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CONVERGENCE AND STABILITY OF EXPLICIT/IMPLICIT SCHEMES FOR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. In this paper an explicit/implicit schemes for parabolic equations with discontinuous coefficients is analyzed. We show that the error of the solution in L^∞ norm and the error of the discrete flux in L^2 norm are in order $O(\tau + h^2)$ and $O(\tau + h^{\frac{3}{2}})$, respectively and the scheme is stable under some weaker conditions, while the difference scheme has the truncation error $O(1)$ at the neighboring points of the discontinuity of the coefficient. Numerical experiments, which are given for both linear and nonlinear problems, show that our theoretical estimates are optimal in some sense. The comparison with some classical scheme is presented.

Key Words. Domain decomposition, parabolic equations, discontinuous coefficient, parallel difference schemes, convergence.

1. Introduction

Multi–material systems are considered in many physical applications, e.g., the heat conduction procedure. When there are several materials in contact with each other, the conductivity coefficient can be varying, and discontinuous on the interface of the contact. Sometimes the conductivity coefficients differ in quantity order very much from one another.

Consider the initial–boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad 0 < t \le T, \tag{1}$$

$$u(0,t) = \beta_1(t), \quad u(1,t) = \beta_2(t), \quad 0 < t \le T, \tag{2}$$

$$u(x,0) = \alpha(x), \quad 0 < x < 1, \tag{3}$$

where the positive function $a(x)$ is the conductive coefficient, and $f(x,t)$ is the source term. We suppose that the functions are piecewise–smooth with discontinuity of first kind at $x = \xi$, where $\xi \in (0,1)$ be a fixed point. Denote $\Omega^- = \{0 \le x \le \xi, 0 \le t \le T\}$, $\Omega^+ = \{\xi \le x \le 1, 0 \le t \le T\}$ and $l = \Omega^- \cap \Omega^+$. The value of a function at $x = \xi$ is denoted by subscript ξ, and the left and right limit at ξ are denoted by subscript L and R respectively. For example, define

$$u_\xi(t) = u(\xi,t), \quad u_L(t) = \lim_{x\to\xi^-} u(x,t), \quad u_R(t) = \lim_{x\to\xi^+} u(x,t).$$

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Assume the following conditions hold.

(I) There are positive constants $\sigma, \sigma_L, \sigma_R$ and C such that

$$a(x) \geq \sigma, \ \ \forall x \in [0,1],$$
$$\sup_{0\leq x\leq \xi} a(x) = \sigma_L, \quad \sup_{\xi\leq x\leq 1} a(x) = \sigma_R, \quad \sup_{(x,t)\in\Omega^-\cup\Omega^+} |f(x,t)| \leq C.$$

(II) $\alpha(x)$, $a(x)$ and $f(x,t)$ are smooth on Ω^- and Ω^+ respectively, but have discontinuity of the first kind on l. And there holds $a_L\frac{\partial \alpha_L}{\partial x} = a_R\frac{\partial \alpha_R}{\partial x}$.

(III) Let $\beta_1(t)$ and $\beta_2(t)$ be smooth functions for $t \in [0,T]$, and the consistent conditions hold, e.g., $\alpha(0) = \beta_1(0)$, $\alpha(1) = \beta_2(0)$.

Then, it is well known (*e.g.*, see [7,8]) that (1)–(3) has an unique weak solution $u = u(x,t)$, which is smooth on Ω^- and Ω^+ respectively, and satisfy the joint condition $u_\xi = u_L = u_R$ and $K_\xi = K_L = K_R$, where K is the flux defined by $K = K(x,t) = a(x)\frac{\partial u}{\partial x}$.

There has been numerous work on numerical solution of the initial-boundary problem (1)–(3). The difficulty lies on the discontinuity of material coefficient. It has been proved in [11,12] that truncation errors for many finite difference schemes are the order $O(1)$ for such discontinuous problems. Samarskii [12,13] studied the classical θ-scheme

$$\partial_t U_j^n = \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}} U_j^{n+\theta}), \ \ j = 1,\cdots,J-1,$$

where $U_j^{n+\theta} = \theta U_j^{n+1} + (1-\theta)U_j^n$. By an energy method, he proved for $\frac{1}{2} \leq \theta \leq 1$ that

$$\|U^n - u^n\|_\infty \leq \begin{cases} C(\tau + h), & \text{if } \theta = 1, \\ C(\tau + h^{\frac{1}{2}}), & \text{if } \frac{1}{2} < \theta < 1, \\ C(\tau^2 + h^{\frac{1}{2}}), & \text{if } \theta = \frac{1}{2}. \end{cases} .$$

For $\frac{1}{2} > \theta > 0$, no similar convergence result for the θ-scheme was obtained. To attain a higher rate of convergence, a modified scheme was proposed in [12,13] by using the harmonic mean over intervals of length h, $a_h(x) = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} a(x+sh)^{-1}ds\right)^{-1}$, and replacing a by a_h in the θ–scheme to obtain

$$\partial_t U_j^n = \partial_x(a_{h,j-\frac{1}{2}}\partial_{\bar{x}} U_j^{n+\theta}), \ \ j = 1,\cdots,J-1.$$

For this modified scheme it was proved that

$$\|U^n - u^n\|_\infty \leq \begin{cases} C(\tau + h^2), & \text{if } \theta = 1, \\ C(\tau + h^{\frac{3}{2}}), & \text{if } \frac{1}{2} < \theta < 1, \\ C(\tau^2 + h^{\frac{3}{2}}), & \text{if } \theta = \frac{1}{2}. \end{cases}$$

An alternative approach is the so-called immersed method, which has been developed for solving elliptic interface problems with finite difference approximations [6] and with finite element approximations [4,10]. The main idea in the immersed type methods is to use the interface conditions in those interface elements. In the recent work [10], an immersed finite element space is introduced. The IFE space is nonconforming and its partition is independent of the interface.

