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A SUPERCONVERGENT FINITE ELEMENT SCHEME FOR THE REISSNER-MINDLIN PLATE BY PROJECTION METHODS

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Abstract. The Reissner-Mindlin model is frequently used by engineers for plates and shells of small to moderate thickness. This model is well known for its "locking" phenomenon so that many numerical approximations behave poorly when the thickness parameter tends to zero. Following the formulation derived by Brezzi and Fortin, we construct a new finite element scheme for the Reissner-Mindlin model using L^2 projections onto appropriately-chosen finite element spaces. A superconvergence result is established for the new finite element solutions by using the L^2 projections. The superconvergence is based on some regularity assumption for the Reissner-Mindlin model and is applicable to any stable finite element methods with regular but non-uniform finite element partitions.

Key Words. finite element methods, superconvergence, the method of least-squares fitting, Reissner-Mindlin plate.

1. Introduction

The Reissner-Mindlin plate is a mathematical model that is frequently used by engineers for plates and shells of small to moderate thickness. To describe the model, we consider a plate or a shell of thickness $t > 0$. Let Ω be the region occupied by the plate. Denote by $w = w(x, y)$ and $\boldsymbol{\phi} = (\phi_1, \phi_2)^t$ the transverse deflection of Ω and the rotation of the fibers normal to Ω, respectively. The Reissner-Mindlin plate model determines w and $\boldsymbol{\phi}$ as the unique solution to the following variational problem: find $(w, \boldsymbol{\phi}) \in H_0^1(\Omega)\times[H_0^1(\Omega)]^2$ such that for all $(v, \boldsymbol{\psi}) \in H_0^1(\Omega)\times[H_0^1(\Omega)]^2$

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \nabla w, \boldsymbol{\psi} - \nabla v) = (g, v), \tag{1.1}$$

where g is the scaled transverse loading function, $\lambda = Ek/2(1 + \nu)$ is the shear modulus with E the Young's modulus, ν the Poisson ratio, k the shear correction factor. The symbol ∇ denotes the gradient operator. $H^1(\Omega)$ is the Sobolev space defined by

$$H^1(\Omega) = \left\{v : \quad v \in L^2(\Omega), \nabla v \in [L^2(\Omega)]^2\right\}.$$

Here $L^2(\Omega)$ is the set of square integrable functions over the domain Ω with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions with vanishing boundary value. The bilinear form $a(\cdot, \cdot)$ in (1.1) is given by

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \frac{E}{12(1 - \nu^2)} \int_\Omega [(1 - \nu)\boldsymbol{\epsilon}(\boldsymbol{\phi}) : \boldsymbol{\epsilon}(\boldsymbol{\psi}) + \nu\nabla \cdot \boldsymbol{\phi}\nabla \cdot \boldsymbol{\psi}],$$

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where $\nabla\cdot$ is the divergence operator, $\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \frac{1}{2}[\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\phi}^t]$, and $-1 < \nu < \frac{1}{2}$.

An obvious numerical procedure for the Reissner-Mindlin model would be a Galerkin finite element method based on the weak formulation (1.1) in which w and $\boldsymbol{\phi}$ are both approximated by continuous piecewise polynomials over a prescribed finite element partition $\mathcal{T}^h$ of Ω. However, such schemes are known to have "locking" difficulty in that the resulting numerical approximations behave poorly when the thickness parameter t tends to zero. Many researchers have been working on the Reissner-Mindlin model by aiming at designing efficient and "locking free" numerical schemes. Among a few of successes, we mention the work of Brezzi and Fortin [3] who derived a formulation for the Reissner-Mindlin model by introducing two variables (the irrotational and solenoidal parts of the transverse shear strain) in addition to the primitive variables (the transverse displacement and the rotation vector) and developed a finite element method which is locking free. Inspired by the work of Brezzi and Fortin, Arnold and Falk [1] developed an efficient triangular element for the Reissner-Mindlin model in the primitive variables using the $P1$ nonconforming linear element for the transverse displacement and conforming linear element with bubbles for the rotation to the Reissner-Mindlin model. They proved that the method converges with an optimal order uniformly with respect to the thickness. For more literature, the reader is referred to [5] [6], [2], [4], [9] and references therein.

