INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 1, Number 2, Pages 111–129

AN H^1-GALERKIN MIXED METHOD FOR SECOND ORDER HYPERBOLIC EQUATIONS

AMIYA K. PANI, RAJEN K. SINHA, AND AJAYA K. OTTA

This paper is dedicated to our beloved teacher Professor Purna Chandra Das on the occasion of his 65th birthday

Abstract. An H^1- Galerkin mixed finite element method is discussed for a class of second order hyperbolic problems. It is proved that the Galerkin approximations have the same rates of convergence as in the classical mixed method, but without LBB stability condition and quasi-uniformity requirement on the finite element mesh. Compared to the results proved for one space variable, the $L^\infty(L^2)$-estimate of the stress is not optimal with respect to the approximation property for the problems in two and three space dimensions. It is further noted that if the Raviart- Thomas spaces are used for approximating the stress, then optimal estimate in $L^\infty(L^2)$-norm is achieved using the new formulation. Finally, without restricting the approximating spaces for the stress, a modification of the method is proposed and analyzed. This confirms the findings in a single space variable and also improves upon the order of convergence of the classical mixed procedure under an extra regularity assumption on the exact solution.

Key Words. Second order wave equation, LBB condition, H^1 Galerkin mixed finite element method, semidiscrete scheme, completely discrete method, optimal error estimates.

1. Introduction

In this paper, we discuss a new mixed finite element method for the following second order hyperbolic initial and boundary value problem

$$u_{tt} - \nabla \cdot (a(x)\nabla u) + c(x)u = f(x,t), \quad (x,t) \in \Omega \times J, \tag{1.1}$$
$$u = 0, \quad (x,t) \in \partial\Omega \times J,$$
$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \; x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^d, d = 2,3$ with boundary $\partial\Omega$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$ and $J = (0,T]$ with $T < \infty$. Assume that the coefficients a, c, initial functions u_0, u_1 and the forcing function f are sufficiently smooth with $a \geq a_0$ for some positive constant a_0, and $c \geq 0 \; \forall x \in \Omega$.

When our primary concern is to obtain both displacement, i.e., u and the stress that is, $\sigma = a\nabla u$, we first split (1.1) into a system of two equations and then apply classical mixed methods, see [5], [8] and [9]. However, this procedure has to satisfy the LBB-stability condition on the approximating subspaces and this restricts the choice of finite element spaces. For example, the Raviart-Thomas

Received by the editors January 20, 2004.
2000 *Mathematics Subject Classification.* 65M60, 65M15, 65M12.

spaces of index $r \geq 1$ are usually used for the standard mixed methods. In order to avoid using LBB-stability condition, we introduce, in this paper, an alternate mixed finite element procedure. The proposed method is a non-symmetric version of least square method and we shall call it as H^1- Galerkin mixed finite element procedure. It takes advantage of the least-square method and yields a better rate of convergence for the stress than the conventional use of linear elements.

In recent years, substantial progress has been made in the least-square mixed methods applied mainly to the elliptic equations, see [4], [7], [10]–[11], [15]–[18], and references, there in. These procedures that circumvent the LBB stability conditions are considered as alternatives to the classical mixed formulations. So far, there has been hardly any analysis for the least square methods applied to parabolic and second order hyperbolic initial and boundary value problems. In an attempt to extend least-square mixed methods to parabolic equations, one of the authors [12] has introduced an H^1-Galerkin mixed procedure ,i.e., a non-symmetric version of least square method and has derived optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms. For more applications of this alternate mixed formulation, see [13]–[14].

In the present article, the proposed mixed method is applied to a system consisting of displacement u and stress σ. The approximating finite element spaces V_h and $\mathbf{W_h}$ are allowed to be of differing polynomial degrees. Hence, estimations have been obtained which distinguish the better approximation properties of V_h and $\mathbf{W_h}$. Compared to classical mixed methods, the present method is not subject to LBB stability condition. While in classical method, the $L^\infty(L^2)$-estimate of the stress is suboptimal, optimal estimate is derived for the problems in one space dimension using this new mixed formulation. It is noted that if the finite element spaces for approximating the stress are of Raviart-Thomas type, then optimal estimates are achieved for the stress. Finally, without restricting the finite dimensional spaces, a modification of the H^1-Galerkin method is proposed and analyzed. Although an extra regularity condition is required on the exact solution, yet an optimal order of convergence with respect to the approximation property for the stress in $L^\infty(L^2)$-norm is established (see, Remarks 2.1, 3.1 and Section 4 below). Moreover, it is noted that the quasi-uniformity condition is not imposed on the finite element mesh for the error estimates in L^2 and H^1-norms.

The layout of the paper is as follows. In Section 2, a second order hyperbolic equation in a single space variable is considered and optimal error estimates are discussed for the semidiscrete case. In two- and three space dimensions, a similar analysis is carried out in Section 3. Moreover, the rates of convergence obtained coincide with those in [5], [8] using classical mixed method. But compared to one dimensional case, the $L^\infty(L^2)$- estimate of the stress in this section is not optimal. It is, further, noted that if the Raviart-Thomas finite element spaces are used for approximating the stress, then we obtain optimal estimate. In Section 4, a modification of H^1- Galerkin mixed finite element method is examined without resricting the approximating spaces for the stress and semidiscrete error estimates are established. In Section 5, a completely discrete scheme is briefly described and *a priori* error bounds are derived only for the modified H^1- Galerkin mixed method.

Throughout this paper, C will denote a generic positive constant which does not depend on the spatial mesh parameter h and time discretization parameter Δt.

2. Hyperbolic Equation in One Space Dimension

In this Section, we consider the following one dimensional second order hyperbolic equation

$$u_{tt} - (au_x)_x + cu = f(x,t), \quad (x,t) \in (0,1) \times J, \tag{2.1a}$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in \overline{J}, \tag{2.1b}$$

and initial conditions

$$u(x,0) = u_0,\ u_t(x,0) = u_1,\ x \in I = (0,1), \tag{2.1c}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}, u_x = \frac{\partial u}{\partial x}$ and $J = (0,T]$ with $T < \infty$. The coefficients a, c are smooth functions of x and a is bounded below by a positive constant say a_0. Moreover, f is a given smooth function which is defined on $(0,1) \times J$.

For H^1-Galerkin mixed finite element procedure, we first split (2.1a) into the following system of two equations

$$u_x = \alpha(x)\sigma, \quad u_{tt} - \sigma_x + cu = f, \tag{2.2}$$

where $\alpha(x) = \frac{1}{a(x)}$.

Denoting by $(\cdot,\cdot)$ the natural inner product on $L^2(I)$, let $H_0^1 = \{v \in H^1(I) : v(0) = v(1) = 0\}$. Further, we use the classical Sobolev spaces $W^{m,p}(I)$, $1 \le p \le \infty$ and call them $W^{m,p}$ with norm $\| \cdot \|_{m,p}$. Now the weak form of equation (2.2) is to determine a pair $\{u, \sigma\} : [0,T] \mapsto H_0^1 \times H^1$ such that

$$(u_x,\, v_x) = (\alpha(x)\sigma,\, v_x)\,, \quad v \in H_0^1, \tag{2.3a}$$

$$(\alpha\sigma_{tt},\, w) + (\sigma_x,\, w_x) = (cu,\, w_x) - (f,\, w_x)\,, \quad w \in H^1. \tag{2.3b}$$

For the first term in (2.3b), we have used integration by parts, the Dirichlet boundary conditions and $u_{tt}(0,t) = u_{tt}(1,t) = 0$.
Let V_h and W_h be finite dimensional subspaces of H_0^1 and H^1, respectively, with the following approximation properties: For $1 \le p \le \infty$, and $k > 0$ integer

$$\inf_{v_h \in V_h} \{\|v - v_h\|_{L^p} + h\|v - v_h\|_{W^{1,p}}\} \le Ch^{k+1}\|v\|_{W^{k+1,p}}, \quad v \in H_0^1 \cap W^{k+1,p}.$$

The above approximation property is also valid for the finite element space W_h with replacing k by r and only requiring $v \in W^{r+1,p}$.

The semidiscrete H^1-Galerkin mixed finite element approximations to (2.3) are defined as $\{u_h, \sigma_h\} : [0,T] \mapsto V_h \times W_h$ satisfying

$$(u_{hx},\, v_{hx}) = (\alpha(x)\sigma_h,\, v_{hx})\,, \quad v_h \in V_h, \tag{2.4a}$$

$$(\alpha\sigma_{htt},\, w_h) + (\sigma_{hx},\, w_{hx}) = (cu_h,\, w_{hx}) - (f,\, w_{hx})\,, \quad w \in W_h \tag{2.4b}$$

with given $(\sigma_h(0), \sigma_{ht}(0))$. Note that (2.4) yields a system of differential algebraic equations (DAEs). Since the stiffness matrix associated with (u_{hx}, v_{hx}) is positve definite, the system of DAEs is of index one. Therefore, the system (2.4a)–(2.4b) is uniquely solvable for a consistent initial condition, see [3].

Following Wheeler [21], we define elliptic projections $\{\tilde{\sigma}_h, \tilde{u}_h\} \in W_h \times V_h$ as

$$A(\sigma - \tilde{\sigma}_h, w_h) = 0 \quad \forall w_h \in W_h, \tag{2.5a}$$

$$(u_x - \tilde{u}_{hx}, v_{hx}) = 0 \quad \forall v_h \in V_h, \tag{2.5b}$$

where $A(w,z) = (w_x,\, z_x) + \lambda\,(w,\, z)$. Here λ is chosen appropriately so that A is H^1- coercive, i.e.,

$$A(z,z) \geq \mu_0 \|z\|_1^2, \;\; z \in H^1,$$

where μ_0 is a positive constant. Moreover, it is not hard to check that $A(.,.)$ is bounded. Let $\rho = \sigma - \tilde{\sigma}_h$ and $\eta = u - \tilde{u}_h$. It is now quite standard to obtain the following estimates for ρ and η:

$$\sum_{i=0}^{2} \|\frac{\partial^i \rho}{\partial t^i}\|_j \leq Ch^{r+1-j} \sum_{i=0}^{2} \|\frac{\partial^i \sigma}{\partial t^i}\|_{r+1}, \; j = 0,1, \tag{2.6a}$$

and

$$\sum_{i=0}^{2} \|\frac{\partial^i \eta}{\partial t^i}\|_j \leq Ch^{k+1-j} \sum_{i=0}^{2} \|\frac{\partial^i u}{\partial t^i}\|_{k+1}, \; j = 0,1. \tag{2.6b}$$

Further, we have for $j = 0, 1$ and $1 \leq p \leq \infty$

$$\|\rho\|_{W^{j,p}} \leq Ch^{r+1-j} \|\sigma\|_{W^{r+1,p}}, \tag{2.7a}$$

and

$$\|\eta\|_{W^{j,p}} \leq Ch^{k+1-j} \|u\|_{W^{k+1,p}}. \tag{2.7b}$$

Note that for $p = \infty$, we require quasiuniformity condition on the finite element mesh.

