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AMERICAN PUT OPTIONS ON ZERO-COUPON BONDS AND A PARABOLIC FREE BOUNDARY PROBLEM

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Abstract. In this paper we study American put options on zero-coupon bonds under the CIR model of short interest rates. The uniqueness of the optimal exercise boundary and the solution existence and uniqueness of a degenerate parabolic free boundary problem are established. Numerical examples are also presented to confirm theoretical results.

Key Words. American put option, zero-coupon bond, optimal exercise boundary, free boundary problem, uniqueness, existence

1. Introduction

In this paper, we shall study American put options on zero-coupon bonds. Since bonds and their options are financial derivatives of interest rates, we need term structure models of interest rates to determine the rational prices of these financial products. In those models, the short rate of interest is considered to be a random process governed by a stochastic differential equation. Here we adopt the CIR model developed by Cox, Ingersoll and Ross in 1985 [8]. The prominent feature of this model is that the short interest rate is never negative. Indeed, the stochastic process $r(t)$ of the short interest rate under the CIR model follows the square-root dynamics:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad t > 0, \tag{1}$$

where $W(t)$ is a standard Brownian motion under the risk-neutral measure Q, κ is the speed of adjustment, θ is the long-term value of interest rate, and σ is a positive constant. It can be shown that $r(t)$ is always positive when $\kappa\theta/\sigma^2 \geq 1/2$ and that $r(t)$ can reach zero when $\kappa\theta/\sigma^2 < 1/2$.

Since the American option can be exercised at any time up to its expiration date, there is an optimal exercise boundary. The optimal exercise boundary will divide the whole domain into two regions. It is optimal to exercise the option in one region but the option should be kept in the other region. American option problems can be treated further by optimal stopping problems and by parabolic free boundary value problems. While there have been extensive studies on American stock options, American bond options have not been paid much attention in theoretical analysis. We refer the interested reader to [7], [9], [15], [19]) and references cited therein in this aspect. In this paper we shall show that there is a unique optimal exercise boundary and the corresponding free boundary problem has a unique weak solution.

The outline of the paper is as follows. As in [12] for American stock put options, we use the optimal stopping problem formulation to investigate properties of

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American put options on zero-coupon bonds in Section 2. Especially, we show the existence and uniqueness of the optimal exercise boundary. In section 3, we study the parabolic free boundary problem by variational method. The difficulty is that the partial differential operator is degenerate. The free boundary problem may be investigated by using weighted Sobolev spaces in the light of the formulation for the finite volume methods in [1] and [3]. With appropriate variable transforms, we are able to remove the degenerate factor and then propose a variational formulation with a coercive bilinear form in the usual Sobolev space. The solution uniqueness follows from the coercivity of the bilinear form. The solution existence is established by considering limit of a sequence of solutions for nonlinear variational problems of the parabolic type. In Section 4, numerical results are presented to confirm our theoretical results.

2. American put option and its optimal exercise boundary

Consider the American put option with exercise price \$$K$ and expiry date T, which is written on a zero-coupon bond with face value \$1 (without loss of generality) and maturity date T^* ($> T$). Recall that American contingent claims can be formulated as optimal stopping problems (see [5], [14] and references cited therein). Denote by $p(r,t)$ the price of the put when $r(t) = r$ at time t. Then (see [7])

$$p(r,t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right) g(r(\tau),\tau) \middle| \mathcal{F}_t\right], \tag{2}$$

where $\{\mathcal{F}_t\}$ is the filtration generated by $W(t)$, $\mathcal{T}_{[t,T]}$ is the set of all stopping times assuming values in $[t,T]$, $g(r,t) = (K - B(r,t;T^*))^+$ is the payoff of the put, $z^+ = \max(z,0)$, and $B(r,t;T^*)$ is the bond price given by

$$B(r,t;T^*) = E\left[\exp\left(-\int_t^{T^*} r(s)ds\right) \middle| \mathcal{F}_t\right]. \tag{3}$$

In [8], the explicit expression of $B(r,t;T^*)$ was found as follows:

$$B(r,t;T^*) = A(T^* - t)e^{-C(T^*-t)r}$$

where $A(t)$ is a smooth and strictly decreasing function, $C(t)$ is a smooth and strictly increasing function and $A(0) = 1$, $C(0) = 0$. Therefore, $B(r,t;T^*)$ is an increasing function of t and a decreasing function of r, which is as expected in practice.

It should be pointed out that the exercise price K must be strictly less than $B(0,T;T^*) = A(T^*-T)$ which is the maximum of bond price $B(r,t;T^*)$ on $[0,\infty)\times[0,T]$. Otherwise, the exercise would be never optimal (see [8] for American call options). In fact, if $K \geq B(0,T;T^*)$, then

$$K \geq B(r,t;T^*),\ r \geq 0,\ 0 \leq t \leq T.$$

Hence it follows from (2) and (3) that

$$
\begin{aligned}
&p(r,t)\\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right)\big(K - B(r(\tau),\tau,T^*)\big)\middle|\mathcal{F}_t\right]\\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right)\left(K - E\left[\exp\left(-\int_\tau^{T^*} r(s)ds\right)\middle|\mathcal{F}_\tau\right]\right)\middle|\mathcal{F}_t\right]\\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} \left(E\left[K\exp\left(-\int_t^\tau r(s)ds\right)\middle|\mathcal{F}_t\right] - B(r,t;T^*)\right)\\
&= K - B(r,t;T^*),\ r\ge 0,\ 0\le t\le T.
\end{aligned}
$$

From now on, we assume that $K < B(0,T;T^*)$.