There are two types of schemes for time-dependent problems in general, implicit and explicit schemes. The former has no restriction on its time stepping. But in each time step one has to solve a global system of equations. The implementation on parallel computers is not straightforward due to its global nature. The latter is easy to program and implement on parallel computers. However, it suffers the severely restricted time stepsize from stability requirement. The classical θ-scheme is explicit for $0 < \theta < 1/2$ and implicit for $1/2 \leq \theta \leq 1$. Recently, a so-called explicit/implicit scheme has been studied by many authors, which is based on the concept of domain decomposition and a combination of implicit scheme and explicit scheme [1-3, 14, 16-18]. In these approaches, a physical domain is divided into several subdomains. The problem is solved in some subdomains implicitly and others explicitly. The main advantage of explicit/implicit scheme is its parallelism. The scheme still give rise to a restriction involving the time step, conductive coefficient and discretization parameter; however, this restriction is much less severe than for a fully explicit scheme, particularly for problems involving in multi-medium with significantly different conductivities. An explicit/implicit scheme based on a block finite difference was proposed in [3] for a heat equation. In this procedure, interface values or fluxes at the subdomain interface are calculated by explicit formulas with space stepsize H, and then interior values are determined by implicit differencing with space stepsize h, where $H = kh$ and k is an integer. It is proved both theoretically and numerically that the error of this scheme is $O(\tau + h^2 + H^3)$ in one-dimensional case. Amitai, et al [1] proposed a new algorithm in which a high order asynchronous explicit scheme is applied only at subdomains' boundary, and then any known higher–order implicit finite difference scheme can be applied with each subdomain. The high order asynchronous explicit scheme is derived based on Green's function. The scheme is specifically suitable for parallel computer. A block hopscotch scheme was presented in [5] for solving linear parabolic PDEs. It has been noted that all these works focus on the problems with smooth conductivity. It is more natural and more convenience to apply the technique for multi-medium problems in which one can solve a sub-problem with a continuous conductivity on each process. However, theoretical analysis on convergence and stability for the explicit/implicit schemes with discontinuous conductivity is unknown.

For positive integers J and N, let $h = 1/J$ and $\tau = T/N$ be space and time step sizes. Let $D = \{(x_j, t^n) | x_j = jh\,, j = 0, 1, \cdots, J\,, t^n = n\tau\,, n = 0, 1, \cdots, N\}$ be the set of net points. Denote $x_{j+\frac{1}{2}} = \frac{1}{2}(x_j + x_{j+1})$, $j = 0, 1, \cdots, J-1$. Let the discontinuity point $x = \xi = x_s + \kappa h$, $(0 < \kappa < 1)$.

For a function $g(x,t)$, let $g_j^n = g(x_j, t^n)$. For the discrete function $U(x,t)$ on D, define the difference quotients

$$\partial_{\bar{x}} U_j^n = \frac{1}{h}\left(U_j^n - U_{j-1}^n\right), \quad \partial_x U_j^n = \frac{1}{h}\left(U_{j+1}^n - U_j^n\right), \quad \partial_t U_j^n = \frac{1}{\tau}\left(U_j^{n+1} - U_j^n\right)$$

and the maximum norm

$$\|U^n\|_\infty = \max_{0 \le j \le J} |U_j^n|.$$

In this paper, we study the following explicit/implicit difference scheme

$$\partial_t U_j^n = \partial_x(a_{j-\frac{1}{2}} \partial_{\bar{x}} U_j^{n+1}) + F_j^{n+1}, \quad j = 1, \cdots, s+1; \; n = 0, 1, \cdots, N-1, \tag{4}$$

$$\partial_t U_j^n = \partial_x(a_{j-\frac{1}{2}} \partial_{\bar{x}} U_j^n) + F_j^n, \quad j = s+2, \cdots, J-1; \; n = 0, 1, \cdots, N-1, \tag{5}$$

$$U_j^0 = \alpha(x_j), \quad j = 0, 1, \cdots, J, \tag{6}$$

$$U_0^n = \beta_1(t^n), \quad U_J^n = \beta_2(t^n), \quad n = 0, 1, \cdots, N, \tag{7}$$

where

$$a_{j-\frac{1}{2}} = a(x_{j-\frac{1}{2}}), \quad j = 1, \cdots, s, s+2, \cdots, J,$$

$$a_{s+\frac{1}{2}} = \left(\frac{\kappa}{a_L} + \frac{1-\kappa}{a_R}\right)^{-1},$$

$$F_j^n = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x,t^n)dx, \ \ j=1,\cdots,J-1.$$

The convergence of above scheme was analyzed in [9], in which they derived the error estimate of order $O(\tau + h^{\frac{3}{2}})$ in L^∞-norm for solution of (1)-(3) and order $O(\tau+h)$ in L^2 norm for flux. No numerical results were given there.

The primary purpose of this paper is to present an optimal error estimate of the explicit/implicit scheme for problems in one-dimensional space with discontinuous conductivity. We prove that the error for the solution of problem is the order $O(\tau+h^2)$ in L^∞ norm and the error for the flux is the order $O(\tau+h^{\frac{3}{2}})$ in L^2 norm, while the truncation error of the scheme is the order $O(1)$. The stability condition of this scheme is much better than for a fully explicit scheme, when the explicit scheme is applied for subdomains with smaller conductive coefficients. Numerical experiments, including on a nonlinear problem and a problem in multi-dimensional space, illustrate that our theoretical estimates are optimal and this method can be applied for a wider range of physical problems.

2. Truncation error.

It is easy to show that, for $j=1,\cdots,s-1,s+2,\cdots,J-1$,

$$F_j^n - f_j^n = O(h^2), \ \ n=0,1,\cdots,N.$$

When $\kappa=\frac{1}{2}$, the above equality is also true for $j=s$ and $s+1$. When $\kappa\in(0,\frac{1}{2})$, there holds

$$\begin{aligned} F_s^n - f_s^n &= \frac{1}{h}\left(\int_{x_{s-\frac{1}{2}}}^{\xi} f(x,t^n)dx + \int_{\xi}^{x_{s+\frac{1}{2}}} f(x,t^n)dx\right) - f(x_s,t^n) \\ &= \left(\frac{1}{2}+\kappa\right) f_L^n + \left(\frac{1}{2}-\kappa\right) f_R^n - f_L^n + O(h) \\ &= \left(\frac{1}{2}-\kappa\right)(f_R^n - f_L^n) + O(h). \end{aligned}$$

Since ξ is not in $(x_{s+\frac{1}{2}}, x_{s+\frac{3}{2}})$ for $\kappa\in(0,\frac{1}{2})$, we have $F_{s+1}^n - f_{s+1}^n = O(h^2)$.