Semidiscrete error estimates. For *a priori* error estimates, we decompose the errors as $\sigma - \sigma_h := (\sigma - \tilde{\sigma}_h) + (\tilde{\sigma}_h - \sigma_h) = \rho + \xi$ and $u - u_h := (u - \tilde{u}_h) + (\tilde{u}_h - u_h) = \eta + \zeta$. Using (2.3a)- (2.3b), (2.4a)- (2.4b) and auxiliary projections (2.5a) - (2.5b), the equations in ξ and ζ may be written as

$$(\zeta_x,\, v_{hx}) = (\alpha\rho,\, v_{hx}) + (\alpha\xi,\, v_{hx})\,, \;\; v_h \in V_h, \tag{2.8a}$$

and for $w_h \in W_h$

$$(\alpha\xi_{tt},\, w_h) + A\,(\xi,\, w_h) = -\,(\alpha\rho_{tt},\, w_h) + \lambda\,(\rho + \xi,\, w_h) + (c\zeta,\, w_{hx}) + (c\eta,\, w_{hx})\,. \tag{2.8b}$$

Theorem 2.1. *With $\sigma_0 = au_{0x}$ and $\sigma_t(0) = au_{1x}$, assume that $\sigma_h(0) = \tilde{\sigma}_h(0)$ and $\sigma_{ht} = P_h\sigma_t(0)$, where P_h is the L^2 projection defined by $(w - P_h w, w_h) = 0, \; w_h \in W_h$. Then there exists a constant $C > 0$ independent of h, such that for $1 < p \leq \infty$*

$$\begin{aligned} \|(u - u_h)(t)\|_{L^p} \;\; &+ \;\; \|(\sigma - \sigma_h)(t)\|_{L^p} \leq Ch^{min(k+1,r+1)} \left[\|\sigma_t(0)\|_{r+1} + \|u\|_{L^\infty(W^{k+1,p})} \right. \\ &+ \;\; \left. \|u_t\|_{L^1(H^{k+1})} + \|\sigma\|_{L^\infty(W^{r+1,p})} + \|\sigma_{tt}\|_{L^1(H^{r+1})} \right]. \end{aligned}$$

Moreover, the following estmates hold for $u - u_h$ and $\sigma - \sigma_h$ in H^1-norm

$$\begin{aligned} \|(u - u_h)(t)\|_1 \;\; &\leq \;\; Ch^{min(k,r+1)} \left[\|\sigma_t(0)\|_{r+1} + \|u\|_{L^\infty(H^{k+1})} \right. \\ &+ \;\; \left. \|u_t\|_{L^1(H^k)} + \|\sigma\|_{L^1(H^{r+1})} + \|\sigma_{tt}\|_{L^1(H^{r+1})} \right], \end{aligned}$$

and

$$\begin{aligned} \|(\sigma - \sigma_h)(t)\|_1 \;\; &\leq \;\; Ch^{min(k+1,r)} \left[\|\sigma_t(0)\|_r + \|u\|_{L^\infty(H^{k+1})} \right. \\ &+ \;\; \left. \|u_t\|_{L^1(H^{k+1})} + \|\sigma\|_{L^\infty(H^{r+1})} + \|\sigma_{tt}\|_{L^1(H^r)} \right]. \end{aligned}$$

Proof. Since estimates of ρ and η can be found out from (2.6a)–(2.6b), it suffices to estimate ξ and ζ. Choose $w_h = \xi_t$ in (2.8b) to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[\|\alpha^{\frac{1}{2}}\xi_t\|^2 + A\,(\xi,\, \xi)] \;\; &= \;\; -\,(\alpha\rho_{tt},\, \xi_t) + \lambda\,(\rho,\, \xi_t) + \lambda\,(\xi,\, \xi_t) \\ &+ \;\; \frac{d}{dt}\,(c\eta,\, \xi_x) - (c\eta_t,\, \xi_x) - ((c\zeta)_x,\, \xi_t)\,. \end{aligned}$$

On integrating with respect to time from 0 to t and applying Young's inequality, it follows that

$$\begin{aligned}\|\xi_t(t)\|^2 + \|\xi(t)\|_1^2 &\leq C\Big[\|\xi_t(0)\|^2 + \|\eta(t)\|\|\xi(t)\|_1 \\ &+ \int_0^t (\|\rho(s)\| + \|\rho_{tt}(s)\| + \|\zeta(s)\|_1)\,\|\xi_t\|\,ds \\ &+ \int_0^t (\|\eta_t(s)\| + \|\xi_t\|)\,\|\xi(s)\|_1\,ds\Big].\end{aligned}$$

Here, we have used $\xi(0) = 0$ as $\sigma_h(0) = \tilde{\sigma}_h(0)$. Setting $|||\xi(t)|||^2 = \|\xi_t(t)\|^2 + \|\xi(t)\|_1^2$, let $t^* \in [0, t]$ be such that

$$|||\xi(t^*)||| = \max_{0\leq s\leq t} |||\xi(s)|||.$$

Then, we have

$$\begin{aligned}|||\xi(t)||| \leq |||\xi(t^*)||| &\leq C\Big[\|\xi_t(0)\| + \|\eta(t)\| + + \int_0^t (\|\rho(s)\| + \|\rho_{tt}(s)\| \\ &+ \|\zeta(s)\|_1 + \|\eta_t(s)\|)\,ds + \int_0^t |||\xi(s)|||\,ds\Big].\end{aligned}$$

In order to estimate $\|\zeta\|_1$, choose $v_h = \zeta$ in (2.8a) and use Poincaré inequality to obtain

$$\|\zeta\|_1 \leq C\|\zeta_x\| \leq C(\|\rho\| + \|\xi\|). \tag{2.9}$$

Further, it follows from L^2- projection of $\sigma_t(0)$ that $\|\xi_t(0)\| \leq Ch^{r+1}\|\sigma_t(0)\|_{r+1}$. An application of Gronwall's lemma with estimates in (2.6) and (2.9) now yields

$$\begin{aligned}|||\xi(t)||| &\leq Ch^{min(k+1,r+1)}\,[\|\sigma_t(0)\|_{r+1} + \|u\|_{k+1} \\ &+ \int_0^t (\|u_t(s)\|_{k+1} + \|\sigma(s)\|_{r+1} + \|\sigma_{tt}(s)\|_{r+1})\,ds\Big].\end{aligned}$$

Hence, we obtain a superconvergence result for ξ in H^1-norm.

Using Sobolev imbedding theorem, $\|w\|_{L^p} \leq C\|w\|_1, w \in H^1$, for $1 \leq p \leq \infty$. Now apply superconvergence result with estimates (2.7a)–(2.7b) and triangle inequality to complete the L^p estimates of $\sigma - \sigma_h$. Using the above superconvergence result in (2.9), we have from estimates (2.7a)-(2.7b)

$$\begin{aligned}\|\zeta(t)\| \leq C\|\zeta(t)\|_1 &\leq C\quad h^{min(k+1,r+1)}\,[\|\sigma_t(0)\|_{r+1} + \|u(t)\|_{k+1} + \|\sigma(t)\|_{r+1} \\ &+ \int_0^t (\|\sigma_{tt}\|_{r+1} + \|\sigma\|_{r+1} + \|u_t\|_{k+1})\,ds\Big].\end{aligned}$$

Apply the triangle inequality with (2.6a) and (2.6b) to complete the first part of the proof. With appropriate changes in the estimation of $|||\xi(t)|||$ and (2.9), we obtain with the help of triangle inequality the last two estimates and this completes the proof. □

For H^1-estimates, it is possible to choose $\sigma_h(0)$ and $\sigma_{ht}(0)$ as L^2-projection of $\sigma(0)$ and $\sigma_t(0)$, respectively.

Below, we use the nonstandard energy formulation of Baker [2] to prove optimal error estimate in L^2-norm using L^2 projection of initial conditions $\sigma(0)$ and $\sigma_t(0)$. More precisely, we assume that $\sigma_h(0) = P_h\sigma(0)$ and $\sigma_{ht}(0)$ is defined as the weighted L^2 projection of $\sigma_t(0)$,i.e.,

$$(\alpha(\sigma_t(0) - \sigma_{th}(0)), w_h) = 0,\ \ w_h \in W_h. \tag{2.10}$$

For nonstandard formulation, set

$$\hat{\phi} = \int_0^t \phi(\tau)\, d\tau.$$

Then integrating with respect to time (2.5a)-(2.5b) and using the following elliptic projections

$$A(\hat{\rho}, \chi) = 0,\ \ \chi \in W_h, \tag{2.11a}$$

$$(\hat{\eta}_x, v_{hx}) = 0,\ \ v_h \in V_h, \tag{2.11b}$$

we obtain the equations in $\hat{\zeta}$ and $\hat{\xi}$ as

$$\left(\hat{\zeta}_x,\, v_{hx}\right) = (\alpha\hat{\rho},\, v_{hx}) + \left(\alpha\hat{\xi},\, v_{hx}\right),\ \ v_h \in V_h, \tag{2.12a}$$

and

$$(\alpha\xi_t,\, w_h) + A\left(\hat{\xi},\, w_h\right) = -\left(\alpha\rho_t,\, w_h\right) + \lambda\left(\hat{\rho} + \hat{\xi},\, w_h\right) + \left(c\hat{\zeta} + \hat{\eta},\, w_{hx}\right). \tag{2.12b}$$

Note that after integrating (2.8b) with respect to time, the terms at $t = 0$ become zero (see, (2.12b)) because of the weighted L^2- projection defined in (2.10).

Theorem 2.2. *With $\sigma_0 = au_{0x}$ and $\sigma_t(0) = au_{1x}$, assume that $\sigma_h(0) = P_h\sigma(0)$ and σ_{ht} is defined by (2.10). Then there exists a constant $C > 0$ independent of h, such that*

$$\begin{aligned} \|(u - u_h)(t)\| \ &+\ \|(\sigma - \sigma_h)(t)\| \leq Ch^{min(k+1,r+1)}\left[\|\sigma_0\|_{r+1}\right. \\ &+\ \left.\|u\|_{L^\infty(H^{k+1})} + \|\sigma\|_{L^\infty(H^{r+1})} + \|\sigma_t\|_{L^1(H^{r+1})}\right]. \end{aligned}$$

Proof. Choose $w_h = \hat{\xi}_t = \xi$ in (2.12b) to have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[\|\alpha^{\frac{1}{2}}\xi\|^2 + A\left(\hat{\xi},\, \hat{\xi}\right)] \ &=\ -\left(\alpha\rho_t,\, \xi\right) + \lambda\left(\hat{\rho} + \hat{\xi},\, \xi\right) \\ &+\ \frac{d}{dt}\left(c\hat{\eta},\, \hat{\xi}_x\right) - \left(c\eta,\, \hat{\xi}_x\right) - \left((c\hat{\zeta})_x,\, \xi\right). \end{aligned}$$

Setting $|||\hat{\xi}(t)|||^2 = \|\xi(t)\|^2 + \|\hat{\xi}(t)\|_1^2$, let at $t = t^*$

$$|||\hat{\xi}(t^*)||| = \max_{0\leq s\leq t} |||\hat{\xi}(s)|||.$$

Then we have

$$\begin{aligned} |||\hat{\xi}(t)||| \leq |||\hat{\xi}(t^*)||| \ \ &\leq C\ \ \left[\|\alpha^{\frac{1}{2}}\xi(0)\| + \|\hat{\eta}\| + \int_0^t (\|\hat{\rho}(s)\| + \|\rho_t(s)\|\right. \\ &+\ \ \left.\|\hat{\zeta}(s)\|_1 + \|\eta(s)\|)\, ds + \int_0^t |||\hat{\xi}(s)|||\, ds\right]. \end{aligned}$$

