tau\in T^h} h_\tau^2\bar{R}_\tau\phi_\tau - \gamma\sum_{l\in\tilde{\varepsilon}_h} h_l R_l\phi_l, \tag{3.30}$$

with parameters $\beta > 0$ and $\gamma > 0$ to be determined later. We have

(3.31)

$$\begin{aligned} |w_n|_1^2 &= \|\nabla(\beta\sum_{\tau\in T^h} h_\tau^2\bar{R}_\tau\phi_\tau - \gamma\sum_{l\in\tilde{\varepsilon}_h} h_l R_l\phi_l)\|_0^2 \\ &\leq \beta^2\sum_{\tau\in T^h} h_\tau^2 C_3^2\|\bar{R}_\tau\|_{0,\tau}^2 + 2\beta\gamma C_3C_4\sum_{\tau\in T^h}\{\sum_{l;\,\omega_l\cap\tau\neq\varnothing} h_\tau h_l^{\frac{1}{2}}\|\bar{R}_\tau\|_{0,\tau}\|R_l\|_{0,l}\} \\ &\quad +\gamma^2\sum_{l\in\tilde{\varepsilon}_h}\{\sum_{l';\,\omega_{l'}\cap\omega_l\neq\varnothing} C_4^2 h_l^{\frac{1}{2}}h_{l'}^{\frac{1}{2}}\|R_l\|_{0,l}\|R_{l'}\|_{0,l'}\} \\ &\leq C\max\{\beta^2,\gamma^2\}\max\{C_3^2,C_4^2\}\Big(\sum_{\tau\in T^h}\int_\tau h_\tau^2\bar{R}_\tau^2 + \sum_{l\in\tilde{\varepsilon}_h}\int_l h_l R_l^2\Big) \\ &\leq C\max\{\beta^2,\gamma^2\}\max\{C_3^2,C_4^2\}\Big(\eta_{4i}^2 + \sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau - \bar{R}_\tau)^2\Big). \end{aligned}$$

Step 2: the estimate of η_{4i}^2. Let $< R_h(Y_h), w_n >= \sum_{\tau\in T^h}(\bar{R}_\tau, w_n)_\tau - \sum_{l\in\tilde{\varepsilon}_h}(R_l, w_n)_l$. Since we know $h_l \leq Ch_\tau$ for all edges l of any element τ, then we have

(3.32)

$$\begin{aligned} &< R_h(Y_h), w_n >= \sum_{\tau\in T^h}(\bar{R}_\tau, w_n)_\tau - \sum_{l\in\tilde{\varepsilon}_h}(R_l, w_n)_l \\ &= \beta\sum_{\tau\in T^h} h_\tau^2(\bar{R}_\tau,\phi_\tau\bar{R}_\tau)_\tau + \gamma\sum_{l\in\tilde{\varepsilon}_h}(R_l,\phi_l R_l)_l - \gamma\sum_{l\in\tilde{\varepsilon}_h}\sum_{\tau;\,\tau\cap\omega_l\neq\varnothing}(\bar{R}_\tau,\phi_l R_l)_\tau \\ &\geq \beta\sum_{\tau\in T^h} h_\tau^2\|\bar{R}_\tau\|_{0,\tau}^2 + \gamma\sum_{l\in\tilde{\varepsilon}_h} h_l\|R_l\|_{0,l}^2 - \gamma\sum_{l\in\tilde{\varepsilon}_h}\{\sum_{\tau;\,\tau\cap\omega_l\neq\varnothing} Ch_\tau C_5 h_l^{\frac{1}{2}}\|\bar{R}_\tau\|_{0,\tau}\|R_l\|_{0,l}\} \\ &\geq \beta\sum_{\tau\in T^h} h_\tau^2\|\bar{R}_\tau\|_{0,\tau}^2 + \gamma\sum_{l\in\tilde{\varepsilon}_h} h_l\|R_l\|_{0,l}^2 - \gamma\sum_{l\in\tilde{\varepsilon}_h}\{\frac{1}{2}h_l\|R_l\|_{0,l}^2 + \frac{1}{2}\sum_{\tau;\,\tau\cap\omega_l\neq\varnothing} C^2h_\tau^2C_5^2\|\bar{R}_\tau\|_{0,\tau}^2\} \\ &\geq (\beta - \frac{\gamma}{2}C^2C_5^2)\sum_{\tau\in T^h} h_\tau^2\|\bar{R}_\tau\|_{0,\tau}^2 + \frac{\gamma}{2}\sum_{l\in\tilde{\varepsilon}_h} h_l\|R_l\|_{0,l}^2. \end{aligned}$$