Similarly, when $\kappa\in(\frac{1}{2},1)$, there hold

$$F_s^n - f_s^n = O(h^2), \ \ F_{s+1}^n - f_{s+1}^n = \left(\frac{1}{2}-\kappa\right)(f_R^n - f_L^n) + O(h).$$

For $(x,t)\in D$, set $e(x,t)=U(x,t)-u(x,t)$. Then $e_j^n = e(x_j,t^n)$ satisfies

$$\partial_t e_j^n = \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}} e_j^{n+1}) + G_j^{n+1}, \ \ j=1,\cdots,s+1; \ n=0,1,\cdots,N-1; \tag{8}$$

$$\partial_t e_j^n = \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}} e_j^{n}) + G_j^{n+1}, \ \ j=s+2,\cdots,J-1; \ n=0,1,\cdots,N-1; \tag{9}$$

$$e_j^0 = 0, \ \ j=0,1,\cdots,J; \tag{10}$$

$$e_0^n = 0, \ \ e_J^n = 0, \ \ n=0,1,\cdots,N\,, \tag{11}$$

where G_j^{n+1} is the truncation error, i.e.,

$$G_j^{n+1} = -\partial_t u_j^n + \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}} u_j^{n+1}) + F_j^{n+1}, \ \ j=1,\cdots,s+1; n=0,1,\cdots,N-1;$$

$$G_j^{n+1} = -\partial_t u_j^n + \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}} u_j^{n}) + F_j^{n}, \ \ j=s+2,\cdots,J-1; n=0,1,\cdots,N-1.$$

By direct calculation we get

$$G_j^{n+1} = O(\tau+h^2), \ \ j=1,\cdots,s-1,s+2,\cdots,J-1; n=0,1,\cdots,N.$$

Let

$$\phi(t) = \begin{cases} \frac{1}{2}a_{s+\frac{1}{2}}(\kappa^2 u_L'' - (1-\kappa)^2 u_R'') + \left(\frac{1}{2}-\kappa\right) K_R', & \kappa \in \left(0, \frac{1}{2}\right] \\ \frac{1}{2}a_{s+\frac{1}{2}}(\kappa^2 u_L'' - (1-\kappa)^2 u_R'') + \left(\frac{1}{2}-\kappa\right) K_R' & \\ + \left(\frac{1}{2}-\kappa\right)(f_R - f_L), & \kappa \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Truncation errors at $j = s$ and $s+1$ are given by

$$G_s^{n+1} = -\phi^{n+1} + O(\tau + h),\ G_{s+1}^{n+1} = \phi^{n+1} + O(\tau + h), n = 0, 1, \cdots, N-1,$$

respectively, where $\phi^{n+1} = \phi(t^{n+1})$. It follows that the truncation error can be rewritten as

$$G_j^{n+1} = p_j^{n+1} + q_j^{n+1} + r_j^{n+1},\ \ j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1,$$

where

$$p_j^{n+1} = O(\tau + h^2),\ \ j = 1, 2, \cdots, J-1;$$
$$q_j^{n+1} = r_j^{n+1} = 0, j = 1, 2, \cdots, s-1, s+2, \cdots, J-1;$$

and

$$q_s^{n+1} = -\phi^{n+1},\ \ q_{s+1}^{n+1} = \phi^{n+1};$$
$$r_s^{n+1} = O(h),\ \ r_{s+1}^{n+1} = O(h).$$

3. Convergence and Stability. The following lemma will be used later.

Lemma 1 (Discrete Green Formula, see [15]) Let $y(x)$ and $z(x)$ be discrete functions on $\{x_j | j = 0, 1, \cdots, J\}$. Then

$$\sum_{j=p}^{q} y_j \partial_x z_j h = -\sum_{j=p+1}^{q+1} \partial_{\bar{x}} y_j z_j h - y_p z_p + y_{q+1} z_{q+1}.$$

Lemma 2 (Discrete Gronwall inequality, see [15]) Let w^n be a discrete function on $\{t^n | n = 0, 1, \cdots, N\}$ satisfying

$$w^n \le C_1 \sum_{k=0}^{n} w^k \tau + C_2,\ \ n = 0, 1, \cdots, N.$$

Then

$$w^n \le C_2 e^{2C_1 n\tau},\ \ n = 0, 1, \cdots, N,$$

where τ is sufficiently small such that $C_1\tau \le \frac{1}{2}$.

Lemma 3 (Discrete maximum principle) Suppose that the discrete function w_j^n on D satisfies

$$\partial_t w_j^n - \partial_x(a_{j-\frac{1}{2}} \partial_{\bar{x}} w_j^{n+1}) \le 0,\ \ j = 1, \cdots, s+1; n = 0, 1, \cdots, N-1; \tag{12}$$

$$\partial_t w_j^n - \partial_x(a_{j-\frac{1}{2}} \partial_{\bar{x}} w_j^{n}) \le 0,\ \ j = s+2, \cdots, J-1; n = 0, 1, \cdots, N-1; \tag{13}$$

$$w_j^0 \le 0,\ \ j = 0, 1, \cdots, J; \tag{14}$$

$$w_0^n \le 0,\ \ w_J^n \le 0,\ \ n = 1, 2, \cdots, N. \tag{15}$$

Assume that

$$\frac{\tau}{h^2}(a_{j+\frac{1}{2}} + a_{j-\frac{1}{2}}) < 1,\ \ j = s+2, \cdots, J-1. \tag{16}$$

Then there holds $w_j^n \le 0$ for $j = 0, 1, \cdots, J; n = 0, 1, \cdots, N$.

Proof. We prove the lemma by induction on n. Let $w_j^n \le 0$ for $j = 0, 1, \cdots, J$. From (13) it follows that

$$w_j^{n+1} \le \frac{\tau}{h^2}\left(a_{j+\frac{1}{2}} w_{j+1}^n + a_{j-\frac{1}{2}} w_{j-1}^n\right) + \left(1 - \frac{\tau}{h^2}(a_{j+\frac{1}{2}} + a_{j-\frac{1}{2}})\right) w_j^n \le 0,$$

for $j = s+2, \cdots, J-1$, provided that (16) holds. Let $0 \le j_1 \le s+2$ such that $w_{j_1}^{n+1} = \max_{0\le j\le s+2} w_j^{n+1}$. If $j_1 = 0$ or $j_1 = s+2$, then $w_j^{n+1} \le 0$ for $j = 1, \cdots, s+1$. If $1 \le j_1 \le s+1$, then from (12)

$$w_{j_1}^{n+1} \le w_{j_1}^n + \frac{\tau}{h^2}\left(a_{j_1+\frac{1}{2}}(w_{j_1+1}^{n+1} - w_{j_1}^{n+1}) - a_{j_1-\frac{1}{2}}(w_{j_1}^{n+1} - w_{j_1-1}^{n+1})\right) \le w_{j_1}^n \le 0,$$

for $j = 1, \cdots, s+1$. The lemma 3 is proved. ■

We denote the discrete flux by

$$V_j^n = a_{j-\frac{1}{2}}\partial_{\bar{x}}U_j^n\,.$$

Its error in a discrete L_2 norm is defined by

$$\|e_x^n\|_a^2 = \sum_{j=1}^{J} a_{j-\frac{1}{2}}|\partial_{\bar{x}}e_j^n|^2 h.$$

Our main result is as follows.