Before we apply Gronwall's lemma, we need to estimate $\|\hat{\zeta}\|_1$. Choose $v_h = \hat{\zeta}$ in (2.12a) and use Poincaré inequality to obtain

$$\|\hat{\zeta}\|_1 \leq C\|\hat{\zeta}_x\| \leq C(\|\hat{\rho}\| + \|\hat{\xi}\|). \tag{2.13}$$

An application of Gronwall's lemma with estimates in (2.13) yields

$$|||\hat{\xi}(t)||| \leq C\left[\|\alpha^{\frac{1}{2}}\xi(0)\| + h^{min(k+1,r+1)}\int_0^t (\|u\|_{k+1} + \|\sigma\|_{r+1} + \|\sigma_t\|_{r+1})\ ds\right].$$

Note that $\|\alpha^{\frac{1}{2}}\xi(0)\| \leq C\|\rho(0)\| \leq Ch^{r+1}\|\sigma(0)\|_{r+1}$. Apply the triangle inequality with (2.6a) and (2.6b) to complete the rest of the proof. □

Remarks 2.1. (i) Compared to Theorem 2.1, the L^2-estimates for $u - u_h$ and $\sigma - \sigma_h$ in Theorem 2.2, require less regularity of the exact solution. Further, the initial approximations are implemented as L^2- projections of the initial functions σ_0 and $\sigma_t(0)$ in stead of computationally more expensive elliptic projections.

Again from Theorem 2.1, $\|u-u_h\|_{L^\infty(H^1)} = O(h^k)$, when $k = r+1$, where as $\|\sigma-\sigma_h\|_{L^\infty(H^1)} = O(h^r)$ in case $r = k+1$. Hence, the order of convergence corresponds to the degree of the polynomials used in the corresponding finite element spaces. For one dimensional self-adjoint two point boundary value problem, similar estimates are obtained in Pehlivanov *et al.* [16].

(ii) For C^0-Lagrange elements with $k = 3$ and $r = 1$, the classical mixed finite element method fails, where as the present method converges with order of convergence $O(h^2)$ for $\|u - u_h\|_1$ and $\|\sigma - \sigma_h\|-$ norms.

(iii) Note that the coupling between u and σ is mainly through c-term. When $c = 0$, we obtain from Theorems 2.1 and 2.2

$$\|\sigma - \sigma_h\| \le Ch^{r+1}.$$

Further from Theorem 2.1, we have the superconvergence result

$$\|\xi(t)\|_1 \le Ch^{r+1}.$$

Then a use of Sobolev imbedding theorem yields $\|\xi(t)\|_{L^\infty} \le Ch^{r+1}$, and hence,

$$\|(\sigma - \sigma_h)(t)\|_{L^\infty} \le Ch^{r+1}.$$

In this case, the degree k for V_h does not influence the L^2 and L^∞- estimates of the error $\sigma - \sigma_h$.

(iv) We now compare the above results, i.e., Theorem 2.1 with Geveci [8], Makridakis [9] and Theorem 2.2 with Cowsar *et al.* [6] (when $\alpha = 1$) or with Cowsar *et al.* [5]. In one space dimension, the finite element space V_h consists of piecewise linear polynomials and the space W_h contains C^0- piecewise quadratic elements. From [8] and [9], the following result holds using mixed projections for approximating the initial conditions :

$$\|(\sigma - \sigma_h)(t)\| \le Ch^2 \left[\|u(t)\|_3 + \int_0^t \|u_{tt}(s)\|_2 \, ds \right].$$

Whereas, with C^0-piecewise linear polynomials for both V_h and W_h, we have from Theorem 2.1 the following estimate

$$\|(\sigma - \sigma_h)(t)\| \le Ch^2 \left[\|u_1\|_3 + \|u(t)\|_3 + \int_0^t \|u_{tt}(s)\|_2 \, ds \right].$$

Note that using Theorem 2.2, we derive

$$\|(\sigma - \sigma_h)(t)\| \le Ch^2 \left[\|u_0\|_3 + \|u(t)\|_3 + \int_0^t \|u_t(s)\|_3 \, ds \right].$$

In case $c = 0$, from (iii), we obtain for the present case with W_h as C^0- piecewise quadratic polynomial space

$$\|(\sigma - \sigma_h)(t)\| \le Ch^3 \left[\|u_1\|_4 + \|u(t)\|_4 + \int_0^t \|u_{tt}(s)\|_3 \, ds \right].$$

Moreover, if $c \neq 0$, and both V_h and W_h contain C^0- piecewise quadratic elements, we have from Theorem 2.1

$$\|u - u_h)(t)\| + \|(\sigma - \sigma_h)(t)\| \le Ch^3 \left[\|u_1\|_4 + \|u(t)\|_4 + \int_0^t \|u_{tt}(s)\|_3 \, ds \right].$$

Similarly from Theorem 2.2, it now follows that

$$\|u-u_h)(t)\| + \|(\sigma-\sigma_h)(t)\| \le Ch^3\left[\|u_1\|_4 + \|u(t)\|_4 + \int_0^t \|u_t(s)\|_4\,ds\right].$$

Thus, we obtain better estimates using higher regularity.

3. Hyperbolic Equation in Several Space Variables

In this Section, we apply H^1-Galerkin method to the problem (1.1) in several space dimension. Introducing $\alpha\sigma = \nabla u$ with $\alpha = \frac{1}{a}$, we rewrite (1.1) as a system

$$\alpha\sigma = \nabla u, \tag{3.1a}$$

and

$$u_{tt} - \nabla\cdot\sigma + cu = f. \tag{3.1b}$$

Let $W = \left\{\mathbf{q} \in (L^2(\Omega))^d : \nabla\cdot\mathbf{q} \in L^2(\Omega)\right\}$ with norm $\|\mathbf{q}\|_{\mathbf{H}(\mathrm{div},\Omega)} = (\|\mathbf{q}\|^2 + \|\nabla\cdot\mathbf{q}\|^2)^{\frac{1}{2}}$. Then weak formulation is now defined to be a pair $\{u,\sigma\} : [0,T] \mapsto H_0^1 \times \mathbf{W}$ satisfying

$$(a\nabla u,\, \nabla v) = (\sigma,\, \nabla v)\,, \quad v \in H_0^1, \tag{3.2a}$$

and

$$(\alpha\sigma_{tt},\, \mathbf{w}) + (\nabla\cdot\sigma,\, \nabla\cdot\mathbf{w}) = (cu,\, \nabla\cdot\mathbf{w}) - (f,\, \nabla\cdot\mathbf{w})\,, \quad \mathbf{w} \in \mathbf{W}. \tag{3.2b}$$

For our subsequent use, we employ the classical Hilbert Sobolev spaces $H^m(\Omega)$ and shall call them as H^m with norm $\|\cdot\|_m$. Let $(H^m)^d = \mathbf{H}^m$ denote the corresponding product space with usual product norm. When $m = 0$, we simply write $\mathbf{H}^0$ as $\mathbf{L}^2$. **Semidiscrete H^1- Galerkin mixed finite element procedure.** Let $\mathfrak{T}_h$ be a partition of Ω into a finite number of elements called simplexes, i.e., $\Omega = \cup_{K\in\mathfrak{T}_h} K$ with $h = max\{\,\mathrm{diam}(K) : K \in \mathfrak{T}_h\}$. Let V_h and $\mathbf{W}_h$, respectively, be finite dimensional subspaces of H_0^1 and $\mathbf{W}$ satisfying the following approximation properties: For $k > 0, r > 0$ integers

$$\inf_{v_h\in V_h}\{\|v-v_h\| + h\|v-v_h\|_1\} \le Ch^{k+1}\|v\|_{k+1},\ v \in H^{k+1}\cap H_0^1,$$

and

$$\inf_{\mathbf{q}_h\in\mathbf{W}_h}\left\{\|\mathbf{q}-\mathbf{q}_h\| + h\|\mathbf{q}-\mathbf{q}_h\|_{\mathbf{H}(\mathrm{div};\Omega)}\right\} \le Ch^{r+1}\|\mathbf{q}\|_{r+1},\ \mathbf{q} \in \mathbf{H}^{r+1}.$$

Standard examples of such spaces are as follows

$$V_h = \left\{v_h \in C^0(\Omega) : v_h|_K \in P_k(K), \forall K \in \mathfrak{T}_h, v_h = 0\, on\, \partial\Omega\right\},$$

and

$$\mathbf{W}_h = \{\mathbf{q}_h \in \mathbf{W} : (\mathbf{q}_h)_i|_K \in P_r(K), i = 1, 2, \cdots, d, \forall K \in \mathfrak{T}_h\},$$

where $P_s(K)$ is the space of polynomials of degree $\le s$ on K. Other examples of approximating spaces can be found in Raviart and Thomas [1]. Note that we also allow the use of isoparametric elements.

The semidiscrete H^1- Galerkin finite element procedure for the system is determined as a pair $\{u_h,\sigma_h\} : [0,T] \mapsto V_h \times \mathbf{W}_h$ satisfying

$$(a\nabla u_h,\, \nabla v_h) = (\sigma_h,\, \nabla v_h)\,, \quad v_h \in V_h, \tag{3.3a}$$

and

$$(\alpha\sigma_{htt},\, \mathbf{w}_h) + (\nabla\cdot\sigma_h,\, \nabla\cdot\mathbf{w}_h) = (cu_h,\, \nabla\cdot\mathbf{w}_h) - (f,\, \nabla\cdot\mathbf{w}_h)\,,\ \ \mathbf{w}_h \in \mathbf{W}_h \tag{3.3b}$$

with appropriately chosen initial pair $\{\sigma_h(0), \sigma_{th}(0)\}$ to be defined later.

Again following Wheeler [21], we define Ritz projection $\tilde{u}_h \in V_h$ of u satisfying

$$(\nabla(u-\tilde{u}_h),\, \nabla v_h) = 0, \quad \forall v_h \in V_h.$$

Further, let $\tilde{\sigma}_h \in \mathbf{W}_h$ denote a standard finite element interpolant of σ. Let $\rho = \sigma - \tilde{\sigma}_h$ and $\eta = u - \tilde{u}_h$. Then, for nonnegative integers k and r

$$\|\eta\| + h\|\nabla\eta\| \le Ch^{k+1}\|u\|_{k+1}, \tag{3.4a}$$

and

$$\|\rho\| + h\|\rho\|_{H(\mathrm{div},\Omega)} \le Ch^{r+1}\|\sigma\|_{r+1}. \tag{3.4b}$$

Let $\sigma - \sigma_h = \sigma - \tilde{\sigma}_h + \tilde{\sigma}_h - \sigma_h = \rho + \xi$ and $u - u_h = u - \tilde{u}_h + \tilde{u}_h - u_h = \eta + \zeta$. For semidiscrete error analysis, we have from (3.3)-(3.4) and auxiliary projections the equations in ξ and ζ as

$$(\nabla\zeta,\, \nabla v_h) = (\alpha(\sigma-\sigma_h),\, \nabla v_h)\,,\ v_h \in V_h, \tag{3.5a}$$

and

$$\begin{aligned}(\alpha\xi_{tt},\, \mathbf{w}_h) + (\nabla\cdot\xi,\, \nabla\cdot\mathbf{w}_h) &= -(\alpha\rho_{tt},\, \mathbf{w}_h) - (\nabla\cdot\rho,\, \nabla\cdot\mathbf{w}_h) \\ &+ (c(\eta+\zeta),\, \nabla\cdot\mathbf{w}_h)\,,\ w_h \in \mathbf{W}_h.\end{aligned} \tag{3.5b}$$

In the following, we obtain semidiscrete error estimates for $\sigma - \sigma_h$ and $u - u_h$.