Then letting $\gamma = 2$ and $\beta = C^2C_5^2 + 1$, we have

$$\sum_{\tau\in T^h} h_\tau^2\|\bar{R}_\tau\|_{0,\tau}^2 + \sum_{l\in\tilde{\varepsilon}_h} h_l\|R_l\|_{0,l}^2 \leq < R_h(Y_h), w_n > . \tag{3.33}$$

Then

(3.34)

$$\begin{aligned}
\eta_{4i}^2 &= \sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar R_\tau+\bar R_\tau\,)^2+\sum_l\int_l h_l R_l^2\\
&\le \sum_{\tau\in T^h} h_\tau^2\|\bar R_\tau\|_{0,\tau}^2+\sum_{l\in\tilde\varepsilon_h} h_l\|R_l\|_{0,l}^2+\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar R_\tau\,)^2\\
&\le< R_h(Y_h),w_n>+\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar R_\tau\,)^2\\
&=\sum_{\tau\in T^h}(R_\tau,w_n)_\tau-\sum_{l\in\tilde\varepsilon_h}(R_l,w_n)_l+\sum_{\tau\in T^h}(R_\tau-\bar R_\tau,w_n)_\tau+\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar R_\tau\,)^2.
\end{aligned}$$

Since we know that for $\forall\, v\in H_0^1(\Omega)$

$$\sum_{\tau\in T^h}\int_\tau(\,\hat f+U_h-\partial_t Y_h)v-\int_\tau\nabla(\hat Y_h)\nabla v=\sum_{\tau\in T^h}(\,R_\tau,v)\tau-\sum_{l\in\tilde\varepsilon_h}(R_l,v)_l$$

and $w_n\in H_0^1(\Omega)$, then

(3.35)

$$\begin{aligned}
\eta_{4i}^2 \;&\le \sum_{\tau\in T^h}\Big\{\int_\tau(\hat f+U_h-\partial_t Y_h)w_n-\int_\tau\nabla(\hat Y_h-Y_h)\nabla w_n-\int_\tau\nabla Y_h\nabla w_n\Big\}\\
&\quad+\sum_{\tau\in T^h}(R_\tau-\bar R_\tau,w_n)_\tau+\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar R_\tau\,)^2\\
&=\sum_{\tau\in T^h}\Big\{\int_\tau\{(\hat f-f)w_n+(U_h-u)w_n+(\partial_t(y-Y_h))w_n+\nabla(y-Y_h)\nabla w_n\}\\
&\quad-\int_\tau\nabla(\hat Y_h-Y_h)\nabla w_n\Big\}+\sum_{\tau\in T^h}(R_\tau-\bar R_\tau,w_n)_\tau+\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar R_\tau\,)^2.
\end{aligned}$$

Step 3: the estimate of $k_i\eta_{4i}^2$. Then for any $\delta>0$, which will be defined below, we have

$$\begin{aligned}
k_i\eta_{4i}^2&=\int_{t_{i-1}}^{t_i}(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta\eta_{4i}^2dt\\
&\le\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\int_\tau\big\{(\hat f-f)+(U_h-u)+\partial_t(y-Y_h)\big\}(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta w_n dt\\
&\quad+\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\int_\tau\big\{\nabla(y-Y_h)-\nabla(\hat Y_h-Y_h)\big\}(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta\nabla w_n dt\\
&\quad+\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\big(R_\tau-\bar R_\tau,(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta w_n\big)_\tau dt\\
&\quad+\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}h_\tau^2\int_\tau(\,R_\tau-\bar R_\tau\,)^2(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta dt.
\end{aligned}\tag{3.36}$$