By taking $\tau = t$ in (2), we get the usual constraint condition for the put price:

$$p(r,t) \ge g(r,t), \quad r\ge 0,\ 0\le t\le T. \tag{4}$$

In the following theorem, we give the other bounds of put price $p(r,t)$.

Theorem 1. *For all* $(r,t)\in[0,\infty)\times[0,T]$*, we have*

$$P_E(r,t) \le p(r,t) \le C_E(r,t) + K - B(r,t;T^*), \tag{5}$$

where $P_E(r,t)$ *and* $C_E(r,t)$ *are the prices of the European put and call options, respectively.*

Proof. For $\tau = T$ in (2), we have

$$p(r,t) \ge E\left[\exp\left(-\int_t^T r(s)ds\right) g(r(T),T)\middle|\mathcal{F}_t\right] \equiv P_{E}(r,t).$$

By using properties of conditional expectation, we get

$$
\begin{aligned}
&p(r,t) + B(r,t;T^*)\\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right)\big(g(r(\tau),\tau) + B(r(\tau),\tau)\big)\middle|\mathcal{F}_t\right]\\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right)\big((B(r(\tau),\tau) - K)^+ + K\big)\middle|\mathcal{F}_t\right]\\
&\le \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right)(B(r(\tau),\tau) - K)^+\middle|\mathcal{F}_t\right] + K\\
&\equiv c_E(r,t) + K,
\end{aligned}
$$

where $c_E(r,t)$ is the price of the American call option which is the same as American call price $C_E(r,t)$ (see [8]). To sum up, we complete the proof of (5). □

Remark 1. *Since* $P_E(r,t)$ *is positive on* $(0,\infty)\times[0,T)$*, it follows from the first inequality of (5) that American put price* $p(r,t)$ *is also positive on* $(0,\infty)\times[0,T)$.

It is easy to show the following put-call parity (see [7])

$$P_E(r,t) = C_E(r,t) + KB(r,t;T) - B(r,t;T^*).$$

By combing this identity with (5), we get

$$P_E(r,t) \le p(r,t) \le P_E(r,t) + K(1 - B(r,t;T)).$$

The above estimates suggest that there is a function $\phi(r,t)$ such that

$$p(r,t) - P_E(r,t) = K\phi(r,t), \quad 0\le \phi(r,t) \le 1 - B(r,t;T).$$

The difference $p(r,t) - P_E(r,t)$ is usually called the early exercise premium. This observation might be useful to derive some analytical approximation of $p(r,t)$. The interested reader is referred to [4] and [13] in this topic about American options on stocks.

We need the following results to show that there is a unique optimal exercise boundary.

Lemma 1. *(i) Put price $p(r,t)$ is a decreasing functions of t. (ii) If $a/\sigma^2 > 1/2$, then $P(r,t) = p(r,t) + B(r,t;T^*)$ is a decreasing function of r.*

Proof. (i) It is not difficulty to check that

$$p(r,t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T-t]}} E\left[\exp\left(-\int_0^\tau r(s)ds\right) g(r(\tau), t+\tau)\right],$$

where $r(s)$ is the solution of (1) with $r(0) = r$. Notice that for $0 \le t_1 < t_2 \le T$, if $\tau \in \mathcal{T}_{[0,T-t_2]}$ then $\tau \in \mathcal{T}_{[0,T-t_1]}$. Hence, we have

$$\begin{aligned} p(r,t_2) &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T-t_2]}} E\left[\exp\left(-\int_0^\tau r(s)ds\right) g(r(\tau), t_2+\tau)\right] \\ &\le \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T-t_1]}} E\left[\exp\left(-\int_0^\tau r(s)ds\right) g(r(\tau), t_2+\tau)\right] \\ &\le \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T-t_1]}} E\left[\exp\left(-\int_0^\tau r(s)ds\right) g(r(\tau), t_1+\tau)\right] \\ &= p(r,t_1), \end{aligned}$$

where we used the fact that $g(r,t)$ is a decreasing function of t. Conclusion (i) is true.

Since

$$P(r,t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds\right) G(r(\tau),\tau)\middle| \mathcal{F}_t\right],$$

where $G(r,t) = g(r,t) + B(r,t;T^*)$ is a decreasing function of r, we only need to prove the process $r(s)$ is an increasing function of $r(t) = r$ in order to verify conclusion (ii). For $r_2 > r_1 \ge 0$, let $r_i(s)$ be the solution of (1) with $r(t) = r_i (i = 1,2)$. The process $X(s) = r_2(s) - r_1(s)$ satisfies formally the long-normal dynamics:

$$dX(s) = X(s)\left(-bdt + \upsilon(s)dW(s)\right), \quad s \in (t,T],$$

where

$$\upsilon(s) = \frac{\sigma}{\sqrt{r_1(s)} + \sqrt{r_2(s)}}.$$

Hence, if Itô integral $\int_t^T \upsilon(s)dW(s)$ is well-defined, then we have by Itô's formula (Theorem 4.1.2 of [18])

$$X(s) = X(t)e^{\Theta(s,t)} > 0, \quad s \in [t,T],$$

where

$$\Theta(s,t) = -b(s-t) - \frac{1}{2}\int_t^s \upsilon^2(s)ds + \int_t^s \upsilon(s)dW(s).$$

Therefore, $r(s)$ is an increasing function of $r(t) = r$.