The objective of this paper is to propose and analyze a modified scheme for the Brezzi-Fortin method [3], which will yield numerical approximations for the Reissner-Mindlin plate model with high order of accuracy. There are two challenges to this task: (1) modification of the Brezzi-Fortin's method, and (2) tedious analysis for the postprocessing projection method of Wang [11]. Our result has potential impact in practical computation for Reissner-Mindlin model in that it can provide an efficient a posteriori error estimator for adaptive grid local refinement.

2. The Brezzi-Fortin Formulation and Approximation

We first introduce some standard notations. Denote by $H^m(\Omega)$ for any integer $m \geq 0$ the Sobolev space:

$$H^m(\Omega) = \left\{v :\ \partial_x^{\alpha_1}\partial_y^{\alpha_2} v \in L^2(\Omega), \alpha_i \geq 0, \alpha_1 + \alpha_2 \leq m\right\}$$

with norm given by

$$\|v\|_s = \left(\sum_{\alpha_1+\alpha_2\leq m} \|\partial_x^{\alpha_1}\partial_y^{\alpha_2} v\|^2 \right)^{\frac{1}{2}}.$$

For non-integer values of m, $H^m(\Omega)$ is defined via the standard interpolation method. Let $\mathcal{D}(\Omega)$ be the linear space of infinitely differentiable functions with compact support on Ω. As usual, $H_0^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_s$. For any function $\boldsymbol{\phi} \in H_0^1(\Omega)$, denote its curl by

$$\nabla\times\boldsymbol{\phi} = \partial_2\phi_1 - \partial_1\phi_2.$$

Denote by $\nabla^\perp$ the formal adjoint of $\nabla\times$ given by

$$\nabla^\perp p = \begin{pmatrix} -\partial_2 p \\ \partial_1 p \end{pmatrix}, \quad p \in H^1(\Omega).$$

Following [1], without loss of generality we may assume that $\lambda = 1$ and

$$a(\boldsymbol{\phi},\, \boldsymbol{\psi}) = (\nabla\boldsymbol{\phi},\, \nabla\boldsymbol{\psi}).$$

The formulation of Brezzi and Fortin [3] for the problem (1.1) was based on the following Helmholtz decomposition of the shear strain vector:

$$t^{-2}(\nabla w - \boldsymbol{\phi}) = \nabla r + \nabla^{\perp} p, \tag{2.1}$$

with $(r, p) \in H_0^1(\Omega) \times H^1(\Omega)/R$. Through a careful calculation, Brezzi and Fortin [3, 1] proved that the solution w and $\boldsymbol{\phi}$ to the Reissner-Mindlin plate model (1.1) can be obtained from the following sequential procedure:

(1) *Find* $r \in H_0^1(\Omega)$ *such that*

$$(\nabla r,\, \nabla s) = (g, s), \tag{2.2}$$

for all $s \in H_0^1(\Omega)$.

(2) *Find* $(\boldsymbol{\phi}, p) \in [H_0^1(\Omega)]^2 \times H^1(\Omega)/R$ *such that*

$$(\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\psi}) - (\nabla^{\perp} p, \boldsymbol{\psi}) = (\nabla r, \boldsymbol{\psi}), \tag{2.3}$$

$$-(\boldsymbol{\phi}, \nabla^{\perp} q) - t^2(\nabla^{\perp} p, \nabla^{\perp} q) = 0, \tag{2.4}$$

for all $\boldsymbol{\psi} \in [H_0^1(\Omega)]^2$ *and* $q \in H^1(\Omega)/R$.