Since w_n is constant with respect to time and (3.31), we obtain that

$$
\begin{aligned}
&\|(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta w_n\|_{L^2(t_{i-1},t_i;H^1(\Omega))}\\
&\le |w_n|_1\big\{\int_{t_{i-1}}^{t_i}(\delta+1)^2(\frac{t-t_{i-1}}{k_i})^{2\delta}dt\big\}^{\frac{1}{2}}=\frac{\delta+1}{\sqrt{2\delta+1}}k_i^{\frac{1}{2}}|w_n|_1\\
&\le C_6\frac{\delta+1}{\sqrt{2\delta+1}}k_i^{\frac{1}{2}}\eta_{4i}+C_6\frac{\delta+1}{\sqrt{2\delta+1}}k_i^{\frac{1}{2}}\big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2\big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.37}
$$

From (3.31) and (3.26), we have

$$
\begin{aligned}
&-\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\int_\tau\nabla(\hat{Y}_h-Y_h)(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta\nabla w_n dt\\
&=-(1-\frac{\delta+1}{\delta+2})k_i\big(\nabla(Y_h^i-Y_h^{i-1}),\nabla w_n\big)\\
&\le|1-\frac{\delta+1}{\delta+2}|C_7k_i^{\frac{1}{2}}\Big(\eta_{4i}+\big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2\big)^{\frac{1}{2}}\Big)k_i^{\frac{1}{2}}|Y_h^i-Y_h^{i-1}|_1\\
&\le|1-\frac{\delta+1}{\delta+2}|C_7k_i^{\frac{1}{2}}\Big(\eta_{4i}+\big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2\big)^{\frac{1}{2}}\Big)\\
&\quad\times\Big\{C\big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}+k_i\eta_{4i}^2\\
&\quad+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\big)\Big\}^{\frac{1}{2}}\\
&\le C_7C^{\frac{1}{2}}|1-\frac{\delta+1}{\delta+2}|k_i\eta_{4i}^2+C_7C^{\frac{1}{2}}|1-\frac{\delta+1}{\delta+2}|k_i\big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2\big)^{\frac{1}{2}}\eta_{4i}\\
&\quad+|1-\frac{\delta+1}{\delta+2}|C_7k_i^{\frac{1}{2}}\Big(\eta_{4i}+\big(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2\big)^{\frac{1}{2}}\Big)\\
&\quad\times\Big\{C\big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}\\
&\quad+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\big)\Big\}^{\frac{1}{2}}.
\end{aligned}
\tag{3.38}
$$

Choosing $\delta=2C_7C^{\frac{1}{2}}-2$, so we have $C_7C^{\frac{1}{2}}|1-\dfrac{\delta+1}{\delta+2}|=\dfrac{1}{2}$. Then from (3.36), (3.37) and (3.38), it follows

(3.39)

$$
\begin{aligned}
\frac{1}{2}k_i\eta_{4i}^2 \quad &\le \int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\int_\tau \big\{(\hat{f}-f)+(U_h-u)+\partial_t(y-Y_h)\big\}(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta w_n dt\\
&+\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\int_\tau \nabla(y-Y_h)(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta \nabla w_n dt\\
&+\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h}\big(R_\tau-\bar{R}_\tau,(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta w_n\big)_\tau dt\\
&+\int_{t_{i-1}}^{t_i}\sum_{\tau\in T^h} h_\tau^2\int_\tau (\,R_\tau-\bar{R}_\tau\,)^2(\delta+1)(\frac{t-t_{i-1}}{k_i})^\delta dt\\
&+C_7C^{\frac{1}{2}}|1-\frac{\delta+1}{\delta+2}|k_i(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2)^{\frac{1}{2}}\eta_{4i}\\
&+|1-\frac{\delta+1}{\delta+2}|C_7k_i^{\frac{1}{2}}\Big(\eta_{4i}+(\sum_{\tau\in T^h}\int_\tau h_\tau^2(R_\tau-\bar{R}_\tau)^2)^{\frac{1}{2}}\Big)\\
&\quad\times\Big\{C\big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}\\
&\qquad+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\big)\Big\}^{\frac{1}{2}}\\
&\le C(\delta')\big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}\\
&\quad+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\big)+\delta' k_i\eta_{4i}^2\\
&\quad+C_8k_i\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar{R}_\tau\,)^2.
\end{aligned}
$$