Recall that the probability density of the interest rate at time s, conditional on its value at time t, is given by (see [8])

$$f(r(s),s;r(t),t) = \phi e^{-(u+v)}\left(\frac{v}{u}\right)^{q/2} I_q\left(2\sqrt{uv}\right),$$

where

$$\phi = \frac{2b}{\sigma^2(1 - e^{-b(s-t)})}, \quad u = \phi r(t) e^{-b(s-t)}, \quad v = \phi r(s),$$

$$q = \frac{2a}{\sigma^2} - 1, \quad a = \kappa\theta, \quad b = \kappa,$$

and $I_q(z)$ is the modified Bessel function of the first kind of order q (see Chapter of [20]).

$$I_q(z) = \left(\frac{z}{2}\right)^q \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(q+n+1)}$$

Then for $a/\sigma^2 > 1/2$, i.e., $q > 0$, we have

$$\begin{aligned}
&E\left[\int_t^T \left(\frac{\sigma}{\sqrt{r_1(s)} + \sqrt{r_2(s)}}\right)^2 ds\right] \le \sigma^2 \int_t^T E\left[\frac{1}{r_2(s)}\right] ds \\
&= \sigma^2 \int_t^T \int_0^\infty \frac{1}{r_2(s)} f(r_2(s), s; r_2(t), t) dr_2(s) ds \\
&= 2 \int_{\frac{2br_2}{\sigma^2\left(e^{b(T-t)}-1\right)}}^\infty \int_0^\infty e^{-(u+v)} \sum_{n=0}^\infty \frac{u^{n-1} v^{n+q-1}}{n!\Gamma(n+q+1)} dv du \\
&= 2 \int_{\frac{2br_2}{\sigma^2\left(e^{b(T-t)}-1\right)}}^\infty e^{-u} \sum_{n=0}^\infty \frac{\Gamma(n+q) u^{n-1}}{n!\Gamma(n+q+1)} du \\
&\le \frac{2}{q} \int_{\frac{2br_2}{\sigma^2\left(e^{b(T-t)}-1\right)}}^\infty \frac{e^{-u}}{u} du + 2 \int_0^\infty e^{-u} \sum_{n=1}^\infty \frac{u^{n-1}}{n!(n+q)} du \\
&= \frac{2}{q} \int_{\frac{2br_2}{\sigma^2\left(e^{b(T-t)}-1\right)}}^\infty \frac{e^{-u}}{u} du + 2 \sum_{n=1}^\infty \frac{1}{n(n+q)} \\
&< \infty.
\end{aligned}$$

The above estimates implies that the Itô integral $\int_t^T \upsilon(s) dW(s)$ is well-defined (see [18]). The proof of conclusion (ii) is completed. □

Remark 2. *It is an open question to show that the Itô integral $\int_t^T \upsilon(s) dW(s)$ is well-defined when $a/\sigma^2 \le 1/2$. But numerical results suggest that $p(r,t) + B(r,t;T^*)$ would also be a decreasing function of r in this case.*

For $t \in [0, T]$, let $\tilde{r}(t)$ be the solution to $B(r, t; T^*) = K$, i.e.,

$$\tilde{r}(t) = \frac{\log(A(T^* - t)/K)}{C(T^* - t)}.$$

Then $\tilde{r}(t)$ is an increasing function and

$$B(r, t; T^*) > K, \quad r < \tilde{r}(t); \qquad B(r, t; T^*) < K, \quad r > \tilde{r}(t).$$

As in the case for American call options ((see [8]), $\tilde{r}(T)$ is the smallest interest rate at which the put should be exercised at its expiration date. For $t \in [0, T)$, define

$$r^*(t) = \inf\{r : \ p(r,t) = g(r,t)\} = \inf\{r : \ P(r,t) = G(r,t)\}.$$

It is apparent that $r^*(t)$ is the smallest value of the interest rate at which the exercise of the put becomes optimal at time t. We shall call $r^*(t)$ the early exercise interest rate which is the optimal exercise boundary mentioned in the introduction. We have the following result for $r^*(t)$.

Theorem 2. *If $a/\sigma^2 > 1/2$, then for each $t \in [0,T)$, we have*

$$p(r,t) > g(r,t), \quad 0 < r < r^*(t), \quad p(r,t) = g(r,t); \quad r \geq r^*(t), \tag{6}$$

which means that there is a unique optimal exercise boundary.