(3) *Find* $w \in H_0^1(\Omega)$ *such that*

$$(\nabla w, \nabla u) = (\boldsymbol{\phi} + t^2 \nabla r, \nabla u). \tag{2.5}$$

for all $u \in H_0^1(\Omega)$.

Observe that the equation in the third step can be reduced to

$$(\nabla w, \nabla u) = (\boldsymbol{\phi}, \nabla u) + t^2(g, u). \tag{2.6}$$

Based on the above sequential formulation, Brezzi and Fortin proposed and analyzed a finite element method which can be described as follows. Let $\mathcal{T}^h$ be a finite element partition of the domain Ω with mesh size h. Assume that the partition $\mathcal{T}^h$ is quasi-uniform; i.e., it is regular and satisfies the inverse assumption [7]. Denote by P_k the space of polynomials of degree no more than k. Let P_T be a finite dimensional space defined on $T \in \mathcal{T}^h$ such that $P_{k_2} \subset P_T \subset C^1(T)$. Associated with the partition $\mathcal{T}^h$, define

$$\begin{aligned}
M_1 &= \{u : \ \ u \in H_0^1(\Omega),\ u|_T \in P_{k_1},\ \forall\, T \in \mathcal{T}^h\},\\
M_2 &= \{\boldsymbol{\psi} : \ \ \boldsymbol{\psi} \in H_0^1(\Omega)^2,\ \boldsymbol{\psi}|_T \in P_T \times P_T,\ \forall\, T \in \mathcal{T}^h\},\\
M_3 &= \{q : \ \ q \in H^1(\Omega),\ \ q|_T \in P_{k_2},\ \ \forall T \in \mathcal{T}^h\},\\
M_4 &= \{u : \ \ u \in H_0^1(\Omega),\ u|_T \in P_{k_3},\ \forall\, T \in \mathcal{T}^h\}.
\end{aligned}$$

For numerical stability consideration, assume that the finite element spaces M_2 and M_3 satisfy the following *inf-sup* condition:

$$\sup_{\eta \in M_2} \frac{(\nabla^{\perp} q, \eta)}{\|\eta\|_1} \ge \beta^* \|q\|, \quad q \in M_3, \tag{2.7}$$

where $\beta^* > 0$ is an absolute constant independent of the mesh parameter h.

The corresponding finite element method of Brezzi and Fortin for the problems (2.2)-(2.5) can be stated as follows:

(1) *Find* $\bar{r}_h \in M_1$ *such that*

$$(\nabla \bar{r}_h,\, \nabla s) = (g, s), \tag{2.8}$$

for all $s \in M_1$.

(2) *Find* $(\bar{\boldsymbol{\phi}}_h, \bar{p}_h) \in M_2 \times M_3/R$ *such that*

$$(\nabla \bar{\boldsymbol{\phi}}_h, \nabla \boldsymbol{\psi}) - (\nabla^{\perp} \bar{p}_h, \boldsymbol{\psi}) = (\nabla \bar{r}_h, \boldsymbol{\psi}), \tag{2.9}$$

$$-(\bar{\boldsymbol{\phi}}_h, \nabla^{\perp} q) - t^2 (\nabla^{\perp} \bar{p}_h, \nabla^{\perp} q) = 0, \tag{2.10}$$

for all $\boldsymbol{\psi} \in M_2$ *and* $q \in M_3/R$.

(3) *Find* $\bar{w}_h \in M_4$ *such that*

$$(\nabla \bar{w}_h, \nabla u) = (\bar{\boldsymbol{\phi}}_h + t^2 \nabla \bar{r}_h, \nabla u). \tag{2.11}$$

for all $u \in M_4$.