Taking $\delta'=\frac{1}{4}$, we have

(3.40)

$$
\begin{aligned}
k_i\eta_{4i}^2 \quad &\le C\Big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}\\
&+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+k_i\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar{R}_\tau\,)^2\Big).
\end{aligned}
$$

(III). Then by (3.26) and (3.40), we have

$$
\begin{aligned}
\frac{k_i}{3}|Y_h^i-Y_h^{i-1}|_1^2 \quad &\le C_9\Big(\|f-\hat{f}\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}+\|u-U_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}\\
&+\|y-Y_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))}+\|\partial_t(y-Y_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))}\\
&+k_i\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,R_\tau-\bar{R}_\tau\,)^2\Big).
\end{aligned}
\tag{3.41}
$$

Combining (3.40) with (3.41), we can prove Lemma 3.5. □

Similarly we can prove the following lower bound estimates.

Lemma 3.6. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (2.3) and (2.10), respectively. Then*

(3.42)
$$
\begin{aligned}
&k_i\eta_{2i}^2 + \frac{k_i}{3}|P_h^{i-1} - P_h^i|_1^2 \\
&\le C_{11}\Big\{\|y - Y_h\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))} + \|p - P_h\|^2_{L^2(t_{i-1},t_i;H^1(\Omega))} + \|\partial_t(p - P_h)\|^2_{L^2(t_{i-1},t_i;H^{-1}(\Omega))} \\
&\quad + k_i \sum_{\tau\in T^h}\int_\tau h_\tau^2\Big(\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h - \overline{\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h}\,\Big)^2\Big\},
\end{aligned}
$$

where
$$
\eta_{2i}^2 = \sum_{\tau\in T^h}\int_\tau h_\tau^2(\,\hat{Y}_h - y_d + \partial_t P_h + \Delta\tilde{P}_h\,)^2 + \sum_l \int_l h_l[\nabla\tilde{P}_h\cdot n]^2.
$$

Lemma 3.7. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (2.3) and (2.10), respectively. Then*

$$
\eta_1^2 \le C_{12}\big\{\|u - U_h\|^2_{L^2(0,T;L^2(\Omega))} + \|p - P_h\|^2_{L^2(0,T;L^2(\Omega))}\big\}. \tag{3.43}
$$

Proof. We have

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(-\Pi_h P_h + P_h)^2 \\
&= \int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)(P_h - p + p - \Pi_h p + \Pi_h p - \Pi_h P_h) \\
&\le C\delta\int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)^2 + C(\delta)\int_{t_{i-1}}^{t_i}\|P_h - p\|^2_{L^2(\Omega)} \\
&\quad + \int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)(p - \Pi_h p).
\end{aligned} \tag{3.44}
$$

Since $u + p = \max\{\overline{p} + \overline{a}, \min\{\overline{p} + \overline{b}, 0\}\} = const$, we have $\Pi_h(u + p) = u + p$. Thus

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)(p - \Pi_h p) \\
&= \int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)(p + u - \Pi_h(p + u) + \Pi_h u - u) \\
&= \int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)(\Pi_h(u - U_h) - (u - U_h)) \\
&\le C\delta\int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(P_h - \Pi_h P_h)^2 + C(\delta)\int_{t_{i-1}}^{t_i}\|u - U_h\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.45}
$$

Combining (3.44) and (3.45), taking $\delta = \frac{1}{3C}$ and summing i from 0 to N, we obtain

(3.46)
$$
\eta_1^2 = \sum_{i=1}^N\int_{t_{i-1}}^{t_i}\sum_{\tau_U}\int_{\tau_U}(-\Pi_h P_h + P_h)^2 \le C_{12}\{\|u - U_h\|^2_{L^2(0,T;L^2(\Omega))} + \|p - P_h\|^2_{L^2(0,T;L^2(\Omega))}\}.
$$

Thus from Lemmas 3.5, 3.6 and 3.7, we get the following lower bound estimates.