Theorem 3.1. *With $\sigma_0 = a\nabla u_0$ and $\sigma_t(0) = a\nabla u_1$, assume that*

$$\|\sigma_0 - \sigma_{0h}\|_{H(\mathrm{div},\Omega)} \le Ch^r\|\sigma_0\|_{r+1},$$

and $\sigma_{ht}(0)$ is either L^2- projection or interpolant of $\sigma_t(0)$. Then there is a constant C independent of h such that

$$\begin{aligned}\|(u-u_h)(t)\| &+ \|(\sigma-\sigma_h)(t)\|_{H(\mathrm{div},\Omega)} \le Ch^{min(k+1,r)}\Big[\|\sigma_0\|_{r+1} + \|\sigma_t(0)\|_r \\ &+ \|u_0\|_{k+1} + \int_0^t (\|u_t\|_{k+1} + \|\sigma\|_r + \|\sigma_t\|_{r+1} + \|\sigma_{tt}\|_r)\, ds\Big].\end{aligned} \tag{3.6}$$

Further, using nonstandard formulation and $\sigma_{ht}(0)$ as the weighted L^2- projection of $\sigma_t(0)$ that is $(\alpha(\sigma_t(0) - \sigma_{ht}(0)), \mathbf{w}_h) = 0,\ \mathbf{w}_h \in \mathbf{W}_h$, the following estimate holds:

$$\begin{aligned}\|(\sigma-\sigma_h)(t)\| &\le Ch^{min(k+1,r)}\Big[\|\sigma_0\|_r + \|\sigma(t)\|_r \\ &+ \int_0^t (\|\sigma(s)\|_{r+1} + \|\sigma_t(s)\|_r + \|u(s)\|_{k+1})\ ds\Big].\end{aligned} \tag{3.7}$$

Proof. Choose $v_h = \zeta$ in (3.5a) and use Poincaré inequality to obtain

$$\|\zeta\|_1 \le C\|\nabla\zeta\| \le C(\|\rho\| + \|\xi\|). \tag{3.8}$$

Further, setting $\mathbf{w}_h = \xi_t$ in (3.5b), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\alpha^{\frac{1}{2}}\xi_t\|^2 &+ \|\nabla\cdot\xi\|^2] = -(\alpha\rho_{tt}, \xi_t) - \frac{d}{dt}(\nabla\cdot\rho - c\eta,\, \nabla\cdot\xi) \\ &+ (\nabla\cdot\rho_t - c\eta_t,\, \nabla\cdot\xi) - (\nabla(c\zeta),\, \xi_t)\,.\end{aligned}$$

For the last term, we have used Gauss divergence theorem and $\zeta \in H^1_0$. Integrate the above equation with respect to time and set $|||\xi(t)|||^2 = \|\xi_t(t)\|^2 + \|\nabla\cdot\xi(t)\|^2$.

Then for $t = t^*$, assume that $|||\xi(t^*)||| = \max_{0\le\tau\le t} |||\xi(\tau)|||$ and apply Cauchy Schwarz inequality to obtain

$$\begin{aligned}|||\xi(t)||| \le |||\xi(t^*)||| &\le C \left[\|\xi_t(0)\| + \|\xi(0)\|_{H(\text{div},\Omega)} + \|\rho(0)\|_{H(\text{div},\Omega)} + \|\eta(0)\|\right] \\ &+ C\Big[\|\rho(t)\|_{H(\text{div},\Omega)} + \|\eta(t)\| + \int_0^t (\|\rho_t(s)\|_{H(\text{div},\Omega)} \\ &+ \|\rho_{tt}(s)\| + \|\zeta(s)\|_1 + \|\eta_t(s)\|)\, ds] + C\int_0^t |||\xi(s)|||\, ds.\end{aligned}$$

Using (3.8), an application of Gronwall's lemma yields

$$\begin{aligned}|||\xi(t)||| &\le Ch^{min(k+1,r)} [\|\sigma_0\|_{r+1} + \|\sigma_t(0)\|_r + \|u_0\|_{k+1} \\ &+ \left.\int_0^t (\|u_t(s)\|_{k+1} + \|\sigma(s)\|_r + \|\sigma_t(s)\|_{r+1} + \|\sigma_{tt}(s)\|_r)\, ds\right].\end{aligned}$$

The triangle inequality with (3.4a) and (3.4b) now completes the estimate of $\sigma - \sigma_h$. For the estimation of $u - u_h$, we now note that $\xi(t) = \xi(0) + \int_0^t \xi_t(s)\, ds$ and hence, we have an estimation of $\|\xi(t)\|$ using the above inequality. From (3.4b)– (3.8), it is straight forward to obtain an estimation for $\|\zeta(t)\|_1$ which is of order $O(h^{\min(k+1,r)})$. Again a use of the triangle inequality completes the proof of the first part.

For the second part, we now rewrite the equation in $\hat{\zeta}$ and $\hat{\xi}$ as

$$\left(\nabla\hat{\zeta},\, \nabla v_h\right) = (\alpha(\hat{\sigma} - \hat{\sigma_h}),\, \nabla v_h)\,,\ v_h \in V_h, \tag{3.9a}$$

and

$$\begin{aligned}(\alpha\xi_t,\, \mathbf{w}_h) &+ \left(\nabla\cdot\hat{\xi},\, \nabla\cdot\mathbf{w}_h\right) = -\,(\alpha\rho_t,\, \mathbf{w}_h) - (\nabla\cdot\hat{\rho},\, \nabla\cdot\mathbf{w}_h) \\ &+ \left(c(\hat{\eta} + \hat{\zeta}),\, \nabla\cdot\mathbf{w}_h\right),\ w_h \in \mathbf{W}_h.\end{aligned} \tag{3.9b}$$

With a choice of $v_h = \hat{\zeta}$ in (3.9a), it follows that

$$\|\hat{\zeta}\|_1 \le C\|\nabla\hat{\zeta}\| \le C(\|\hat{\rho}\| + \|\hat{\xi}\|). \tag{3.10}$$

Moreover, choose $\mathbf{w}_h = \xi = \hat{\xi}_t$ in (3.9b) to find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\alpha^{\frac{1}{2}}\xi\|^2 &+ \|\nabla\cdot\hat{\xi}\|^2] = -(\alpha\rho_t, \xi) - \frac{d}{dt}\left(\nabla\cdot\hat{\rho} - c\hat{\eta},\, \nabla\cdot\hat{\xi}\right) \\ &+ \left(\nabla\cdot\rho - c\eta,\, \nabla\cdot\hat{\xi}\right) - \left(\nabla(c\hat{\zeta}),\, \xi\right).\end{aligned}$$

Assume that $|||\hat{\xi}(t^*)||| = \max_{0\le\tau\le t} |||\hat{\xi}(\tau)|||$, for some $\tau = t^* \in [0, t]$ and apply Cauchy Schwarz inequality to obtain

$$\begin{aligned}|||\hat{\xi}(t)||| &\le |||\hat{\xi}(t^*)||| \le C\Big[\|\xi(0)\| + \|\hat{\rho}(t)\|_{H(\text{div},\Omega)} + \|\hat{\eta}(t)\| \\ &+ \int_0^t \left(\|\rho_t\| + \|\rho\|_{H(\text{div},\Omega)} + \|\eta\| + \|\hat{\zeta}\|_1\right) ds\Big] + C\int_0^t |||\hat{\xi}(s)|||\, ds.\end{aligned}$$

Using (3.10), an application of Gronwall's lemma yields

$$\|\xi(t)\| \le Ch^{min(k+1,r)}\left[\|\sigma_0\|_r + \int_0^t (\|u(s)\|_{k+1} + \|\sigma_t(s)\|_r + \|\sigma(s)\|_{r+1})\, ds\right].$$

The triangle inequality with (3.4a) and (3.4b), now completes the rest of the proof. □

Remarks 3.1 (i) Estimates (3.6) indicates that for $k = r$, the error estimate $\|\nabla\cdot(\sigma - \sigma_h)(t)\|$ is optimal in the stated norm. However for $k \ne r$, this estimate

distinguishes the better approximation properties of V_h or $\mathbf{W}_h$. When $k+1=r$ in (3.7), we can decrease the influence of V_h on the rate of convergence for σ_h.
(ii) We now compare (3.7) with the order of convergence of the mixed finite element method presented in [5]–[6]. Assume that$(V_h, \mathbf{W}_h)$is a pair of Raviart Thomas spaces. Then components of $\mathbf{W}_h$ contains incomplete polynomials of degree $r = k+1$ on each finite element $K \in \mathfrak{T}_h$. From [5]–[6], we obtain the following estimate

$$\|(u-u_h)(t)\| + \|(\sigma-\sigma_h)(t)\| \leq C(\sigma, p)h^r.$$

Hence, the rate of convergence coincides with (3.7) when $k+1=r$, but for the present method the LBB condition has been avoided. Moreover, quasi-uniformity assumption is not required in our analysis.
(iii) Compared to the results in this Section, the results obtained in one dimensional situation (see, Section 2) are quite sharp in the sense that the estimate of the stress in $L^\infty(L^2)$-norm is optimal. The main reason for this is that in Section 2, we have used elliptic projection (Wheeler's technique) for the stress (see, (2.5a)), whereas in case of several spatial variables interpolant $\tilde{\sigma}$ of the stress σ is used as an intermediate projection. Therefore, we have only derived optimal estimates for $\|\nabla\cdot(\sigma-\sigma_h)\|$ and suboptimal estimates in L^2-norm. However, if we choose the finite dimensional space $\mathbf{W}_h$ as one of the Raviart-Thomas spaces RT_r with index r (i.e., the components of $\mathbf{W}_h$ consists of incomplete polynomials of degree $r+1$ on each finite elements) or Brezzi-Douglas-Marini spaces of index r, i.e., BDM_r, see [1] for both RT and BDM spaces, it is possible to improve the $L^\infty(L^2(\Omega))$-estimate of $\sigma-\sigma_h$. Now in stead of using finite element interpolant as an auxiliary function, set $\tilde{\sigma}_h = \Pi_h\sigma$, where the Raviart-Thomas projection $\Pi_h\sigma : \mathbf{H}(\text{div};\Omega) \mapsto \mathbf{W}_h$ is defined by

$$(\nabla\cdot(\sigma-\Pi_h\sigma), \nabla\cdot\mathbf{w}_h) = 0, \qquad \mathbf{w}_h \in \mathbf{W}_h. \tag{3.11}$$