The numerical approximation obtained from the above finite element procedure has the following error estimates:

Theorem 2.1. *Let* $(r, \boldsymbol{\phi}, p, w)$ *and* $(\bar{r}_h, \bar{\boldsymbol{\phi}}_h, \bar{p}_h, \bar{w}_h)$ *be the solutions of (2.2)-(2.5) and (2.8)-(2.11) respectively. Assume that the* inf-sup *condition (2.7) is satisfied for the finite element spaces* M_2 *and* M_3. *Then, there there exists a constant* C *independent of* h *such that*

$$\|r - \bar{r}_h\|_1 \le C \inf_{u \in M_1} \|r - u\|_1, \tag{2.12}$$

$$\begin{aligned} \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}_h\|_1 + \|p - \bar{p}_h\| + t\|p - \bar{p}_h\|_1 \quad &\le C \left(\inf_{\boldsymbol{\psi} \in M_2} \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_1 \right. \\ &\left. + t \inf_{q \in M_3} \|p - q\|_1 + \inf_{q \in M_3} \|p - q\| + \|\nabla(r - \bar{r}_h)\| \right), \end{aligned} \tag{2.13}$$

and

$$\|w - \bar{w}_h\|_1 \le C \left(\inf_{v \in M_4} \|w - v\|_1 + \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}_h\| + t^2 \|\nabla(r - \bar{r}_h)\| \right). \tag{2.14}$$

For a detailed proof, readers are referred to either the original paper by Brezzi and Fortin [3] or a subsequent discussion by Arnold and Falk [1].

For the purpose of error analysis, we state a regularity result for the solution of a general problem which seeks $(r, \boldsymbol{\phi}, p, w) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times H^1(\Omega)/R \times H_0^1(\Omega)$ such that

$$(\nabla r, \nabla s) = (g, s) \qquad \forall s \in H_0^1(\Omega), \tag{2.15}$$

$$(\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\psi}) - (\nabla^{\perp} p, \boldsymbol{\psi}) = (f, \boldsymbol{\psi}) \qquad \forall \boldsymbol{\psi} \in [H_0^1(\Omega)]^2, \tag{2.16}$$

$$-(\boldsymbol{\phi}, \nabla^{\perp} q) - t^2 (\nabla^{\perp} p, \nabla^{\perp} q) = (\kappa, q) \qquad \forall q \in H^1(\Omega)/R, \tag{2.17}$$

$$(\nabla w, \nabla u) = (\ell, u) \qquad \forall u \in H_0^1(\Omega). \tag{2.18}$$

It is not hard to see the following regularity result [3, 1].

Proposition 2.1. *Let* r, ϕ, p, w *be the solution of (2.15)-(2.18). Assume that the domain* Ω *is sufficiently regular. Then, there exists a constant* C *such that*

$$\begin{aligned} \|r\|_s + \|\boldsymbol{\phi}\|_s + \|p\|_{s-1} + t\|p\|_s + \|w\|_s \le C \left(\|g\|_{s-2} \right. \\ \left. + \|f\|_{s-2} + \|\kappa\|_{s-1} + \|\ell\|_{s-2} \right), \end{aligned} \tag{2.19}$$

for any fixed $s \ge 1$, *provided that the data* g, f, κ, *and* ℓ *are smooth enough so that the right-hand side of (2.19) is well-defined.*

3. A Remark to the Brezzi-Fortin Formulation

The Brezzi-Fortin scheme (2.8)-(2.11) is based on the Brezzi-Fortin formulation (2.2)-(2.5) for the Reissner-Mindlin plate model. As pointed out in the previous section, the equation (2.5) can be replaced by (2.6) in the reformulation of the Reissner-Mindlin plate problem, yielding a variational form that does not explicitly depend upon the variable r. Consequently, the last step (2.11) in the Brezzi-Fortin scheme can be replaced by the following one:

(3') *Find* $\tilde{w}_h \in M_4$ *such that*

$$(\nabla \tilde{w}_h, \nabla u) = (\bar{\boldsymbol{\phi}}_h, \nabla u) + t^2(g, u). \tag{3.1}$$

for all $u \in M_4$.