Theorem 3.2. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (2.3) and (2.10), respectively. Then,*

(3.47)
$$\begin{aligned}\sum_{i=1}^{4}\eta_i^2+\eta_6^2 \quad &\leq C\big\{\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}+\|y-Y_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y-Y_h\|^2_{L^2(0,T;H^1(\Omega))}\\ &+\|\partial_t(y-Y_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|p-P_h\|^2_{L^\infty(0,T;L^2(\Omega))}+\|p-P_h\|^2_{L^2(0,T;H^1(\Omega))}\\ &+\|\partial_t(p-P_h)\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|f-\hat{f}\|^2_{L^2(0,T;H^{-1}(\Omega))}\\ &+\sum_{i=1}^{N}k_i\sum_{\tau\in T^h}\int_\tau h_\tau^2(\hat{Y}_h-y_d+\partial_tP_h+\Delta\tilde{P}_h-\overline{\hat{Y}_h-y_d+\partial_tP_h+\Delta\tilde{P}_h}\,)^2\\ &+\sum_{i=1}^{N}k_i\sum_{\tau\in T^h}\int_\tau h_\tau^2(\,\hat{f}+U_h-\partial_tY_h+\Delta\hat{Y}_h-\overline{\hat{f}+U_h-\partial_tY_h+\Delta\hat{Y}_h}\,)^2\big\},\end{aligned}$$

where $\bar{v}$ is the integral average value of v on the element τ such that $\bar{v}|_\tau=\frac{\int_\tau v}{\int_\tau 1}$.

4. Numerical Experiments

In this section, we carry out some numerical experiments by using AFEpack software package (see [21]). We show that the derived error estimates developed in Section 3 can be effectively used in adaptive finite element approximation of the control problem.

To solve the optimal control numerically we used the following iterations: (The proof of its convergence can be found in [12]) Consider

$$\min_{u\in K^h} J_h(u), \tag{4.1}$$

where $J_h(u)$ is a convex functional on U^h and K^h is a convex subset of U^h. The iterative scheme reads $(n=1,2,\cdots)$

$$\begin{cases} b(u_{n+\frac{1}{2}},v)=b(u_n,v)-\rho_n(J_h'(u_n),v), \qquad \forall\, v\in U^h,\\ u_{n+1}=P_K^b(u_{n+\frac{1}{2}}),\end{cases} \tag{4.2}$$

where $b(u,\,v)=(u,\,v)$, the projection operator $P^b_K U^h\to K^h$: for given $w\in U^h$ find $P^b_K w\in K^h$ such that ([12])

$$b(P^b_Kw-w,P^b_Kw-w)=\min_{u\in K^h}b(u-w,u-w). \tag{4.3}$$

An application of (4.2) to the discretized control problem (2.10) yields the following algorithm

(4.4)
$$\begin{cases} b(U_{n+\frac{1}{2}},v)=b(U_n,v)-\rho_n\int_0^T(U_n+P_n,v)dt, \quad U_{n+\frac{1}{2}},\ U_n\ \in U^h,\ \forall\ v\in U^h,\\ \displaystyle\int_0^T\big((\frac{\partial Y_n}{\partial t},w)+a(Y_n,w)\big)dt+(Y_n(0)-y_0,w(0))=\int_0^T(f+U_n,w)dt,\ \forall\ w\in W^h,\\ \displaystyle\int_0^T\big(-(\frac{\partial P_n}{\partial t},q)+a(q,P_n)\big)dt+(P_n(T),q(T))=\int_0^T(Y_n-y_d,q)dt, \quad \forall\ q\in W^h,\\ U_{n+1}=P^b_K(U_{n+\frac{1}{2}}),\end{cases}$$

where we have omitted the subscript h. In our examples, the iteration parameter ρ_n is set $\rho_n = 0.5$.