Proof. We first claim that $r^*(t) > \tilde{r}(t)$ for all $t \in [0,T)$. Suppose that $r^*(t) \leq \tilde{r}(t)$ for some $t \in [0,T)$. Then $p(r^*(t),t) = g(r^*(t),t) = 0$, which contradicts that the put price is positive (see Remark 1). By the definition of $r^*(t)$ and (4), we have that $P(r^*(t),t) = G(r^*(t),t) = K$ and $P(r,t) \geq G(r,t) = K$ for $r > r^*(t)$. Thus, by conclusion (ii) of Lemma 1, we have

$$P(r,t) > G(r,t), \quad 0 < r < r^*(t); \quad P(r,t) = G(r,t) = K, \quad r \geq r^*(t), \tag{7}$$

which implies (6). □

Remark 3. *By viewing Remark 2, we may assume that $p(r,t) + B(r,t;T^*)$ is a decreasing function of r when $a/\sigma^2 \leq 1/2$. Under such assumption, (7) hold for $a/\sigma^2 \leq 1/2$.*

3. Solution existence and uniqueness of the free boundary problem

It is well-known that put price $p(r,t)$ and optimal exercise boundary $r^*(t)$ are the solution of the following free boundary problem (see [7] and references cited therein):

$$p_t + Lp = 0, \quad p(r,t) > g(r,t), \quad 0 \leq r < r^*(t), \ 0 \leq t < T, \tag{8}$$

$$p(r^*(t),t) = g(r^*(t),t), \quad p_r(r^*(t),t) = g_r(r^*(t),t), \quad 0 \leq t < T, \tag{9}$$

$$p(r,t) = g(r,t), \quad r > r^*(t), \ 0 \leq t \leq T, \tag{10}$$

$$p(r,T) = g(r,T), \quad r \geq 0, \tag{11}$$

where

$$Lp = \frac{1}{2}\sigma^2 r p_{rr} + (a - br)p_r - rp.$$

In this section, we shall prove this free boundary problem has a unique weak solution.

Remark 4. *Let $D = \{(r,t) : \ 0 \leq r \leq r^*(t), \ 0 \leq t \leq T\}$. Suppose that p_t, $p_r(r,t)$, p_{rt}, p_{rr}, and p_{rrr} are continuous on D. It follows from the maximum principle (Theorem 2.1 of [10]) that $p_r \geq 0$ on D. Thus $p(r,t)$ is an increasing function of r on D. Notice that $p(r,t) = g(r,t)$ is an increasing function of r outside $[0,\infty) \times [0,T] \backslash D$. So the put price $p(r,t)$ is an increasing function of r.*

Let R be an upper bound of $r^*(t)$ and

$$\tilde{p}(r,t) = p(r,T-t), \quad \tilde{g}(r,t) = g(r,T-t).$$

Then we can rewrite free boundary problem (8)–(11) into the following linear complementarity problem:

$$\tilde{p}_t - L\tilde{p} \geq 0, \ p \geq \tilde{g}, \ 0 < r < R, \ 0 < t \leq T, \tag{12}$$

$$(\tilde{p}_t - L\tilde{p})(\tilde{p} - \tilde{g}) = 0, \ 0 < r < R, \ 0 < t \leq T, \tag{13}$$

$$\tilde{p}(R,t) = \tilde{g}(R,t), \ 0 \leq t \leq T, \tag{14}$$

$$\tilde{p}(r,0) = \tilde{g}(r,0), \ 0 \leq r \leq R. \tag{15}$$

In order to remove that degenerate factor r in the second order derivative term of L, we consider the following transforms:

$$x = \sqrt{r}, \quad \tilde{p}(r,t) = x^{-\alpha} e^{\gamma t} u(x,t), \tag{16}$$

where α and γ are positive constants to be determined to have a coercive bilinear form. We have by calculation

$$\tilde{p}_t - L\tilde{p} = x^{-\alpha} e^{\gamma t} \left(u_t - \frac{\sigma^2}{8} u_{xx} + c_1(x) u_x + c_2(x) u \right),$$

where

$$c_1(x) = \xi_1 x^{-1} + \xi_2 x, \quad c_2(x) = \xi_3 x^{-2} + x^2 + \gamma - \frac{\alpha b}{2},$$
$$\xi_1 = \frac{\sigma^2}{8}\left(1 + 2\alpha - \frac{4a}{\sigma^2}\right), \quad \xi_2 = \frac{b}{2}, \quad \xi_3 = \frac{\sigma^2 \alpha}{8}\left(\frac{4a}{\sigma^2} - \alpha - 2\right),$$

Hence (12)–(15) become

$$u_t - \frac{\sigma^2}{8} u_{xx} + c_1(x) u_x + c_2(x) u \ge 0, \quad u \ge F(x,t), \quad (x,t) \in \Omega_T, \tag{17}$$

$$\left(u_t - \frac{\sigma^2}{8} u_{xx} + c_1(x) u_x + c_2(x) u \right)(u - F(x,t)) = 0, \quad (x,t) \in \Omega_T, \tag{18}$$

$$u(0,t) = F(0,t), \quad 0 \le t \le T, \tag{19}$$

$$u(X,t) = F(X,t), \quad 0 \le t \le T, \tag{20}$$

$$u(x,0) = F(x,0), \quad 0 \le x \le X, \tag{21}$$

where $\Omega_T = \Omega \times (0,T)$, $\Omega = (0,X)$, $X = \sqrt{R}$, and $F(x,t) = x^{\alpha} e^{-\gamma t} \tilde{g}(x^2, t)$.