With $\rho = \sigma - \Pi_h\sigma$, the following error estimates hold, see ([1],[19]):

$$\|\rho(t)\| \leq Ch^{r+1}\|\sigma(t)\|_{r+1},$$

and

$$\|\nabla\cdot\rho(t)\| \leq Ch^{r+1}\|\nabla\cdot\sigma(t)\|_{r+1} \leq Ch^{r+1}\|\sigma(t)\|_{r+2}.$$

Since Π_h commutes with the time derivatives, we obtain

$$\|\rho_t(t)\| \leq Ch^{r+1}\|\sigma_t(t)\|_{r+1}.$$

Now write

$$\sigma-\sigma_h = (\sigma-\Pi_h\sigma) + (\Pi_h\sigma-\sigma_h) := \rho+\xi.$$

The term $-(\nabla\cdot\rho, \nabla\cdot\mathbf{w}_h)$ in $(3.5b)$ now vanishes and hence, the equation in ξ can be written as

$$\begin{aligned}(\alpha\xi_{tt}, \mathbf{w}_h) + (\nabla\cdot\xi, \nabla\cdot\mathbf{w}_h) &= -(\alpha\rho_{tt}, \mathbf{w}_h)\\ &+ (c(\eta+\zeta), \nabla\cdot\mathbf{w}_h), \; w_h \in \mathbf{W}_h.\end{aligned} \tag{3.12}$$

Proceeding exactly as in the proof of Theorem 3.1, and using the initial approximation $\sigma_h(0) = \Pi_h\sigma(0)$, we, finally, obtain as compared to (3.6)

$$\begin{aligned}\|(u-u_h)(t)\| &+ \|(\sigma-\sigma_h)(t)\| \leq Ch^{\min(k+1,r+1)}\left(\|\sigma_0\|_{r+1} + \|\sigma_t(0)\|_{r+1} + \|u_0\|_{k+1}\right.\\ &+ \left.\|u_t\|_{L^1(H^{k+1})} + \|\sigma\|_{L^1(H^{r+1})} + \|\sigma_{tt}\|_{L^1(H^{r+1})}\right).\end{aligned}$$

Moreover, we estimate $\sigma-\sigma_h$ in $H(\text{div}\,;\Omega)$ as

$$\begin{aligned}\|(\sigma-\sigma_h)(t)\|_{H(\text{div},\Omega)} &\leq Ch^{\min(k+1,r+1)}\left(\|\sigma_0\|_{r+1} + \|u_0\|_{k+1}\right.\\ &+ \left.\|\sigma\|_{L^1(H^{r+2})} + \|u_t\|_{L^1(H^{k+1})} + \|\sigma_{tt}\|_{L^1(H^{r+1})}\right).\end{aligned}$$

Using non-standard energy formulation, it is easy to derive the following error estimate:

$$\begin{aligned}\|(\sigma-\sigma_h)(t)\| &\le Ch^{\min(k+1,r+1)}\left(\|\sigma_0\|_{r+1}\right.\\ &+ \left.\|u_t\|_{L^1(H^{k+1})}+\|\sigma\|_{L^1(H^{r+1})}+\|\sigma_t\|_{L^1(H^{r+1})}\right).\end{aligned}$$

Note that the $L^\infty(L^2)$-estimate of $\sigma-\sigma_h$ is optimal in the stated norm if $k=r$ and this is achieved, provided we use $\mathbf{W}_h$ as the Raviart-Thomas spaces of index r or the BDM spaces of index r. However, it is possible to use other classical mixed finite element spaces ([1]) for approximating σ that preserves L^2-optimality for the error $\sigma-\sigma_h$.

4. Modified H^1- Galerkin Mixed Finite Element Procedure

In this Section, we propose a modified Galerkin method to obtain optimal estimates for the stress in $L^\infty(L^2)$-norm without restricting the finite element space $\mathbf{W}_h$.

With $\alpha\sigma=\nabla u$, write (1.1) as

$$\begin{aligned} u_{tt}-\nabla\cdot\sigma+cu=f,\ (x,t)\in\Omega\times J,\\ \nabla\times(\alpha\sigma)=0,\ (x,t)\in\Omega\times J,\\ (\mathbf{n}\wedge\alpha\sigma)=0,\ (x,t)\in\partial\Omega\times J,\\ u(0)=u_0,\quad u_t(0)=u_1,\quad x\in\Omega,\end{aligned}$$

where $\mathbf{n}$ is the outward normal and $\wedge$ denotes the exterior product.

For the weak formulation, let $\mathbf{W}=\{\mathbf{w}\in(H^1)^d:\mathbf{n}\wedge\alpha\mathbf{w}=0 \text{ on } \partial\Omega,\ d=2,3\}$. Using Gauss divergence theorem, we now seek a pair $\{u,\sigma\}:[0,T]\mapsto H_0^1\times\mathbf{W}$ such that

$$(\nabla u,\nabla v)=(\alpha\sigma,\nabla v),\ \ v\in H_0^1, \tag{4.1a}$$

$$(\alpha\sigma_{tt},\mathbf{w})+A(\sigma,\mathbf{w})=(-f+cu,\nabla\cdot\mathbf{w}),\ \mathbf{w}\in\mathbf{H}^1, \tag{4.1b}$$

where

$$A(\boldsymbol{\phi},\mathbf{w})=(\nabla\cdot\boldsymbol{\phi},\nabla\cdot\mathbf{w})+(\nabla\times\alpha\boldsymbol{\phi},\nabla\times\alpha\mathbf{w}).$$

For Galerkin procedure, we consider a finite dimensional space V_h as in Section 3 and then define

$$\begin{aligned}\mathbf{W}_h &= \{\mathbf{w}_h\in C(\bar\Omega)^d:(\mathbf{w}_h)_i\,|_K\in P_r(K),\ i=1,\cdots,d,\ \forall K\in\mathfrak{T}_h,\\ &\quad (\mathbf{n}\wedge\alpha\mathbf{w}_h)=0,\ \text{at the nodes on}\,\partial\Omega\}.\end{aligned}$$

Since $(\mathbf{n}\wedge\alpha\mathbf{w}_h)=0$ only at the boundary nodes, the finite element space $\mathbf{W}_h$ is not a subspace of $\mathbf{W}$ and hence, it results in a non-conforming method. Note that the above finite dimensional spaces satisfy the same approximation properties as in Section 3 (see, [11] for $r=1$ and [15]). A modified H^1- Galerkin mixed finite element approximation is determined as a pair $\{u_h,\sigma_h\}:[0,T]\mapsto V_h\times\mathbf{W}_h$ such that

$$(\nabla u_h,\nabla v_h)=(\alpha\sigma_h,\nabla v_h),\ \ v\in V_h, \tag{4.2a}$$

$$(\alpha\sigma_{htt},\mathbf{w}_h)+A(\sigma_h,\mathbf{w}_h)=(-f+cu_h,\nabla\cdot\mathbf{w}_h),\ \ \mathbf{w}_h\in\mathbf{W}_h, \tag{4.2b}$$

with $\sigma_h(0)$ and $\sigma_{ht}(0)$to be defined later.

Now define auxiliary projections $\{\tilde u_h,\tilde\sigma_h\}\in V_h\times\mathbf{W}_h$ as

$$(\nabla(u-\tilde u_h),\nabla v_h)=0,\ v_h\in V_h, \tag{4.3a}$$

$$A_1(\sigma-\tilde\sigma_h,\mathbf{w}_h)=0,\ \ \mathbf{w}_h\in\mathbf{W}_h, \tag{4.3b}$$

where

$$A_1(\boldsymbol{\phi}, \mathbf{w}_h) = A(\boldsymbol{\phi}, \mathbf{w}_h) + (\boldsymbol{\phi}, \mathbf{w}_h).$$

When the domain Ω is convex or the boundary $\partial\Omega$ is of class $C^{1,1}$ or Ω is a curvilinear polygon (or polytope) of class $C^{1,1}$ with no concave angles, then there is a positive constant μ_0 independent of h such that following estimate holds

$$\left(\|\mathbf{q}_h\|^2_{\mathbf{H}(\mathrm{div},\Omega)} + \|\nabla \times (\alpha \mathbf{q}_h)\|^2\right) \geq \mu_0 \|\mathbf{q}_h\|_1^2,$$

for all $\mathbf{q}_h \in \mathbf{W}_h$ and for small h, see pp. 509-510 of [15]. Thus, $A_1(\cdot,\cdot)$ satisfies the coercivity condition

$$A_1(\boldsymbol{\phi}_h, \boldsymbol{\phi}_h) \geq \mu_0 \|\boldsymbol{\phi}_h\|_1^2, \quad \boldsymbol{\phi}_h \in \mathbf{W}_h.$$

Let $u - \tilde{u}_h = \eta$ and $\sigma - \tilde{\sigma}_h = \rho$. With an appropriate modification of the analysis of Pehlivanov and Carey [15], the following estimates for ρ and its temporal derivatives are easy to derive.

$$\sum_{l=0}^{2} \|\frac{\partial^l \rho}{\partial t^l}\|_j \leq C h^{r+1-j} \sum_{l=0}^{2} \|\frac{\partial^l \sigma}{\partial t^l}\|_{r+1}, \quad j = 0, 1. \tag{4.4}$$

Note that the related difficulties with non-conforming finite element method will mainly show up in the error estimates of ρ.

For semidiscrete error estimates, we now split the errors $u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h) = \eta + \zeta$ and $\sigma - \sigma_h = (\sigma - \tilde{\sigma}_h) + (\tilde{\sigma}_h - \sigma_h) = \rho + \xi$. Below, we state and prove our main theorem in this section.