Example 1. The first example is the following control problem on $\Omega \times (0, T] = (0, 1)^2 \times (0, 1]$:

$$\begin{aligned} &\min \frac{1}{2}\int_0^1 \int_\Omega (y - y_d)^2 + \frac{1}{2}\int_\Omega (u - u_0)^2 dt \\ &s.t. \quad \frac{\partial y}{\partial t} - \Delta y = u + f, \quad 0 \le \int_\Omega u \le 1, \end{aligned} \tag{4.5}$$

and the data and solutions are:

$$\begin{aligned} \mu(x) &= \sin \pi x_1 \sin \pi x_2, \\ \nu(t) &= \sin \pi t, \\ p(x,t) &= \mu(x)\nu(t), \\ u_0(x,t) &= \begin{cases} 0.5, & x_1 + x_2 > 1.0, \\ 0.0, & x_1 + x_2 \le 1.0, \end{cases} \\ u(x,t) &= u_0 - p + \max\{\overline{p - u_0} + 0, \min\{\overline{p - u_0} + 1, 0\}\}, \\ y(x,t) &= \mu(x)\nu(t), \\ y_d(x,t) &= y(x,t) + \tfrac{\partial p}{\partial t} + \triangle p, \\ f(x,t) &= \tfrac{\partial y}{\partial t} - \triangle y - u. \end{aligned} \tag{4.6}$$

We compute Example 1 on a uniform mesh and an adaptive mesh, respectively. In this example, the control has a discontinuous line introduced by u_0 so that it has much weaker global regularity than the co-state. The state and co-state are approximated by the piecewise linear elements, while piecewise discontinuous constant elements are used to approximate the control.

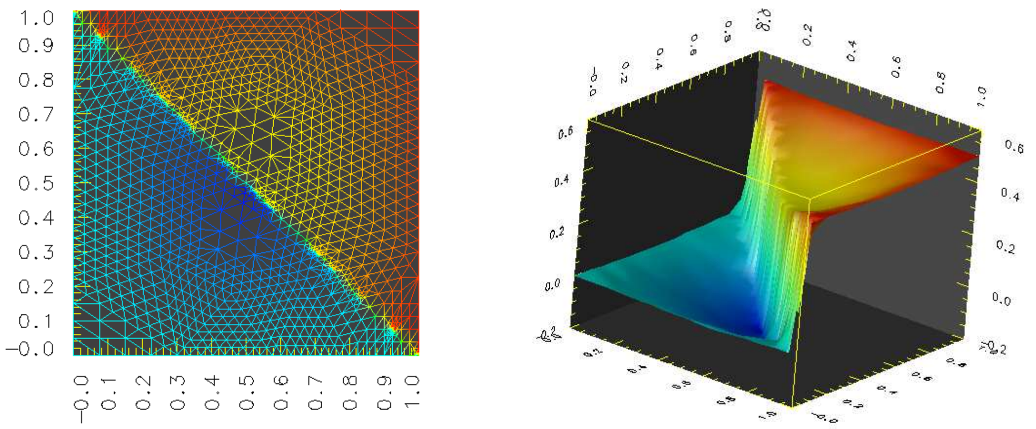
In Table 1, the mesh information is displayed with $L^2(0, T; L^2(\Omega))$ approximation errors for the control and the states. The adaptive time steps are 16 given by the code, and we also use 16 uniform time steps in uniform mesh computation. The summation of the nodes, sides, elements and degree of freedoms(DOFs) for uniform mesh and adaptive mesh from step 1 to step 16 are shown. It can be clearly seen that on adaptive mesh one may use more fewer nodes, sides, elements and DOFs in the state variables. Since the main computational loads in solving the control problem come from repeatedly solving the state and the co-state equations, the adaptive mesh can substantially save much computation.

Table 1: Piecewise discontinuous constant element approximation for the control

		on uniform mesh			on adaptive mesh		
		u	y	p	u	y	p
mesh info	# nodes	10625	10625	10625	7964	1286	1286
	# sides	30192	30192	30192	20784	3135	3135
	# elements	19584	19584	19584	12837	1866	1866
	# DOFs	19584	10625	10625	12837	1286	1286
$L^2(0, T; L^2(\Omega))$ error		2.87e-01	2.51e-02	2.53e-02	2.85e-01	6.07e-02	6.20e-02

And in Figure 1, we can see the adaptive mesh for the control and the approximation value of the control at $t = 1$, respectively. The location of the jump is reflected in the adaptive mesh for the control.