For $\Omega = (0,X)$, denote by $H_0^1(\Omega)$ the closure of all smooth functions with compact support in Ω in the usual Sobolev space $H^1(\Omega)$ ([17]). The norm of $v \in H^1(\Omega)$ is denoted by $\|v\|_{1,\Omega}$. We also use $(\cdot,\cdot)$ and $\|\cdot\|_{0,\Omega}$ to denote the inner product and the norm on $L^2(\Omega)$, respectively. Define the bilinear form

$$a(\phi,\psi) = \frac{\sigma^2}{8}(\phi_x, \psi_x) + (c_1 \phi_x + c_2 \phi, \psi).$$

It can be checked by integration by parts that (see [2])

$$\|x^{-1}\phi\|_{0,\Omega} \le 2\|\phi_x\|_{0,\Omega}, \quad \forall \phi \in H_0^1(\Omega). \tag{22}$$

So we have

$$a(\phi,\psi) \le \gamma_1 \|\phi\|_{1,\Omega} \|\psi\|_{1,\Omega}, \quad \forall \phi \in H_0^1(\Omega),$$

where γ_1 is a positive constant, that is, $a(\cdot,\cdot)$ is bounded.

Since it is always easier to deal with homogeneous boundary conditions, letting $w = u - F$, we have the following variational problem for (17)–(21): Find $w \in W(0,T)$ with $w(0) = 0$ such that for a.e. $t \in (0,T]$, $w(t) \in \Pi$ and

$$(w_t, v - w) + a(w, v - w) \ge f(t, v - w), \quad \forall v \in \Pi \tag{23}$$

where

$$W(0,T) = \{v : \ v \in L^2(0,T; H_0^1(\Omega)), \ v_t \in L^2(0,T; H^{-1}(\Omega))\},$$
$$\Pi = \{v : \ v \in H_0^1(\Omega), \ v \ge 0\}, \quad f(t,v) = -(F_t, v) - a(F, v).$$

Since $F_t \in L^\infty(\Omega \times (0,T))$, in order to have $f(t,v)$ well-defined, we should require that $F \in H_E^1(\Omega) = \{\phi \in H^1(\Omega) : \phi(0) = 0\}$ for all $t \in [0,T]$, which is true if $\alpha > 1/2$ or $K \le B(0,0;T^*) = A(T^*)$. Under such conditions, we have

$$f(t,v) = -(\phi, v) + (\psi \delta(x - \tilde{x}(t)), v), \tag{24}$$

where $\tilde{x}(t) = \sqrt{\tilde{r}(T-t)}$, and

$$\phi(x,t) = Ke^{-\gamma t}x^{2+\alpha}H(x-\tilde{x}(t)), \quad \psi(x,t) = \frac{\sigma^2}{4}KC(T-t)e^{-\gamma t}x^{1+\alpha}.$$

Expression (24) of $f(t,v)$ means that $f(t,v)$ is well-defined regardless of any restrictions on α and K.

For a solution $w(x,t)$ of the variational problem (23), we shall call

$$p(r,t) = r^{-\frac{\alpha}{2}}e^{\gamma t}w(\sqrt{r}, T-t) + g(r,t)$$

a weak solution of the free boundary problem (8)–(11). Next, we shall show that the variational problem (23) has a unique solution. To this end, we specify α and γ as follows:

$$\gamma = \frac{(1+2\alpha)b}{4}$$

and $\alpha \in (0,1)$ is chosen in the following way:

(1) $\alpha \in (0, 1/2)$, if $0 < \rho < 3/8$,
(2) $\alpha \in (4\rho - 3/2,\ 1/2)$, if $3/8 \le \rho < 1/2$,
(3) $\alpha \in (1/2,\ 4\rho - 3/2)$, if $1/2 < \rho \le 3/4$,
(4) $\alpha \in (1/2,\ 2\rho - 1/2 - \sqrt{(4\rho-1)(4\rho-3)}/2)$, if $\rho > 3/4$.

where $\rho = a/\sigma^2$.

Lemma 2. *For α and γ determined above, if $a/\sigma^2 \ne 1/2$, then the bilinear form $a(\cdot,\cdot)$ is coercive on $H_0^1(\Omega)$, i.e., there is a positive constant γ_0 such that*

$$a(v,v) \ge \gamma_0 \|v_x\|_{0,\Omega}^2, \quad \forall v \in H_0^1(\Omega). \tag{25}$$

Proof. With $\gamma = (1+2\alpha)b/4$, for $v \in H_0^1(\Omega)$, integration by parts gives

$$a(v,v) = \frac{\sigma^2}{8}\left(\|v_x\|_{0,\Omega}^2 + (-\nu + 2\alpha\nu - \alpha^2)\|x^{-1}v\|_{0,\Omega}^2\right) + \|xv\|_{0,\Omega}^2,$$

where $\nu = 2\rho - 1/2$. If α is chosen such that

$$-\nu + 2\alpha\nu - \alpha^2 \le 0, \quad 1 + 4(-\nu + 2\alpha\nu - \alpha^2) > 0,$$

then it follows from (22) that (25) holds with $\gamma_0 = 1 + 4(-\nu + 2\alpha\nu - \alpha^2)$. The proof is completed by solving the above system of inequalities. □

For any $\epsilon \in (0,1)$, let $\phi^\epsilon(x) = x^{\epsilon+1/2}(X - x)$. The we have $v^\epsilon \in H_0^1(\Omega)$ and

$$\lim_{\epsilon \to 0^+} \frac{a(\phi^\epsilon, \phi^\epsilon)}{\|\phi^\epsilon\|_{1,\Omega}^2} = -\frac{\sigma^2}{8}(2\alpha - 1)^2$$

when $a/\sigma^2 = 1/2$. Hence, the bilinear form $a(\cdot,\cdot)$ is not coercive for any α. However, for $\alpha = 1/2$, we still have

$$a(v,v) \ge \|xv\|_{0,\Omega}^2, \quad \forall v \in H_0^1(\Omega).$$

Thus the solution uniqueness of (23) follows (see the proof of the following Theorem 1), but the existence of the solution is an open question.