Theorem 1 *Suppose the conditions* **(I)–(III)** *are fulfilled. Then*

$$(i) \qquad\qquad \|e^n\|_\infty \le O(\tau + h^2)$$

and

$$(ii) \qquad\qquad \|e_x^n\|_a \le O(\tau + h^{3/2})$$

if the condition (16) is satisfied.

Remarks In the scheme (4)–(7) only two subdomains are used for simplicity; however, the arguments employed in this paper can be easily extended to the case of multiple subdomains.

The restriction condition (16) obtained here for the discontinuous problem is similar to those conditions in classical work for smooth problems. It is worth to emphasize in the discontinuous case that, for the consideration of stability, we should use explicit scheme only in those subdomains on which the coefficient $a(x)$ is small, and implicit scheme in those subdomains on which $a(x)$ is large.

Let $\tilde{f}(x,t)$, $\tilde{\alpha}(x)$, $\tilde{\beta}_1(t)$ and $\tilde{\beta}_2(t)$ satisfy (I) and (II). Let $\tilde{U}_j^n$ be the solution of (4)–(7) with the corresponding initial and boundary data $\tilde{\alpha}$ and $\tilde{\beta}_i$ $(i = 1,2)$. Denote

$$B = \max\left\{\max_{0\le j\le J}|\alpha(x_j) - \tilde{\alpha}(x_j)|, \max_{0\le n\le N}|\beta_1(t^n) - \tilde{\beta}_1(t^n)|, \max_{0\le n\le N}|\beta_2(t^n) - \tilde{\beta}_2(t^n)|\right\}.$$

We have the following result.

Theorem 2 *Under the same conditions of Theorem 1, the difference scheme (4)–(7) is L^∞ stable, i.e., there holds*

$$\max_{j,n}|U_j^n - \tilde{U}_j^n| \le T\max_{j,n}|F_j^n - \tilde{F}_j^n| + B.$$

4. Proof of Theorem 1 and 2. First we prove Theorem 1. Denote $\chi_s = \chi_{\{j=s,s+1\}}$ be the characteristic function of the discrete point set $\{j = s, s+1\}$.

Let P_j, Q_j^n and R_j be the solutions of the following difference systems respectively

$$(17) \qquad \begin{cases} -\partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}P_j) = 1, \quad j = 1,2,\cdots,J-1, \\ P_0 = P_J = 0, \end{cases}$$

$$(18)\qquad \begin{cases} -\partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}Q_j^n) = q_j^n, & j = 1, 2, \cdots, J-1, \\ Q_0^n = Q_J^n = 0, \end{cases}$$

$$(19)\qquad \begin{cases} -\partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}R_j) = \chi_s, & j = 1, \cdots, J-1 \\ R_0 = R_J = 0 \end{cases}.$$

A straightforward calculation yields explicit expressions of P_j, Q_j^n and R_j as follows

$$P_1 = \frac{1}{a_{\frac{1}{2}}} \cdot \frac{\sum_{k=2}^{J} \frac{k-1}{a_{k-\frac{1}{2}}}}{\sum_{k=1}^{J} \frac{1}{a_{k-\frac{1}{2}}}} h^2,$$

$$P_j = \left(\frac{\sum_{k=2}^{J} \frac{k-1}{a_{k-\frac{1}{2}}}}{\sum_{k=1}^{J} \frac{1}{a_{k-\frac{1}{2}}}} \cdot \sum_{k=1}^{j} \frac{1}{a_{k-\frac{1}{2}}} - \sum_{k=2}^{j} \frac{k-1}{a_{k-\frac{1}{2}}} \right) h^2, \quad j = 2, \cdots, J.$$

$$Q_k^n = -\frac{\phi^n h^2 \sum_{m=1}^{k} \frac{1}{a_{m-\frac{1}{2}}}}{a_{s+\frac{1}{2}} \sum_{m=1}^{J} \frac{1}{a_{m-\frac{1}{2}}}}, \quad k = 1, \cdots, s$$

$$Q_k^n = \frac{\phi^n h^2 \sum_{m=k+1}^{J} \frac{1}{a_{m-\frac{1}{2}}}}{a_{s+\frac{1}{2}} \sum_{m=1}^{J} \frac{1}{a_{m-\frac{1}{2}}}}, \quad k = s+1, \cdots, J-1$$

$$R_k = \frac{\sum_{m=1}^{k} \frac{1}{a_{m-\frac{1}{2}}}}{\sum_{m=1}^{J} \frac{1}{a_{m-\frac{1}{2}}}} \cdot \left(2 \sum_{m=s+2}^{J} \frac{1}{a_{m-\frac{1}{2}}} + \frac{1}{a_{s+\frac{1}{2}}} \right) h^2, \quad k = 1, \cdots, s,$$

$$R_k = \frac{\sum_{m=k+1}^{J} \frac{1}{a_{m-\frac{1}{2}}}}{\sum_{m=1}^{J} \frac{1}{a_{m-\frac{1}{2}}}} \cdot \left(2 \sum_{m=1}^{s} \frac{1}{a_{m-\frac{1}{2}}} + \frac{1}{a_{s+\frac{1}{2}}} \right) h^2, \quad k = s+1, \cdots, J-1.$$

Then we can get

$$P_j = O(1), \quad Q_j^n = O(h^2), \quad \partial_t Q_j^n = O(h^2), \quad R_j = O(h).$$

Set $E_j^n = e_j^n - C_1 P_j(\tau + h^2) - Q_j^n - C_2 R_j h$, where C_1 and C_2 are constants to be determined. Note that

$$\partial_t E_j^n - \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}E_j^{n+1}) = p_j^{n+1} - \partial_t Q_j^n - C_1(\tau+h^2) + r_j^{n+1} - C_2 h\chi_s, \quad j = 1, \cdots, s+1;$$

$$\partial_t E_j^n - \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}E_j^{n}) = p_j^{n+1} - \partial_t Q_j^n - C_1(\tau+h^2) + r_j^{n+1} - C_2 h\chi_s, \quad j = s+2, \cdots, J-1.$$