Theorem 4.1. *Assume that $\sigma_h(0) = \tilde{\sigma}_h(0)$ with $\sigma(0) = a\nabla u_0$ so that $\xi \equiv 0$. Further, let $\sigma_{ht}(0)$ be L^2-projection of $\sigma_t(0)$, where $\sigma_t(0) = a\nabla u_1$. Then there exists a positive constant C independent of h such that*

$$\begin{aligned}\|(u-u_h)(t)\| &+ \|(\sigma-\sigma_h)(t)\| + h\|(u-u_h)(t)\|_1 \leq Ch^{min(k+1,r+1)}\left[\|\sigma_t(0)\|_{r+1}\right.\\ &+ \left.\|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^1(H^{k+1})} + \|\sigma\|_{L^\infty(\mathbf{H}^{r+1})} + \|\sigma_{tt}\|_{L^1(\mathbf{H}^{r+1})}\right].\end{aligned}$$

Further, the following estimate holds

$$\begin{aligned}\|(\sigma-\sigma_h)(t)\|_1 \leq \quad C \quad & h^{min(k+1,r)}\left[\|\sigma_t(0)\|_r + \|u\|_{L^\infty(H^{k+1})} + \|\sigma\|_{L^\infty(\mathbf{H}^{r+1})}\right.\\ + \quad & \left.\|u_t\|_{L^1(H^{k+1})} + \|\sigma_{tt}\|_{L^1(\mathbf{H}^r)}\right].\end{aligned}$$

Proof. From (4.1)-(4.3) we have

$$(\nabla\zeta, \nabla v_h) = (\alpha(\rho+\xi), \nabla v_h), \quad v_h \in V_h, \tag{4.5a}$$

and for $\mathbf{w}_h \in \mathbf{W}_h$,

$$(\alpha\xi_{tt}, \mathbf{w}_h) + A_1(\xi, \mathbf{w}_h) = -(\alpha\rho_{tt}, \mathbf{w}_h) + (\rho+\xi, \mathbf{w}_h) + (c(\zeta+\eta), \nabla\cdot\mathbf{w}_h). \tag{4.5b}$$

Choose $v_h = \zeta$ in (4.5a) and use $\|\zeta\| \leq C\|\nabla\zeta\|$ as $\zeta \in H_0^1$ to obtain

$$\|\zeta\|_1 \leq C\|\nabla\zeta\| \leq C(\|\rho\| + \|\xi\|). \tag{4.6}$$

Further, setting $\mathbf{w}_h = \xi_t$ in (4.5b), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\alpha^{\frac{1}{2}}\xi_t\|^2 &+ A_1(\xi,\xi)] = -(\alpha\rho_{tt}, \xi_t) + \frac{d}{dt}(c\eta, \nabla\cdot\xi)\\ &- (c\eta_t, \nabla\cdot\xi) + (\rho+\xi, \xi_t) - (\nabla(c\zeta), \xi_t).\end{aligned}$$

On integration with respect to time, we now define $|||\xi(t)|||^2 = \|\xi_t(t)\|^2 + \|\xi(t)\|_1^2$. Assume that there is some $t^* \in [0,t]$ such that $|||\xi(t^*)||| = \max_{0\le\tau\le t} |||\xi(\tau)|||$. Then apply Cauchy Schwarz inequality to obtain

$$\begin{aligned} |||\xi(t)||| \le |||\xi(t^*)||| \quad &\le C \quad \Big[\|\xi_t(0)\| + \|\eta(t)\| + \int_0^t (\|\rho(s)\| + \|\rho_{tt}(s)\| \\ &+ \quad \|\zeta(s)\|_1 + \|\eta_t(s)\|)\, ds] + C\int_0^t |||\xi(s)|||\, ds. \end{aligned}$$

Using (4.6), an application of Gronwall's lemma yields

$$\begin{aligned} |||\xi(t)||| \quad &\le \quad Ch^{min(k+1,r+1)}\,[\|\sigma_t(0)\|_{r+1} + \|u(t)\|_{k+1} \\ &+ \quad \int_0^t (\|u_t(s)\|_{k+1} + \|\sigma(s)\|_{r+1} + \|\sigma_{tt}(s)\|_{r+1})\, ds\Big]. \end{aligned}$$

The triangle inequality with (3.4a)–(3.4b), now completes the L^2- estimates of $\sigma - \sigma_h$. For the estimation of $u - u_h$, note that from the superconvergence result for ξ in H^1-norm, we obtain an estimation of $\|\xi(t)\|$. From (4.4)– (4.8), it is straight forward to obtain an estimation for $\|\zeta(t)\|_1$ which is of order $O(h^{\min(k+1,r+1)})$. Again a use of the triangle inequality completes the proof of the estimate $\|u - u_h\|_1$. Finally, an appropriate modification of the estimate $|||\xi|||$ with $\|\rho\|_1$ completes the H^1-estimate of $\sigma - \sigma_h$. □

Below, we again recall the nonstandard energy formulation of Baker [2] and prove optimal error estimate in L^2-norm using L^2-projection of the initial conditions. More precisely, we shall assume that $\sigma_h(0) = P_h\sigma(0)$ and $\sigma_{ht}(0)$ is defined as weighted L^2 projection of $\sigma_t(0)$

$$(\alpha(\sigma_t(0) - \sigma_{th}(0)), \mathbf{w}_h) = 0,\ \mathbf{w}_h \in \mathbf{W}_h. \tag{4.7}$$

Integrating with respect to time (4.5a)-(4.5b) and using the following elliptic projections

$$A_1(\hat\rho, \mathbf{w}_h) = 0,\ \mathbf{w}_h \in \mathbf{W}_h, \tag{4.8a}$$

$$(\nabla\hat\eta,\, \nabla v_h) = 0,\ v_h \in V_h, \tag{4.8b}$$

we obtain the equations in $\hat\zeta$ and $\hat\xi$ as

$$\left(\nabla\hat\zeta,\, \nabla v_h\right) = \left(\alpha(\hat\rho + \hat\xi),\, \nabla v_h\right),\ v_h \in V_h, \tag{4.9a}$$

and for $\mathbf{w}_h \in \mathbf{W}_h$

$$(\alpha\xi_t,\, \mathbf{w}_h) + A_1\left(\hat\xi,\, \mathbf{w}_h\right) = -\,(\alpha\rho_t,\, \mathbf{w}_h) + (\hat\rho + \hat\xi, \mathbf{w}_h) + \left(c(\hat\zeta + \hat\eta),\, \nabla\cdot\mathbf{w}_h\right). \tag{4.9b}$$

Theorem 4.2. *With $\sigma_0 = a\nabla u_0$ and $\sigma_t(0) = a\nabla u_1$, assume that $\sigma_h(0) = P_h\sigma(0)$ and $\sigma_{ht}(0)$ is defined by (4.7). Then there exists a constant $C > 0$ independent of h, such that*

$$\begin{aligned} \|(u - u_h)(t)\| \quad &+ \quad \|(\sigma - \sigma_h)(t)\| \le Ch^{min(k+1,r+1)}\,[\|\sigma_0\|_{r+1} + \|u\|_{k+1} + \|\sigma\|_{r+1} \\ &+ \quad \int_0^t (\|u\|_{k+1} + \|\sigma(s)\|_{r+1} + \|\sigma_t(s)\|_{r+1})\, ds\Big]. \end{aligned}$$

Proof. Setting $v_h = \hat\zeta$ in (4.9a), it follows from Poincaré inequality that

$$\|\hat\zeta\|_1 \le C\|\nabla\hat\zeta\| \le C(\|\hat\rho\| + \|\hat\xi\|). \tag{4.10}$$

Moreover, choose $\mathbf{w}_h = \xi = \hat{\xi}_t$ in (3.8b) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\alpha^{\frac{1}{2}}\xi\|^2 &+ A_1(\hat{\xi},\hat{\xi})] = -(\alpha\rho_t,\, \xi) + \frac{d}{dt}\left(c\hat{\eta},\, \nabla\cdot\hat{\xi}\right) \\ &- \left(c\eta,\, \nabla\cdot\hat{\xi}\right) + + \left(\hat{\rho}+\hat{\xi},\, \xi\right) - \left(\nabla(c\hat{\zeta}),\, \xi\right).\end{aligned}$$

For some $t^* \in [0,t]$, assume that $|||\hat{\xi}(t^*)||| = \max_{0\le\tau\le t} |||\hat{\xi}(\tau)|||$ and apply Cauchy Schwarz inequality to obtain

$$\begin{aligned}|||\hat{\xi}(t)||| &\le |||\hat{\xi}(t^*)||| \le C[\|\xi(0)\| + \|\hat{\eta}(t)\| \\ &+ \int_0^t \left(\|\rho_t\| + \|\hat{\rho}\| + \|\eta(s)\| + \|\hat{\zeta}(s)\|_1\right) ds] + C\int_0^t |||\hat{\xi}(s)|||\, ds.\end{aligned}$$

Using (4.10), an application of Gronwall's lemma yields

$$\|\xi(t)\| \le Ch^{min(k+1,r+1)}\left[\|\sigma_0\|_{r+1} + \int_0^t \left(\|u(s)\|_{k+1} + \|\sigma_t(s)\|_{r+1} + \|\sigma(s)\|_{r+1}\right) ds\right].$$

The triangle inequality with (4.3a)–(4.3b), now completes the rest of the proof. □

Remarks 4.1. (i) With $k = r$, we have $\|u - u_h\|_{L^\infty(L^2)} + \|\sigma - \sigma_h\|_{L^\infty((L^2)^d)} = O(h^{r+1})$.

(ii) Compared to [5] – [6], the present analysis yields an optimal estimate of $\sigma - \sigma_h$ in $L^\infty(L^2)$-norm (with respect to the approximation property of the finite lement spaces) under higher regularity assumption on the exact solution. If $c = 0$ in (3.1), we have again $\|\sigma - \sigma_h\| = O(h^{r+1})$ even if $k < r$.

(iii) When $d = 2$, i.e., $\Omega \subset R^2$ and $\sigma_h(0) = \tilde{\sigma}_h(0)$, then using Sobolev imbedding theorem and superconvergence estimates for $\nabla\zeta$ and $\nabla\xi$, (see, the proof of the Theorem 4.1) we have

$$\begin{aligned}\|\xi(t)\|_{L^\infty} &\le Ch^{min(k+1,r+1)}(\log\frac{1}{h})^{\frac{1}{2}}\left[\|\sigma_t(0)\|_{r+1} + \|u\|_{L^\infty(H^{k+1})}\right. \\ &+ \left.\|\sigma\|_{L^1(\mathbf{H}^{r+1})} + \|u_t\|_{L^1(H^{k+1})} + \|\sigma_{tt}\|_{L^1(\mathbf{H}^{r+1})}\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\|\zeta(t)\|_{L^\infty} &\le Ch^{min(k+1,r+1)}(\log\frac{1}{h})^{\frac{1}{2}}\left[\|\sigma_t(0)\|_{r+1} + \|u\|_{L^\infty(H^{k+1})}\right. \\ &+ \left.\|\sigma\|_{L^\infty(\mathbf{H}^{r+1})} + \|u_t\|_{L^1(H^{k+1})} + \|\sigma_{tt}\|_{L^1(\mathbf{H}^{r+1})}\right].\end{aligned}$$

Therefore, quasi-optimal maximum norm estimates for $(u - u_h)$ and $(\sigma - \sigma_h)$ can be obtained, provided L^∞-estimates for $(u - \tilde{u}_h)$ and $(\sigma - \tilde{\sigma}_h)$ are available. Note that for C^0-piecewise linear elements, i.e., for $k = 1$, the following estimate

$$\|(u - \tilde{u}_h)(t)\|_{L^\infty} \le Ch^2 \log(\frac{1}{h})\|u(t)\|_{W^{2,\infty}},$$

holds true (see, [20]) under an additional assumption that the finite element mesh satisfies quasi-uniformity condition. Compared to [8] and [9], the above additional result is derived for the proposed modified method.