For $a/\sigma^2 > 1/4$, we may consider the following variable transforms:

$$x = \sqrt{r}, \quad p(r,t) = x^{-\alpha}e^{\beta x^2 + \gamma t}u(x,t)$$

with

$$\alpha = \frac{2a}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{b}{\sigma^2}, \quad \gamma = \frac{ab}{\sigma^2}.$$

Then we have a variational inequality problem with a symmetric bilinear form, which is also coercive if $a/\sigma^2 \ne 1/2$. Although α may be larger than 1, this formulation also gives very good approximations of option prices.

Theorem 3. *For $a/\sigma^2 \neq 1/2$ and α and γ are chosen as in Lemma 2, (23) has a unique solution $w(x,t)$ in $W(0,T)$.*

Proof. The uniqueness of the solution $w(x,t)$ follows from the coercivity (25) of bilinear form $a(\cdot,\cdot)$. In the following, we prove the solution existence.

For a positive integer n, define

$$H_n(z) = \begin{cases} 0, & x \le 0, \\ nx, & 0 < x < \frac{1}{n}, \\ 1, & x \ge \frac{1}{n}. \end{cases}$$

By using the usual approach to quasi-linear parabolic equations (see [16]), we can show that the following variational problem has a unique solution $w^n \in W(0,T)$ with $w^n(0) = 0$:

$$(w_t^n, v) + a(w^n, v) + (\phi H_n(w^n), v) = \psi(t)v(\tilde{x}(t)), \quad \forall v \in H_0^1(\Omega). \tag{26}$$

Furthermore, $\{w^n\}$ is a bounded function sequence in $W(0,T)$. Let

$$p = \begin{cases} 0, & w^n \ge 0, \\ v, & w^n < 0. \end{cases}$$

Then $p \in H_0^1(\Omega)$ and

$$w_t^n p = p_t p, \quad w_x^n p_x = (p_x)^2, \quad w_x^n p = p_x p, \quad w^n p = p^2, \quad H_n(w^n)p = 0.$$

Hence, for $v = p$ in 26, we get

$$(p_t, p) + a(p,p) = \psi(t)p(\tilde{x}(t)) \le 0.$$

Thus, $p = 0$, that is, w^n is nonnegative. So w^n is a bounded non-negative function sequence in $W(0,T)$.

Recall that a bounded sequence in a Hilbert space has a weakly convergent subsequence and a bounded sequence in the dual space of a separable Banach space has a weakly star convergent subsequence. We may assume that $\{w^n\}$ weakly converges to a nonnegative function w in $W(0,T)$ and that $\{H_n(w^n)\}$ weakly star converges to G in $L^\infty((\Omega)\times(0,T))(= (L^1((\Omega\times(0,T)))')$. It is clear that $0 \le G \le 1$. It follows from Lemma 5.1 of [2] that

$$G(x,t) = 1, \quad \text{if } w(x,t) > 0. \tag{27}$$

Then by letting $n \to \infty$ in (26), we have

$$(w_t, v) + a(w,v) + (\phi G, v) = \psi(t)(\delta(x - \tilde{x}(t)), v), \quad \forall v \in H_0^1(\Omega),$$

i.e., for $v \in H_0^1(\Omega)$,

$$(w_t, v - w) + a(w, v - w) + (\phi G, v - w) = \psi(t)(\delta(x - \tilde{x}(t)), v - w).$$

Notice that ϕ is a nonnegative function and that from (27)

$$G(v - w) \le v - w, \quad \forall v \in \Pi.$$

We have

$$(w_t, v - w) + a(w, v - w) + (\phi, v - w) \ge \psi(t)(\delta(x - \tilde{x}(t)), v - w), \quad \forall v \in \Pi.$$

Therefore, (23) has a solution w in $W(0,T)$. □

The above proof is motivated by [2] in which the American option problem is formulated into a quasi-linear parabolic problem. We may have a similar problem: Find $w \in W(0,T)$ with $w(0) = 0$ such that

$$(w_t, v) + a(w,v) + (\phi H(w), v) = \psi(t)(\delta(x - \tilde{x}(t)), v), \quad \forall v \in \Pi.$$

It is easy to see that the solution of this problem is also the solution of the variational inequality problem (23). One could show that the above problem has a unique solution in $W(0,T)$ by verifying $G = H(w)$ in the above proof. It should be pointed out that Theorem 3 may also be proved by using the approaches in [6] and [11] for general parabolic variational inequalities.

4. Examples

In this section, we shall confirm the theoretical results numerically. The finite element method for variational problem (23) developed in [1] will be employed in computation. We brief the Crank–Nicholson scheme and its convergence for completeness in the following. The convergence analysis for the general finite element schemes and the related numerical examples can be found in [1].