By choosing C_1 and C_2 sufficiently large, we obtain

$$\partial_t E_j^n - \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}E_j^{n+1}) \leq 0, \quad j = 1, \cdots, s+1;$$

$$\partial_t E_j^n - \partial_x(a_{j-\frac{1}{2}}\partial_{\bar{x}}E_j^{n}) \leq 0, \quad j = s+2, \cdots, J-1.$$

Obviously there are $E_j^0 \leq 0$ $(j = 0, 1, \cdots, J)$ and $E_0^n \leq 0, E_J^n \leq 0$ $(n = 0, 1, \cdots, N)$. By using lemma 3 we get

$$E_j^n \leq 0, \quad j = 0, 1, \cdots, J; n = 0, 1, \cdots, N.$$

Then

$$e_j^n \leq C_1 P_j(\tau + h^2) + Q_j^n + C_2 R_j h, \ j = 0, 1, \cdots, J; n = 0, 1, \cdots, N.$$

Similarly we can obtain

$$e_j^n \geq -(C_1 P_j(\tau + h^2) + Q_j^n + C_2 R_j h), \ j = 0, 1, \cdots, J; n = 0, 1, \cdots, N.$$

The part (i) of Theorem 1 is proved.

Now we prove (ii) of theorem 1. Multiplying (8)–(9) by $\partial_t e_j^n h$ and summing up them over $j = 1, 2, \cdots, J-1$, we have

$$\sum_{j=1}^{J-1} |\partial_t e_j^n|^2 h = \sum_{j=1}^{J-1} \partial_x (a_{j-\frac{1}{2}} \partial_{\bar{x}} e_j^{n+1}) \partial_t e_j^n h - \tau \sum_{j=s+2}^{J-1} \partial_t \partial_x (a_{j-\frac{1}{2}} \partial_{\bar{x}} e_j^n) \partial_t e_j^n h$$

$$+ \sum_{j=1}^{J-1} G_j^{n+1} \partial_t e_j^n h \equiv I + II + III. \tag{20}$$

¿From lemma 1 and the boundary conditions (11), it follows that

$$I = -\sum_{j=1}^{J} a_{j-\frac{1}{2}} \partial_{\bar{x}} e_j^{n+1} \partial_t \partial_{\bar{x}} e_j^n h = -\frac{1}{2\tau} \left(\|e_x^{n+1}\|_a^2 - \|e_x^n\|_a^2 \right) - \frac{\tau}{2} \sum_{j=1}^{J} a_{j-\frac{1}{2}} |\partial_t \partial_{\bar{x}} e_j^n|^2 h,$$

where the following elementary identity is used,

$$g_j^{n+1}(g_j^{n+1} - g_j^n) = \frac{1}{2} \left(|g_j^{n+1}|^2 - |g_j^n|^2 + |g_j^{n+1} - g_j^n|^2 \right).$$

By lemma 1 there holds

$$II = \tau \left(\sum_{j=s+3}^{J} a_{j-\frac{1}{2}} |\partial_{\bar{x}} \partial_t e_j^n|^2 h + a_{s+\frac{3}{2}} \partial_t \partial_{\bar{x}} e_{s+2}^n \partial_t e_{s+2}^n \right).$$

It follows that

$$I + II \leq -\frac{1}{2\tau} \left(\|e_x^{n+1}\|_a^2 - \|e_x^n\|_a^2 \right) - \frac{\tau}{2} \sum_{j=1}^{s+2} a_{j-\frac{1}{2}} |\partial_{\bar{x}} \partial_t e_j^n|^2 h$$

$$+ \frac{\tau}{2} \sum_{j=s+3}^{J} a_{j-\frac{1}{2}} |\partial_{\bar{x}} \partial_t e_j^n|^2 h + \tau a_{s+\frac{3}{2}} \left(\frac{1}{2h^2} |\partial_t e_{s+2}^n|^2 + \frac{1}{2} |\partial_{\bar{x}} \partial_t e_{s+2}^n|^2 \right) h.$$

Note that

$$\frac{\tau}{2} \sum_{j=s+3}^{J} a_{j-\frac{1}{2}} |\partial_{\bar{x}} \partial_t e_j^n|^2 h \leq \frac{\tau}{h^2} \sum_{j=s+3}^{J-1} (a_{j+\frac{1}{2}} + a_{j-\frac{1}{2}}) |\partial_t e_j^n|^2 h + \frac{\tau}{h^2} a_{s+\frac{5}{2}} |\partial_t e_{s+2}^n|^2 h.$$

Then

$$I + II \leq -\frac{1}{2\tau} \left(\|e_x^{n+1}\|_a^2 - \|e_x^n\|_a^2 \right) + \frac{\tau}{h^2} \sum_{j=s+2}^{J-1} (a_{j+\frac{1}{2}} + a_{j-\frac{1}{2}}) |\partial_t e_j^n|^2 h. \tag{21}$$

It remains to estimate the last term III in (20). We have

$$III = \sum_{j=1}^{J-1} p_j^{n+1} \partial_t e_j^n h + \sum_{j=1}^{J-1} q_j^{n+1} \partial_t e_j^n h + \sum_{j=1}^{J-1} r_j^{n+1} \partial_t e_j^n h \equiv III_1 + III_2 + III_3. \tag{22}$$

For any $\varepsilon > 0$, $|ab| \leq \varepsilon |a|^2 + \frac{1}{4\varepsilon} |b|^2$. Then

$$III_1 \leq \frac{\varepsilon}{8} \sum_{j=1}^{J-1} |\partial_t e_j^n|^2 h + \frac{2}{\varepsilon} \sum_{j=1}^{J-1} |p_j^{n+1}|^2 h = \frac{\varepsilon}{8} \sum_{j=1}^{J-1} |\partial_t e_j^n|^2 h + O(\tau^2 + h^4), \tag{23}$$

and

$$III_2 = -\phi^{n+1} \partial_t e_s^n h + \phi^{n+1} \partial_t e_{s+1}^n h$$

$$= \frac{1}{\tau}\left[\phi^{n+1}(e_{s+1}^{n+1} - e_s^{n+1}) - \phi^n(e_{s+1}^n - e_s^n)\right] h - \partial_t \phi^n \partial_x e_s^n h^2$$

$$\leq \frac{1}{\tau}\left[\phi^{n+1}(e_{s+1}^{n+1} - e_s^{n+1}) - \phi^n(e_{s+1}^n - e_s^n)\right] h + a_{s-\frac{1}{2}}|\partial_x e_s^n|^2 h + O(h^3)\,. \tag{24}$$