5. Fully Discrete Scheme

In this Section, we briefly describe a fully discrete scheme for approximating a pair of solutions $\{u, \sigma\}$ of (1.1) and discuss *a priori* error bounds.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a given partition of the time interval [0,T] with step length $\Delta t = T/N$, for some positive integer N. We use the following notation related to functions defined at discrete time levels. For a smooth function ϕ on [0,T], let

$$\phi^n = \phi(t_n), \quad \phi^{n+\frac{1}{2}} = \frac{1}{2}(\phi^{n+1} + \phi^n), \quad \partial_t \phi^{n+\frac{1}{2}} = \frac{\phi^{n+1} - \phi^n}{\Delta t},$$

$$\partial_t^2 \phi^n = \frac{\partial_t \phi^{n+\frac{1}{2}} - \partial_t \phi^{n-\frac{1}{2}}}{\Delta t} \text{ and } \phi^{n;\frac{1}{4}} = \frac{1}{4}(\phi^{n+1} + 2\phi^n + \phi^{n-1}) = \frac{1}{2}(\phi^{n+\frac{1}{2}} + \phi^{n-\frac{1}{2}}).$$

Let U^n and $\mathbf{Z}^n$, respectively, be the approximations of u and σ at $t = t_n$ which we define through the following implicit scheme. We now determine a sequence of pairs $\{U^n, \mathbf{Z}^n\} \in V_h \times \mathbf{W}_h,\ n = 0, \cdots, N$, satisfying

$$(5.1a) \qquad (U^0, v_h) = (u_0, v_h), \ \ (\alpha \mathbf{Z}^0, \mathbf{w}_h) = (\alpha\sigma(0), \mathbf{w}_h), \ \ \mathbf{w}_h \in \mathbf{W}_h,$$

(5.1b)
$$(\frac{2\alpha}{\Delta t}\partial_t \mathbf{Z}^{\frac{1}{2}}, \mathbf{w}_h) + A(\mathbf{Z}^{\frac{1}{2}}, \mathbf{w}_h) = (-f^{\frac{1}{2}} + cU^{\frac{1}{2}}, \nabla\cdot\mathbf{w}_h) + (\frac{2\alpha}{\Delta t}\sigma_t(0), \mathbf{w}_h), \ \ \mathbf{w}_h \in \mathbf{W}_h,$$

$$(5.1c) \qquad (\nabla U^{n+\frac{1}{2}}, \nabla v_h) = (\alpha \mathbf{Z}^{n+\frac{1}{2}}, \nabla v_h), \ \ v \in V_h, \quad n \geq 0,$$

$$(5.1d) \ \ (\alpha\partial_t^2 \mathbf{Z}^n, \mathbf{w}_h) + A(\mathbf{Z}^{n;\frac{1}{4}}, \mathbf{w}_h) = (-f^{n;\frac{1}{4}} + cU^{n;\frac{1}{4}}, \nabla\cdot\mathbf{w}_h), \ \ \mathbf{w}_h \in \mathbf{W}_h, \ \ n \geq 1.$$

Here, $\sigma(0) = a\nabla u_0$ and $\sigma_t(0) = a\nabla u_1$. Existence of a unique pair of solution $\{U^n, Z^n\}$ of the above resulting linear system is a straightforward consequence of the coercivity of $A(\cdot,\cdot)$ and bounded below property of α.

For the fully discrete error estimates, we split the errors $u(t_n) - U^n := (u(t_n) - \tilde{u}_h(t_n)) + (\tilde{u}_h(t_n) - U^n) = \eta^n + \zeta^n$ and $\sigma(t_n) - \mathbf{Z}^n := (\sigma(t_n) - \tilde{\sigma}_h(t_n)) + (\tilde{\sigma}_h(t_n) - \mathbf{Z}^n) = \rho^n + \xi^n$. Since the estimates of η^n and ρ^n can be found out easily from (3.4a) and (4.4) at $t = t_n$. it is enough to estimate ζ^n and ξ^n. Note that the equations in ζ^n and ξ^n may be written as

$$(5.2a) \qquad (\nabla\zeta^{n+\frac{1}{2}}, \nabla v_h) = (\alpha(\rho^{n+\frac{1}{2}} + \xi^{n+\frac{1}{2}}), \nabla v_h), \ \ v_h \in V_h, \ \ n \geq 0,$$

$$\begin{aligned}(\frac{2\alpha}{\Delta t}\partial_t \xi^{\frac{1}{2}}, \mathbf{w}_h) + A_1(\xi^{\frac{1}{2}}, \mathbf{w}_h) \ &= \ -(\frac{2\alpha}{\Delta t}\partial_t\rho^{\frac{1}{2}}, \mathbf{w}_h) - (2\tau^0, \mathbf{w}_h) + (\xi^{\frac{1}{2}} + \rho^{\frac{1}{2}}, \mathbf{w}_h) \\ (5.2b) \qquad\qquad &+ \ (c(\zeta^{\frac{1}{2}} + \eta^{\frac{1}{2}}), \nabla\cdot\mathbf{w}_h), \ \ \mathbf{w}_h \in \mathbf{W}_h,\end{aligned}$$

and

$$\begin{aligned}(\alpha\partial_t^2\xi^n, \mathbf{w}_h) \ &+ \ A_1(\xi^{n;\frac{1}{4}}, \mathbf{w}_h) = -(\alpha\partial_t^2\rho^n, \mathbf{w}_h) - (\tau^n, \mathbf{w}_h) + (\xi^{n;\frac{1}{4}} + \rho^{n;\frac{1}{4}}, \mathbf{w}_h) \\ (5.2c) \qquad &+ \ (c(\zeta^{n;\frac{1}{4}} + \eta^{n;\frac{1}{4}}), \nabla\cdot\mathbf{w}_h), \ \mathbf{w}_h \in \mathbf{W}_h, \ n \geq 1,\end{aligned}$$

where $\tau^0 = \alpha[\frac{1}{2}\sigma_{tt}^{\frac{1}{2}} + \frac{1}{\Delta t}(\sigma_t(0) - \partial_t\sigma^{\frac{1}{2}})]$ and $\tau^n = (\sigma_{tt})^{n;\frac{1}{4}} - \partial_t^2\sigma(t_n)$.

Now define $\hat{\xi}^0 = 0$ and $\hat{\xi}^n = \Delta t\sum_{j=0}^{n-1}\xi^{j+\frac{1}{2}}$. Then $\Delta t\partial_t\hat{\xi}^{n+\frac{1}{2}} = \xi^{n+\frac{1}{2}}$, $\xi^{n;\frac{1}{4}} = (\xi^{n+\frac{1}{2}} + \xi^{n-\frac{1}{2}})/2$ and $\Delta t\sum_{j=1}^{J}\xi^{j;\frac{1}{4}} = \hat{\xi}^{J+\frac{1}{2}} - \frac{\Delta t}{2}\xi^{\frac{1}{2}}$. Below, we prove the main theorem of this Section.

Theorem 5.1. *Assume that $\mathbf{Z}^0$ satisfies (5.1a). Then there exists a positive constant C independent of h and Δt such that for small Δt*

$$\begin{aligned}\max_{0\leq J\leq N}\|\sigma(t_J) - \mathbf{Z}^J\| \ &\leq Ch^{min(k+1,r+1)}\big[\|\sigma_0\|_{r+1} + \|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^\infty(H^{k+1})} \\ &+ \|\sigma\|_{L^\infty(\mathbf{H}^{r+1})} + \|\sigma_t\|_{L^2(\mathbf{H}^{r+1})}\big] + C(\Delta t)^2\sum_{l=0}^{4}\|\frac{\partial^l\sigma}{\partial t^l}\|_{L^1(\mathbf{L}^2)}.\end{aligned}$$

Proof. Multiplying (5.2c) by Δt and then summing from $n = 1$ to m, we obtain using (5.2b)

$$\begin{aligned}(\alpha \partial_t \xi^{m+\frac{1}{2}}, \mathbf{w}_h) \quad &+ \quad A_1(\hat{\xi}^{m+\frac{1}{2}}, \mathbf{w}_h) = -(\alpha \partial_t \rho^{m+\frac{1}{2}}, \mathbf{w}_h) - \Delta t \sum_{n=0}^{m} (\tau^n, \mathbf{w}_h)\\ &+ \quad (c(\hat{\zeta}^{m+\frac{1}{2}}), \nabla \cdot \mathbf{w}_h) + (c(\hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \mathbf{w}_h)\\ &+ \quad (\hat{\xi}^{m+\frac{1}{2}} + \hat{\rho}^{m+\frac{1}{2}}, \mathbf{w}_h), \quad \mathbf{w}_h \in \mathbf{W}_h.\end{aligned}$$

Choose $\mathbf{w}_h = \xi^{m+\frac{1}{2}} = \Delta t \partial_t \hat{\xi}^{m+\frac{1}{2}}$ in the above equation, and apply Gauss divergence theorem for the last but one term on the right hand side to have

$$\begin{aligned}(\alpha \partial_t \xi^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}}) \quad &+ \quad \Delta t A_1(\hat{\xi}^{m+\frac{1}{2}}, \partial_t \hat{\xi}^{m+\frac{1}{2}}) = -(\alpha \partial_t \rho^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}}) - \Delta t \sum_{n=0}^{m} (\tau^n, \xi^{m+\frac{1}{2}})\\ &- \quad (\nabla(c\hat{\zeta}^{m+\frac{1}{2}}), \xi^{m+\frac{1}{2}}) + \Delta t (c(\hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \partial_t \hat{\xi}^{m+\frac{1}{2}})\\ &+ \quad \Delta t (\hat{\xi}^{m+\frac{1}{2}} + \hat{\rho}^{m+\frac{1}{2}}, \partial_t \hat{\xi}^{m+\frac{1}{2}}).\end{aligned} \tag{5.3}$$

For the first two terms on the left hand side, we note that

$$(\alpha \partial_t \xi^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}}) = \frac{1}{2\Delta t} \left[\|\alpha^{\frac{1}{2}} \xi^{m+1}\|^2 - \|\alpha^{\frac{1}{2}} \xi^m\|^2 \right],$$

and

$$A_1(\hat{\xi}^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}}) = \frac{1}{2\Delta t} [A_1(\hat{\xi}^{m+1}, \hat{\xi}^{m+1}) - A_1(\hat{\xi}^m, \hat{\xi}^m)].$$

In order to estimate the last term on the right hand side of (5.3), we rewrite it as

$$(c(\hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \partial_t \hat{\xi}^{m+\frac{1}{2}}) = \partial_t (c(\hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \hat{\xi}^{m+\frac{1}{2}}) - (c(\partial_t \hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \hat{\xi}^{m+\frac{1}{2}}),$$

and hence, summing from $m = 0$ to J with $J + 1 \leq N$

$$\begin{aligned}\Delta t \sum_{m=0}^{J} (c(\hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \partial_t \hat{\xi}^{m+\frac{1}{2}}) \quad &= \quad (c(\hat{\eta}^{J+\frac{1}{2}}), \nabla \cdot \hat{\xi}^{J+\frac{1}{2}}) - (c(\hat{\eta}^{\frac{1}{2}}), \nabla \cdot \hat{\xi}^{\frac{1}{2}})\\ &- \quad \Delta t \sum_{m=0}^{J} (c(\partial_t \hat{\eta}^{m+\frac{1}{2}}), \nabla \cdot \hat{\xi}^{m+\frac{1}{2}}).\end{aligned}$$