Figure 1: The adaptive mesh for the control and the approximation value of the control at t=1



Example 2. The second example is the following control problem on $\Omega \times (0, T] = (0, 1)^2 \times (0, 1]$:

$$
\begin{aligned}
&\min \frac{1}{2}\int_0^1 \int_\Omega (y - y_d)^2 + \frac{1}{2}\int_\Omega (u - u_0)^2 dt \\
&s.t. \quad \frac{\partial y}{\partial t} - \Delta y = u + f, \quad 0 \le \int_\Omega u \le 1,
\end{aligned}
\tag{4.7}
$$

and the data and solutions are:

$$
\begin{aligned}
\mu(x) &= \sin \pi x_1 \sin \pi x_2, \\
\nu(t) &= \sin \pi t, \\
p(x,t) &= \mu(x)\nu(t), \\
u_0(x,t) &= \begin{cases} 0.5, & x_1 + x_2 > t, \\ 0.0, & x_1 + x_2 \le t, \end{cases} \\
u(x,t) &= u_0 - p + \max\{\overline{p - u_0} + 0, \min\{\overline{p - u_0} + 1, 0\}\}, \\
y(x,t) &= \mu(x)\nu(t), \\
y_d(x,t) &= y(x,t) + \tfrac{\partial p}{\partial t} + \triangle p, \\
f(x,t) &= \tfrac{\partial y}{\partial t} - \triangle y - u.
\end{aligned}
\tag{4.8}
$$

We compute Example 2 on a uniform mesh and an adaptive mesh, respectively. In this example, the control has a discontinuous line which moves with time, so that it also has much weaker global global regularity than the co-state. The state and co-state are approximated by the piecewise linear elements, while piecewise discontinuous constant elements are used to approximate the control.

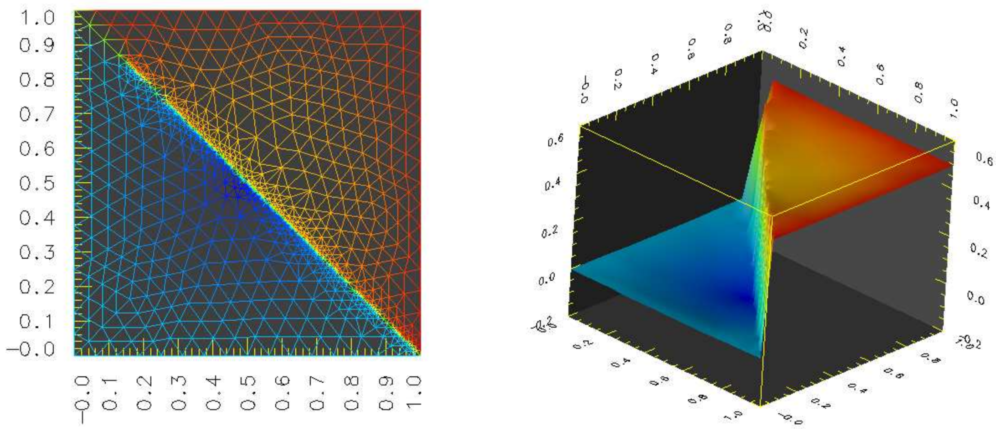
In Table 2, the mesh information is displayed with $L^2(0, T; L^2(\Omega))$ approximation errors for the control and the states. The adaptive time steps are 28 by the code, and we also use 28 uniform time steps in uniform mesh computation. Similarly as Example 1, we can see that the adaptive mesh can substantially save much computation from Table 2.

Table 2: Piecewise discontinuous constant element approximation for the control

		on uniform mesh			on adaptive mesh		
		u	y	p	u	y	p
mesh info	# nodes	30933	30933	30933	20108	1378	1378
	# sides	90048	90048	90048	55798	3303	3303
	# elements	59136	59136	59136	35719	1954	1954
	# DOFs	59136	30933	30933	35719	1378	1378
$L^2(0,T;L^2(\Omega))$ error		2.52e-01	1.95e-02	1.95e-02	2.49e-01	6.86e-02	7.34e-02

And in Figure 2, we can see the adaptive mesh for the control and the approximation value of the control at $t = 1$, respectively. The location of the jump is reflected in the adaptive mesh for the control at $t = 1$.

Figure 2: The adaptive mesh for the control and the approximation value of the control at t=1



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