Let $h = X/N$ and $x_i = ih$ for $i = 0, 1, \ldots, N$, where N is a positive integer. Denote by V_h the piecewise linear finite element space in $H_0^1(\Omega)$ under the above partition of interval $[0, X]$. Let $\Pi_h = \{v \in V_h : v \geq 0\}$. For another positive integer M, let $\tau = T/M$ and $t_m = m\tau$ for $m = -1, 0, 1, \ldots, M$. Then the finite element approximations to (23) is: Find $w_h^m \in \Pi_h$ for $m = 1, 2, \ldots, M$, such that

$$(\delta_\tau w_h^m, v - w_h^m) + a\left(w_h^{m-\frac{1}{2}}, v - w_h^m\right) \geq f\left(t_{m-\frac{1}{2}}, v - w_h^m\right) \tag{28}$$

for all $v \in \Pi_h$, where $w_h^0 = 0$, and

$$\delta_\tau w_h^m = \frac{w_h^m - w_h^{m-1}}{\tau}, \quad t_{m-\frac{1}{2}} = t_m - \frac{1}{2}\tau, \quad w_h^{m-\frac{1}{2}} = \frac{w_h^m + w_h^{m-1}}{2}.$$

Here the time derivative is discretized by the Crank–Nicholson scheme. Note that the discrete problem (28) has a positive definite stiffness matrix as the consequence of Lemma 2 if $\rho \neq 1/2$ or $\rho = \alpha = 1/2$. Concerning the stability and convergence of (28), we have the following results according to the general framework for parabolic variational inequalities in §6.3.3 of [11].

Theorem 4. *For $\rho \neq 1/2$, the finite element method (28) is stable when $\frac{\tau}{h^2} < \frac{\gamma_0}{8\gamma_1^2}$ and approximate solution $w_{h\tau}(x,t) = \sum_{m=0}^{M} w_h^m(x)\chi_{[t_{m-1},t_m]}(t)$ has a convergent subsequence in $L^2(0,T; H_0^1(\Omega))$ as $h \to 0$, $\tau \to 0$, $\frac{\tau}{h^2} \to 0$, where $\chi_{[t_{m-1},t_m]}$ is the characteristic function of interval $[t_{m-1}, t_m]$.*

In the following examples, we consider 1-year put options on a 5-year zero-coupon bond with face value \$100. The exercise price K of the option is \$70. For $\theta = 0.08$ and $\kappa = 0.4$, we take $\sigma = 0.5$ (Case I) and $\sigma = 0.1$ (Case II). Then we have two special situations: $\rho = \kappa\theta/\sigma^2 = 0.128 < 0.5$ for Case I and $\rho = 3.2 > 0.5$ for Case II. The upper bound of the early exercise interest rates is chosen as $R = 0.5$, $M = 2000$, and $N = 1000$.

In Figure 1 – Figure 4, we display graphs of put price $p(r,t)$ and put price plus bond price $P(r,t)$. As expected from the results in Section 2, we observed that $p(r,t)$ is an increasing function of r and a decreasing function of t and that $P(r,t)$ is a decreasing function of r. But the left picture of Figure 3 shows that $P(r,t)$ is not a monotone function of t. Figure 5 shows that early exercise prices are concave downward functions but not a monotone function. We observed that the early exercise prices in Case I and Case II behave quite different. More numerical experiments have shown that as a/σ^2 increases, the shape of the early exercise prices changes from the shape as in the left picture of Figure 5 to the shape as in the right picture of Figure 5. It seems that $r^*(t)$ is only a concave downward function. As expected, we also observed that $r^*(t) > \tilde{r}(t)$ for $t \in [0, T)$.