Moreover,

$$III_3 = r_s^{n+1}\partial_t e_s^n h + r_{s+1}^{n+1}\partial_t e_{s+1}^n h \leq \frac{\varepsilon}{8}\left(|\partial_t e_s^n|^2 + |\partial_t e_{s+1}^n|^2\right) h + O(h^3). \tag{25}$$

Combining (20)–(25) yields

$$\frac{1}{2\tau}\left(\|e_x^{n+1}\|_a^2 - \|e_x^n\|_a^2\right) - \frac{1}{\tau}\left[\phi^{n+1}(e_{s+1}^{n+1} - e_s^{n+1}) - \phi^n(e_{s+1}^n - e_s^n)\right] h$$

$$+(1-\varepsilon)\sum_{j=1}^{s+1}|\partial_t e_j^n|^2 h + \sum_{j=s+2}^{J-1}\left[1 - \varepsilon - \frac{\tau}{h^2}(a_{j+\frac{1}{2}} + a_{j-\frac{1}{2}})\right]|\partial_t e_j^n|^2 h$$

$$\leq C\|e_x^n\|_a^2 + O(\tau^2 + h^3). \tag{26}$$

Notice the initial data (10). It follows from (26) that, for $n = 0, 1, \cdots, N-1$,

$$\frac{1}{2}\|e_x^{n+1}\|_a^2 \leq \phi^{n+1}\partial_{\bar{x}} e_{s+1}^{n+1} h^2 + C\sum_{k=1}^{n+1}\|e_x^k\|_a^2 \tau + O(\tau^2 + h^3). \tag{27}$$

Obviously there holds

$$\left|\phi^{n+1}\partial_{\bar{x}} e_{s+1}^{n+1}\right| h^2 \leq \frac{1}{4} a_{s+\frac{1}{2}}|\partial_{\bar{x}} e_{s+1}^{n+1}|^2 h + O(h^3).$$

Substituting the above inequality into (27), we get

$$\|e_x^{n+1}\|_a^2 \leq C\sum_{k=1}^{n+1}\|e_x^k\|_a^2\tau + O(\tau^2 + h^3). \tag{28}$$

Therefore, by lemma 2, it follows that

$$\|e_x^{n+1}\|_a^2 \leq O(\tau^2 + h^3), \quad n = 0, 1, \cdots, N-1. \tag{29}$$

The proof of Theorem 1 is completed.■

For the proof of theorem 2, we set $v_j^n = U_j^n - \tilde{U}_j^n$, $g_j^n = F_j^n - \tilde{F}_j^n$, and $g = \max_{j,n}|g_j^n|$. Introduce an auxiliary function $w_j^n = v_j^n - (gt^n + B)$. It is easy to see that w_j^n satisfy the condition of lemma 3. Then $w_j^n \leq 0$ for $j = 0, 1, \cdots, J$ and $n = 0, 1, \cdots, N$. Again, let $\bar{w}_j^n = v_j^n + (gt^n + B)$, then $\bar{w}_j^n \geq 0$ for $j = 0, 1, \cdots, J$ and $n = 0, 1, \cdots, N$. So it follows that

$$\max_{j,n}|U_j^n - \tilde{U}_j^n| \leq T\max_{j,n}|F_j^n - \tilde{F}_j^n| + B.$$

Theorem 2 is proved. ■

3. More general cases.

In this section, we extend the explicit/implicit scheme (4)-(7) to the following nonlinear parabolic PDEs in two-dimensional space.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(a(x,y)\frac{\partial u}{\partial y}\right) + f(u_x, u_y, u, x, y, t), \quad (x,y)\in\Omega,\ \ 0 < t \le T, \tag{30}$$

$$u(x,y,0) = \alpha(x,y), \ \ (x,y)\in\Omega, \tag{31}$$

$$u(x,y,t) = \beta(x,y,t), \ \ (x,y,t)\in\partial\Omega\times[0,T] \tag{32}$$

where $\Omega = (0,1)\times(0,1)$. Denote $\Omega^- = \{(x,y,t)|0 \le x \le \xi, 0 \le y \le 1, 0 \le t \le T\}$, $\Omega^+ = \{(x,y,t)|\xi \le x \le 1, 0 \le y \le 1, 0 \le t \le T\}$,

$$u_\xi(y,t) = u(\xi,y,t), \ \ u_L(y,t) = \lim_{x\to\xi^-} u(x,y,t), \ \ u_R(y,t) = \lim_{x\to\xi^+} u(x,y,t).$$

Analogous to the one–dimensional problem, we assume that the known data are smooth on Ω^- and Ω^+ respectively, but have discontinuity of first kind on $x = \xi$. Also assume the initial flux is continuous at $x = \xi$. It follows that there holds

$$\lim_{x\to\xi^-} a(x,y)\frac{\partial u}{\partial x}(x,y,t) = \lim_{x\to\xi^+} a(x,y)\frac{\partial u}{\partial x}(x,y,t).$$

Denote $x_i = ih_1, y_j = jh_2$, where $i = 0,1,\cdots,I$; $j = 0,1,\cdots,J$; $Ih_1 = 1$, $Jh_2 = 1$. Let $x_{i+\frac12} = \frac12(x_i + x_{i+1})$, $y_{j+\frac12} = \frac12(y_j + y_{j+1})$. The discontinuous point $x = \xi = x_s + \kappa h_1$, $0 < \kappa < 1$, $1 < s < J-1$.

For a function $f(x,y)$ on Ω, let $f_{ij} = f(x_i,y_j)$, $f_{i+\frac12,j} = f(x_{i+\frac12},y_j)$, $f_{i,j+\frac12} = f(x_i,y_{j+\frac12})$. Define the difference operator

$$\partial_t U_{ij}^n = \frac{1}{\tau}(U_{ij}^{n+1} - U_{ij}^n), \ \ \partial_{\bar y} U_{ij}^n = \frac{1}{h_2}(U_{ij}^n - U_{ij-1}^n), \ \ \partial_y U_{ij}^n = \frac{1}{h_2}(U_{ij+1}^n - U_{ij}^n),$$

and $\partial_{\bar x} U_{ij}^n$ and $\partial_x U_{ij}^n$ similarly.