Now, let

$$|||\xi|||_{0;J+1} = \max_{0 \leq n \leq J+1} |||\xi^n|||,$$

where

$$|||\xi^n|||^2 = \|\xi^n\|^2 + \Delta t \|\hat{\xi}^n\|_1^2.$$

Multiplying (5.3) by Δt and again summing from $m = 0$ to $m = J$, we find using coercivity condition for $A(\cdot, \cdot)$ that

$$\begin{aligned}|||\xi^{J+1}|||^2 \quad &\leq \quad C(a_0, \mu_0)[\|\xi^0\| + \|\hat{\eta}^{J+\frac{1}{2}}\| + \|\hat{\eta}^{\frac{1}{2}}\| + (\Delta t)^2 \sum_{m=0}^{J} (\sum_{n=0}^{m} \|\tau^n\|)\\ &+ \quad \Delta t \sum_{m=0}^{J} (\|\partial_t \rho^{m+\frac{1}{2}}\| + \|\hat{\rho}^{m+\frac{1}{2}}\| + \|\partial_t \hat{\eta}^{m+\frac{1}{2}}\|)\\ &+ \quad \Delta t \sum_{m=0}^{J} \|\hat{\zeta}^{m+\frac{1}{2}}\|] |||\xi|||_{0;J+1},\end{aligned}$$

and hence,

$$|||\xi^{J+1}|||^2 \leq C(a_0,\mu_0)[\|\xi^0\| + \|\hat{\eta}^{J+\frac{1}{2}}\| + \|\hat{\eta}^{\frac{1}{2}}\| + \Delta t \sum_{m=0}^{J} \|\tau^m\|$$

$$+ \Delta t \sum_{m=0}^{J} (\|\partial_t \rho^{m+\frac{1}{2}}\| + \|\hat{\rho}^{m+\frac{1}{2}}\| + \|\partial_t \hat{\eta}^{m+\frac{1}{2}}\|) + \Delta t \sum_{m=0}^{J} \|\hat{\zeta}^{m+\frac{1}{2}}\|]. \tag{5.4}$$

Note that

$$\Delta t \sum_{m=0}^{J} \|\partial_t \rho^{m+\frac{1}{2}}\| \leq \|\rho_t\|_{L^\infty(\mathbf{L^2})} \leq Ch^{r+1}\|\sigma_t\|_{L^\infty(\mathbf{H}^{r+1})}.$$

Observe that for $m \geq 1$, $\tau^m = (\sigma_{tt})^{m;\frac{1}{4}} - \partial_t^2 \sigma^m$ may be rewritten as

$$\tau^m = \frac{1}{12}\int_{-\Delta t}^{\Delta t} (|s| - \Delta t)(3 - 2(1 - \frac{|s|}{\Delta t})^2)\frac{\partial^4 \sigma}{\partial t^4}(t_m + s)\, ds,$$

and, therefore,

$$\|\tau^m\| \leq C\Delta t \int_{t_{m-1}}^{t_{m+1}} \|\frac{\partial^4 \sigma}{\partial t^4}\|\, ds.$$

Further, for $m = 0$

$$\|\tau^0\| \leq C\Delta t \|\frac{\partial^3 \sigma}{\partial t^3}\|_{L^\infty(0,t_{\frac{1}{2}},\mathbf{L}^2)} \leq C\Delta t \left(\|\frac{\partial^3 \sigma}{\partial t^3}\|_{L^1(\mathbf{L}^2)} + \|\frac{\partial^4 \sigma}{\partial t^4}\|_{L^1(\mathbf{L}^2)}\right).$$

Thus, we obtain

$$\Delta t \sum_{m=0}^{J} \|\tau^m\| \leq C(\Delta t)^2 \left(\|\frac{\partial^3 \sigma}{\partial t^3}\|_{L^1(\mathbf{L}^2)} + \|\frac{\partial^4 \sigma}{\partial t^4}\|_{L^1(\mathbf{L}^2)}\right).$$

For the estimation of the last term on the right hand side of (5.4), we now sum (5.2a) from $n = 0$ to $n = m$ and then set $v_h = \hat{\zeta}^{m+1}$ to have

$$\|\hat{\zeta}^{m+1}\|_1 \leq C\|\nabla \hat{\zeta}^{m+1}\| \leq C\left(\|\hat{\rho}^{m+1}\| + \|\hat{\xi}^{m+1}\|\right).$$

Here, we have used Poincaré inequality. Since $\|\xi^0\| \leq \|\sigma(0) - \mathbf{Z}^0\| + \|\sigma(0) - \tilde{\sigma}_h(0)\|$, it follows that

$$\|\xi^0\| \leq Ch^{r+1}\|\sigma(0)\|_{r+1}.$$

On substituting the above estimates in (5.4) and replacing $\|\hat{\xi}^{m+1}\|$ by $|||\xi|||_{0;m+1}$, we easily conclude that

$$(1 - C\Delta t)|||\xi|||_{0;J+1} \leq C\left\{(\Delta t)^2 \sum_{l=0}^{4} \|\frac{\partial^l \sigma}{\partial t^l}\|_{L^1(\mathbf{L}^2)} + h^{r+1}\left(\|\sigma_0\|_{r+1} + \|\sigma\|_{L^\infty(\mathbf{H}^{r+1})}\right.\right.$$
$$\left. + \|\sigma_t\|_{L^\infty(\mathbf{H}^{r+1})}\right) + h^{k+1}\left(\|u\|_{L^\infty(\mathbf{H}^{k+1})} + \|u_t\|_{L^\infty(\mathbf{H}^{k+1})}\right)$$
$$\left. + \sum_{m=0}^{J} |||\xi|||_{0;m}\right\}.$$

Choose Δt so that $(1 - C\Delta t) > 0$. An application of discrete Gronwall's lemma completes the estimate of $\|\xi^{J+1}\|$. Finally, a use of triangle inequality completes the rest of the proof. □

Remarks 5.1. (i) Setting $v_h = \zeta^{n+\frac{1}{2}}$ in (5.2a), we obtain a superconvergence result for $\zeta^{n+\frac{1}{2}}$ in H^1-norm. Then use of triangle inequality yields

$$\|u(t_{n+\frac{1}{2}}) - U^{n+\frac{1}{2}}\|_j \leq C(u,\sigma)h^{\min(k+1-j,r+1)}, \quad j = 0, 1.$$

If one desires in stead approximation of u at $t = t_n$, it is clear that $\hat{U}^n = (U^{n+\frac{1}{2}} + U^{n-\frac{1}{2}})/2$ furnishes such an approximation which is of same order of accuracy.
(ii) To estimate $\|\xi\|_1$, choose $\mathbf{w}_h = \partial_t \xi^{n+1/2} + \partial_t \xi^{n-1/2}$ in (5.2c). Then use identities

$$A_1(\xi^{n;\frac{1}{4}}, \partial_t \xi^{n+1/2} + \partial_t \xi^{n-1/2}) = \frac{1}{4\Delta t}\left[A_1(\xi^{n+1}, \xi^{n+1}) - A_1(\xi^n, \xi^n)\right],$$

$$(\partial_t^2 \xi^n, \partial_t \xi^{n+1/2} + \partial_t \xi^{n-1/2}) = \frac{1}{\Delta t}\left[\|\partial_t \xi^{n+1/2}\|^2 - \|\partial_t \xi^{n-1/2}\|^2\right]$$

and the standard energy arguments similar to one described above will now yield an eatimate for $\xi^{n+\frac{1}{2}}$ in H^1-norm. Again apply triangle inequality to complete the error estimate of the stress in H^1-norm.

References

[1] Brezzi, F. and Fortin,M., Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[2] Baker, G. A., *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal., 13 (1976), pp. 564–576.

[3] Brennan, K. E., Campbell, S. L. and Petzold, L. R., Numerical Solution of Initial Value Problems in Differential-Algebraic Equations, Elsevier, 1989.

[4] Carey, G. F. and Shen, Y., *Convergence studies of least-square finite elements for first order systems*, Comm. Appl. Numer. Methods, 5 (1989), pp.427–434.

[5] Cowsar, L. C., Dupont, T. F. and Wheeler, M. F., *A priori estimates for mixed finite element methods for the wave equation*, Comp. Methods Appl. Mech. Engrg., 82 (1990), pp. 205–222.

[6] Cowsar, L. C., Dupont, T. F. and Wheeler, M. F., *A priori estimates for mixed finite element approximations of second-order hyperbolic equations with absorbing boundary conditions*, SIAM J. Numer. Anal., 33 (1996), pp. 492–504.

[7] Fix, G. J., Gunzburger, M. D. and Nicolaides, R. A., *On the mixed finite element methods for first order elliptic systems*, Numer. Math., 37 (1981), pp.29–48.

[8] Geveci, T., *On the application of mixed finite element methods to the wave equation*, Math.Model Numer. Anal., 22 (1988), pp.243–250.

[9] Makridakis, C. G., *On mixed finite element methods for linear elastodynamics*, Numer. Math., 61 (1992), pp. 235–260.

[10] Neittaanmäki, P. and Saranen, J., *On finite element approximation of the gradient for solution of Poisson equation*, Numer. Math.,37 (1981), pp.333–337.

[11] Neittaanmäki, P. and Saranen, J., *On the finite element approximation of vector fields by curl and divergence*, Math. Methods Appl. Sci.,3 (1981), pp.328–335.

[12] Pani, A. K., *An H^1- Galerkin mixed finite element method for parabolic partial differential equations*, SIAM J. Numer. Anal. 25 (1998), pp. 712–727.

[13] Pani, A. K. and Fairweather, G., *H^1-Galerkin mixed finite element methods for parabolic integro-differential equations*, IMA J. Numer. Anal. 22 (2002), pp. 231–252.

[14] Pani, A. K. and Fairweather, G., *An H^1-Galerkin mixed finite element method for an evolution equation with a positive type memory term*, SIAM J. Numer. Anal. 40 (2002), pp. 1475-1490.

[15] Pehlivanov, A. I. and G. F. Carey, G. F., *Error estimates for least-squares mixed finite elements*, M^2AN Mathematical Modeling and Numerical Analysis, 28 (1994), pp. 499-516.

[16] Pehlivanov, A. I., G. F. Carey, G. F., Lazarov, R. D. and Shen, Y., *Convergence analysis of least-square mixed finite elements*, Computing, 51 (1993), pp. 111–123.

[17] Pehlivanov, A. I., G. F. Carey, G. F. and Lazarov, R. D., *Least-square mixed finite elements for second order elliptic problems*, SIAM J. Numer. Anal., 31 (1994), pp. 1368–1377.

[18] Pehlivanov, A. I., Carey, G. F. and Vassilevski, P. S., *Least-square mixed finite elements for non-selfadjoint elliptic problems: I. Error estimates*, Numer. Math., 72 (1996), pp. 501–522.

[19] Raviart, P. A. and Thomas, J. M., A mixed finite element method for second order elliptic problems,Lecture Notes in Mathematics, Vol. 606, Springer Verlag, 1977, pp. 293–315.

[20] Scott, R., *Optimal L^∞ estimates for the finite element method on irregular meshes*, Math. Comp., 30 (1976), pp. 681–697.

[21] Wheeler, M. F., *A priori L^2-error estimates for Galerkin approximations to parabolic differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–749.

Industrial Mathematics Group, Department of Mathematics, Indian Institute of Technology, Bombay, Powai, Mumbai- 400 076, India
E-mail: akp@math.iitb.ac.in

Department of Mathematics, Indian Institute of Technology, Guwahati, North Guwahati, Guwahati - 781039, India

Department of Mathematics, Government College, Rourkela, Rourkela-769 005, Orissa, India