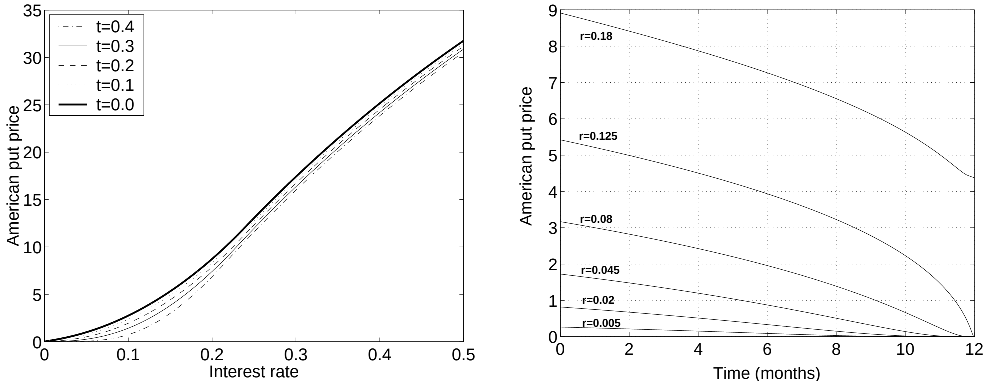

FIGURE 1. American put prices: $\sigma = 0.5$

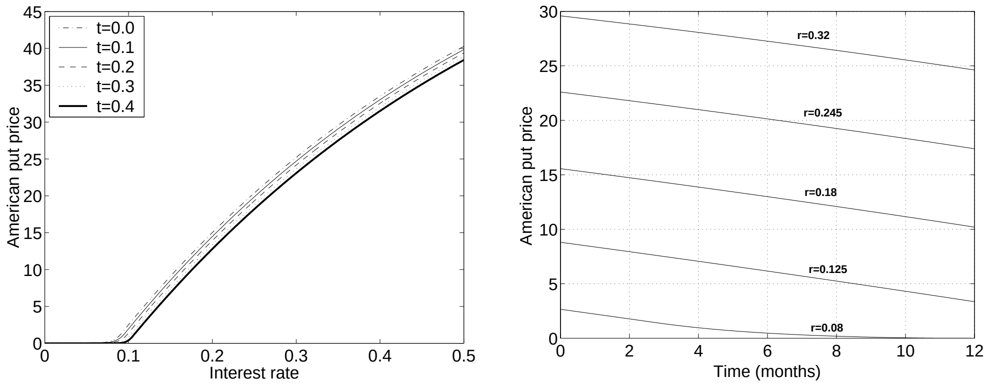

FIGURE 2. American put prices: $\sigma = 0.1$

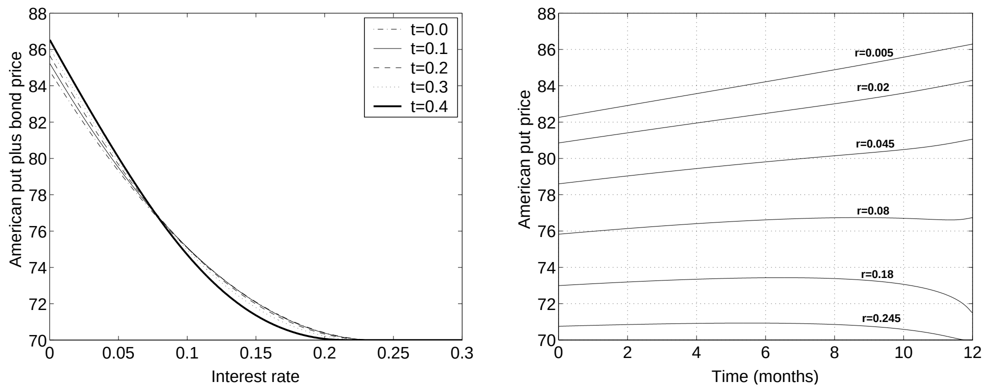

FIGURE 3. American put plus bond prices: $\sigma = 0.5$

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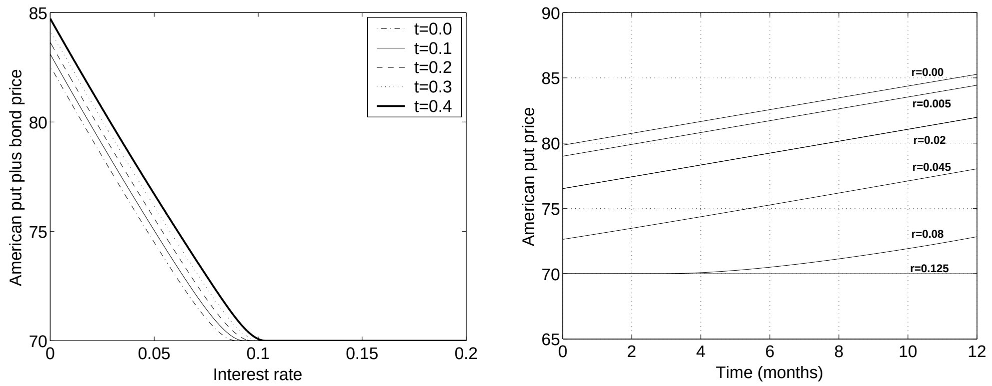

FIGURE 4. American put plus bond prices: $\sigma = 0.1$

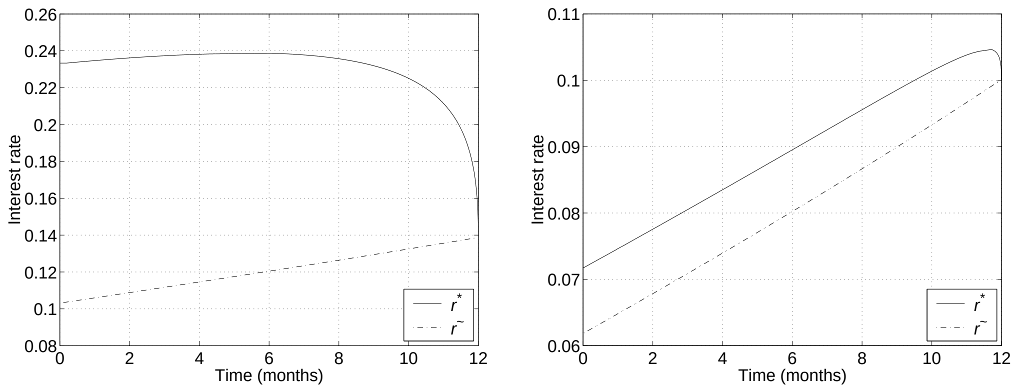

FIGURE 5. Early exercise prices for $\sigma = 0.5$ and $\sigma = 0.1$

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