Construct an explicit/implicit scheme for (30)–(32) as follows:

$$\partial_t U_{ij}^n = \partial_x(a_{i-\frac12,j}\partial_{\bar x} U_{ij}^{n+1}) + \partial_y(a_{i,j-\frac12}\partial_{\bar y} U_{ij}^{n+1}) + F_{ij}^{n+1}, \quad i = 1,2,\cdots,s+1; j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1 \tag{33}$$

$$\partial_t U_{ij}^n = \partial_x(a_{i-\frac12,j}\partial_{\bar x} U_{ij}^{n}) + \partial_y(a_{i,j-\frac12}\partial_{\bar y} U_{ij}^{n}) + F_{ij}^{n}, \quad i = s+2,\cdots,J-1; j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1 \tag{34}$$

with the discrete version of the initial–boundary condition (31)–(32), where

$$\begin{aligned} &a_{i-\frac12,j} = a(x_{i-\frac12},y_j) \qquad i = 1,\cdots,s,s+2,\cdots,I; j = 1,\cdots,J-1;\\ &a_{s+\frac12,j} = \left(\tfrac{\kappa}{a_{Lj}} + \tfrac{1-\kappa}{a_{Rj}}\right)^{-1}, \quad a_{Lj} = a_L(y_j), \quad a_{Rj} = a_R(y_j), \quad j = 1,\cdots,J-1,\\ &a_{i,j-\frac12} = a(x_i,y_{j-\frac12}) \qquad i = 1,\cdots,I-1; j = 1,\cdots,J-1, \end{aligned}$$

and

$$F_{ij}^n = \frac{1}{h_1h_2}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{x_{i-\frac12}}^{x_{i+\frac12}} f(u_x^n, u_y^n, u^n, x, y, t^n)dxdy\,.$$

By the same method as mentioned in section 2, the truncation error G_{ij}^{n+1} of the difference scheme (33)–(34) can be given by

$$G_{ij}^{n+1} = O(\tau + h_1^2 + h_2^2), \ \ i = 1,\cdots,s-1,s+2,\cdots,I-1; j = 1,\cdots,J-1;$$

$$G_{sj}^{n+1} = -G_{s+1j}^{n+1} + O(\tau + h_1 + h_2) = O(1), \ \ j = 1,\cdots,J-1.$$

TABLE 1. Numerical results at $T = 1$ with $r = \tau/h^2$, $h = 1/J$ (example 1)

J	$r = 1.0$		$r = 20.0$	
	$\|e^n\|_\infty$	$\|e^n\|_\infty/(\tau+h^2)$	$\|e^n\|_\infty$	$\|e^n\|_\infty/(\tau+h^2)$
25	6.905D-3	2.2	1.181D-2	0.35
50	1.883D-3	2.4	3.313D-3	0.39
100	4.298D-4	2.1	6.960D-4	0.33
200	1.225D-4	2.4	2.061D-4	0.39

However, it seems difficult to present the optimal error estimates for the nonlinear case. We shall present some numerical examples in next section.

4. Numerical experiments

In this section, we apply the explicit/implicit scheme to three examples. All numerical computations were done on a SUN SPARC station 1 with double precision.

Example 6.1 (linear equation). First, we consider a simple linear problem defined in (1)-(3) with

$$a(x) = \begin{cases} 0.1 - 0.09x, & x \in (0, \frac{2}{3}] \\ 0.01, & x \in (\frac{2}{3}, 1) \end{cases},$$

$$f(x,t) = \begin{cases} 0.09\pi \exp(-0.1\pi^2 t)(\cos \pi x - \pi x \sin \pi x), & x \in (0, \frac{2}{3}] \\ 0.06\pi^2 \exp(-0.1\pi^2 t) \sin 4\pi x, & x \in (\frac{2}{3}, 1) \end{cases},$$

$$\alpha(x) = \begin{cases} \sin \pi x, & x \in (0, \frac{2}{3}] \\ \sin 4\pi x, & x \in (\frac{2}{3}, 1) \end{cases},$$

and $\beta_1(t) = \beta_2(t) = 0$. The exact solution of the problem is

$$u(x,t) = \begin{cases} \exp(-0.1\pi^2 t) \sin \pi x, & x \in (0, \frac{2}{3}] \\ \exp(-0.1\pi^2 t) \sin 4\pi x, & x \in (\frac{2}{3}, 1) \end{cases}.$$

We set $r = \tau/h^2$. The scheme is tested with $r = 1.0$ and $r = 20.0$, respectively. Numerical results for different values of h (or J) and $T = 1$ are presented in Table 1. The scheme is stable for both $r = 1.0$ and $r = 20.0$. It can be observed from Table 1 that $\|e^n\|_\infty/(\tau + h^2)$ approximately is a constant in different stepsizes, which confirm our theoretical analysis in Theorem 1. To test the error on flux, we have to take a different way due to the restriction (16). We solve the problem with a very small τ such that the error on the discretization of time direction can be ignored. Numerical results with $\tau = 10^{-5}$ are presented in Table 2.

We also solve the linear problem by using the θ-scheme given in [12,13]. The comparison with the θ-scheme with $\theta = 0$ and $\theta = 1$ is given in Tables 3-4. Theoretically, all three schemes are of first-order accuracy in time discretization. Compared with the classical θ-scheme the explicit/implicit scheme gives better accuracy for all the values of r and h. When $r = 10$, the forward Euler scheme ($\theta = 0$) is unstable. We present in Figure 1 numerical results with different values of r to examine the stability of scheme. It is obvious that in this case, the scheme is stable until $r = 50$.

Example 6.2 (nonlinear problem). The second example is a nonlinear parabolic PDE defined by

$$u_t = (a(x)u_x)_x + uu_x + f(x,t)$$

TABLE 2. Numerical results at $T = 1$ with $\tau = 10^{-5}$, $h = 1/J$ (example 1)

J	$\|e_x^n\|_a$	$\|e_x^n\|_a/(\tau + h^{\frac{3}{2}})$
25	4.982D-3	0.62
50	1.781D-3	0.63
100	4.563D-4	0.45

TABLE 3. The comparison of the error $\|e^n\|_\infty$ at $T = 1$ for $h = 0.01$ (example 1).