Note that if $M_4 \subset M_1$, then the scheme (3.1) is the same as the original Brezzi-Fortin scheme (2.11). In general, (3.1) is easier, simpler, and more accurate than (2.11).

With the above modification, the new approximation w_h of w has the following simplified and more accurate error estimate:

$$\|w - \tilde{w}_h\|_1 \le C \left(\inf_{v \in M_4} \|w - v\|_1 + \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}_h\| \right). \tag{3.2}$$

The proof for (3.2) is straightforward and is left as an exercise.

4. A Finite Element Scheme based on the Least-Squares Surface Fitting

The post-processing technique introduced by Wang [11] is to project (in L^2 space) the finite element solution to another finite element space with different and carefully-designed approximation properties. The projection space is constructed as a finite element space associated with a coarser mesh with higher order of polynomials than the originally used finite element space in the numerical discretization. This is essentially a coarsening procedure which eliminates the dominating error of high oscillation modes in the solution.

The objective of this section is to propose a modification of the Brezzi-Fortin scheme by using the projection method [11]. To this end, let $\mathcal{T}_{\tau_i}$ $(i = 1, 2, 3, 4)$ be four finite element partitions with mesh size τ_i $(i = 1, 2, 3, 4)$ such that $h << \tau_i$. Assume that τ_i and h have the following relation:

$$\tau_i = h^{\alpha_i},\ i = 1, 2, 3, 4. \tag{4.1}$$

with $\alpha_i \in (0, 1)$. We will see that α_i plays an important role in achieving a superconvergence for the Brezzi-Fortin's finite element approximation of the Reissner-Mindlin plate model.

Let R_{τ_1}, Φ_{τ_2}, P_{τ_3} and W_{τ_4} be four finite element spaces consisting of piecewise polynomials of degree n_1, n_2, n_3, and n_4 associated with the partition $\mathcal{T}_{\tau_i}$, $i = 1, 2, 3, 4$, respectively. Define $\mathcal{R}_{\tau_1}$, Ψ_{τ_2}, $\mathcal{P}_{\tau_3}$ and $\mathcal{W}_{\tau_4}$ to be the four L^2 projection operators from $L^2(\Omega)$ onto the finite element spaces R_{τ_1}, Φ_{τ_2}, P_{τ_3} and W_{τ_4} respectively. Our modified Brezzi-Fortin finite element scheme is given as follows.

(1) *Find* $r_h \in M_1$ *such that*

$$(\nabla r_h,\ \nabla s) = (g, s), \tag{4.2}$$

for all $s \in M_1$.

(2) *Find* $(\boldsymbol{\phi}_h, p_h) \in M_2 \times M_3/R$ *such that*

$$(\nabla\boldsymbol{\phi}_h, \nabla\boldsymbol{\psi}) - (\nabla^\perp p_h, \boldsymbol{\psi}) = (\nabla\mathcal{R}_{\tau_1} r_h, \boldsymbol{\psi}), \tag{4.3}$$

$$-(\boldsymbol{\phi}_h, \nabla^\perp q) - t^2(\nabla^\perp p_h, \nabla^\perp q) = 0, \tag{4.4}$$

for all $\boldsymbol{\psi} \in M_2$ *and* $q \in M_3/R$.

(3) *Find* $w_h \in M_4$ *such that*

$$(\nabla w_h, \nabla u) = (\Psi_{\tau_2}\boldsymbol{\phi}_h, \nabla u) + t^2(g, u). \tag{4.5}$$

for all $u \in M_4$.

Analogous to Theorem 2.1, we have the following result.