	θ-scheme(Samarskii)		explicit/implicit
r	$\theta = 0$	$\theta = 1$	scheme
1	4.440D-3	4.477D-3	4.298D-4
5	4.365D-3	4.552D-3	4.264D-4
10	blow-up	4.646D-3	5.163D-4

TABLE 4. The comparison of $\|e^n\|_\infty$ at $T = 1$ for $h = 0.005$ (example 1)

	θ-scheme(Samarskii)		explicit/implicit
r	$\theta = 0$	$\theta = 1$	scheme
1	1.988D-3	1.979D-3	1.225D-4
5	2.005D-3	1.962D-3	1.401D-4
10	blow-up	1.941D-3	1.621D-4

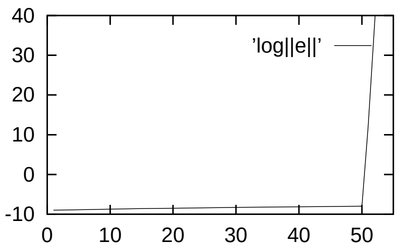

FIGURE 1. The plot of $log\|e^n\|_\infty$ with $T = 1$ and $h = 0.01$ versus r (example 1)

where

$$a(x) = \begin{cases} 0.04, & x \in (0, \frac{2}{3}] \\ 0.01, & x \in (\frac{2}{3}, 1) \end{cases},$$

$$f(x,t) = \begin{cases} -\frac{\pi}{2}\exp(-0.08\pi^2 t)\sin 2\pi x, & x \in (0, \frac{2}{3}] \\ 0.12\pi^2\exp(-0.04\pi^2 t)\sin 4\pi x - 2\pi\exp(-0.08\pi^2 t)\sin 8\pi x, & x \in (\frac{2}{3}, 1) \end{cases},$$

$$\alpha(x) = \begin{cases} \sin \pi x, & x \in (0, \frac{2}{3}] \\ \sin 4\pi x, & x \in (\frac{2}{3}, 1) \end{cases},$$

TABLE 5. Numerical results at $T = 1$ with $r = \tau/h^2$, $h = 1/J$ (example 2)

	$r = 1.0$		$r = 10.0$	
J	$\|e^n\|_\infty$	$\|e^n\|_\infty/(\tau+h^2)$	$\|e^n\|_\infty$	$\|e^n\|_\infty/(\tau+h^2)$
25	3.158D-2	9.9	2.926D-2	1.7
50	8.259D-3	10.3	7.707D-3	1.8
100	2.002D-3	10.0	1.843D-3	1.7
200	5.119D-4	10.2	4.709D-4	1.7

and $\beta_1(t) = \beta_2(t) = 0$. The exact solution of the problem is

$$u(x,t) = \begin{cases} \exp(-0.04\pi^2 t)\sin\pi x, & x \in (0, \frac{2}{3}] \\ \exp(-0.04\pi^2 t)\sin 4\pi x, & x \in (\frac{2}{3}, 1) \end{cases}.$$

The scheme used here for the nonlinear problem is obtained from (4)-(7) with the lower order term

$$u_j \frac{u_{j+1} - u_{j-1}}{2h} + F_j^n$$

which is similar to the approximation in (33)-(34).

We present the error of solution in Table 5. It seems that our error estimates in Theorem 1 are also true for nonlinear problems, although theoretical analysis is only given for linear cases.

Example 6.3 (2-D problem). Finally, we consider the two-dimensional problem (30)–(32) with

$$a(x,y) = \begin{cases} 0.1, & (x,y) \in (0, \frac{1}{3}) \times (0,1) \\ 0.01, & (x,y) \in (\frac{1}{3}, 1) \times (0,1) \end{cases},$$

$$f(x,y,t) = \begin{cases} 0, & (x,y,t) \in (0, \frac{1}{3}] \times (0,1) \times (0,T) \\ -0.81\pi^2 \exp(-0.2\pi^2 t)\sin 10\pi x \sin \pi y, & (x,y,t) \in (\frac{1}{3}, 1) \times (0,1) \times (0,T) \end{cases},$$

$$\alpha(x,y) = \begin{cases} \sin\pi x \sin\pi y, & (x,y) \in (0, \frac{1}{3}] \times (0,1) \\ \sin 10\pi x \sin\pi y, & (x,y) \in (\frac{1}{3}, 1) \times (0,1) \end{cases},$$

and $\beta(x,y,t) = 0$. The exact solution of the problem is

$$u(x,y,t) = \begin{cases} \exp(-0.2\pi^2 t)\sin\pi x \sin\pi x, & (x,y,t) \in (0, \frac{1}{3}] \times (0,1) \times (0,T) \\ \exp(-0.2\pi^2 t)\sin 10\pi x \sin\pi y, & (x,y,t) \in (\frac{1}{3}, 1) \times (0,1) \times (0,T) \end{cases}.$$

The error measure used here is

$$\|e^n\|_\infty = \max_{i,j} |e_{ij}^n|$$

$$\|e^n\|_a = \sqrt{\sum_{j=1}^{J}\sum_{i=1}^{I} \left[a_{i-\frac{1}{2},j}(\partial_{\bar{x}} e_{ij}^n)^2 + a_{i,j-\frac{1}{2}}(\partial_{\bar{y}} e_{ij}^n)^2\right] h_1 h_2}\,.$$

We present numerical results in Table 6 where we take $h = h_1 = h_2$. Some features can be observed from Table 6 which are similar to those in Example 1.

5. Conclusions

We have proved convergence and stability of the explicit/implicit scheme for parabolic equations with discontinuous coefficients. The convergent rates are proved to be sharp. Numerical experiments indicate that the explicit/implicit scheme indeed is of the convergence rates given in theorem 1, and can be extended to apply for nonlinear parabolic equations and two dimensional parabolic problems.

TABLE 6. Numerical results at $T = 0.1$ with $r = \tau/h^2$, $h = 1/J$ (example 3)

J	$r = 1.0$		$r = 10.0$	
	$\|e^n\|_\infty$	$\|e^n\|_\infty/(\tau+2h^2)$	$\|e^n\|_\infty$	$\|e^n\|_\infty/(\tau+2h^2)$
25	8.191D-2	17.1	7.815D-2	4.1
50	1.721D-2	14.3	1.746D-2	3.6
100	5.404D-3	18.0	5.459D-3	4.5
200	1.380D-3	18.4	1.426D-3	4.8

¿From the restriction condition (16) of the meshstep for the explicit/implicit scheme we can see that if one uses implicit schemes in the domain where the conductive coefficient is large, and uses explicit schemes in the domain where the conductive coefficient is small, then the stability condition of the explicit/implicit scheme is less restrictive than that of the fully explicit scheme. Moreover, in principle we can construct parallel schemes with several explicit/implicit blocks such as (4)–(5). The convergence and stability results can be obtained in the same way as the discussion above.

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