Theorem 4.1. *Let* $(r, \boldsymbol{\phi}, p, w)$ *and* $(r_h, \boldsymbol{\phi}_h, p_h, w_h)$ *be the solutions of (2.2)-(2.4), (2.6) and (4.2)-(4.5), respectively. If the inf-sup condition (2.7) is satisfied for the finite element spaces* M_2 *and* M_3*, then there exists a constant* C *independent of* h *such that*

$$\|r - r_h\|_1 \le C \inf_{u\in M_1} \|r - u\|_1, \tag{4.6}$$

$$\begin{aligned}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|p - p_h\| + t\|p - p_h\|_1 \le C \Bigg(&\inf_{\psi\in M_2} \|\boldsymbol{\phi} - \psi\|_1 \\ + \; t \inf_{q\in M_3} \|p - q\|_1 + \inf_{q\in M_3} \|p - q\| &+ \|\nabla(r - \mathcal{R}_{\tau_1} r_h)\|\Bigg),\end{aligned} \tag{4.7}$$

and

$$\|w - w_h\|_1 \le C \left(\inf_{v\in M_4} \|w - v\|_1 + \|\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h\| \right). \tag{4.8}$$

The above error estimates do not imply any superconvergence for the Reissner-Mindlin plate problem. However, they do indicate that the error pollution in the sequential procedure (from r to $\boldsymbol{\phi}$ then to w) can be significantly reduced by using the projection method. For example, the error estimate (4.7) shows that the accuracy for $\boldsymbol{\phi}_h$ is dominated by the approximation property of the finite element spaces M_2 and M_3, provided that $\|\nabla(r - \mathcal{R}_{\tau_1} r_h)\|$ is arbitrarily small. The same statement can be made for the numerical approximation w_h from the error estimate (4.8).

5. Superconvergence by L^2 Projections

The goal of this section is to show that an appropriately-defined L^2 projection of the finite element approximation $(r_h, \boldsymbol{\phi}_h, p_h, w_h)$ obtained from (4.2)-(4.5) is superconvergent to the exact solution $(r, \boldsymbol{\phi}, p, w)$ of the system (2.2)-(2.4) and (2.6). The results are stated in Theorem 5.2 for the approximation of ϕ (the rotation of the plate fibers normal to Ω) and Theorem 5.4 for the approximation of w (the transverse deflection of Ω).

With that said, we begin our detailed and tedious analysis as follows.

Lemma 5.1. *Assume that the regularity (2.19) holds true with* $1 \le s \le \min(2, k_1 + 1)$ *and* $R_{\tau_1} \subset H^{s-2}(\Omega)$*. Then there exists a constant* C *independent of* h *and* τ_1 *such that*

$$\|\mathcal{R}_{\tau_1} r - \mathcal{R}_{\tau_1} r_h\| \le C h^{k_1+s-1+\alpha_1 \min(0,2-s)} \|r\|_{k_1+1} \tag{5.1}$$

where $\alpha_1 \in (0, 1)$ *is as defined in (4.1).*

Proof. See Wang [11].

If the solution of (2.2), r, is sufficiently smooth, then

$$\|r - \mathcal{R}_{\tau_1} r\| \le C\tau_1^{n_1+1}\|r\|_{n_1+1} = Ch^{\alpha_1(n_1+1)}\|r\|_{n_1+1}. \tag{5.2}$$

Combining (5.1) with (5.2), we obtain the following result.

Theorem 5.1. *Assume that the regularity (2.19) holds true with* $1 \le s \le \min(2, k_1+1)$ *and* $R_{\tau_1} \subset H^{s-2}(\Omega)$. *Then, we have*

$$\|r - \mathcal{R}_{\tau_1} r_h\| + h^{\alpha_1}\|\nabla_{\tau_1}(r - \mathcal{R}_{\tau_1} r_h)\| \le Ch^{\beta_1}\left(\|r\|_{n_1+1} + \|r\|_{k_1+1}\right) \tag{5.3}$$

where $\alpha_1 = \frac{(k_1+s-1)}{n_1+1-\min(0,2-s)}$ *and* $\beta_1 = \alpha_1(n_1+1)$ *and the gradient* ∇_{τ_1} *should be taken element by element over* $\mathcal{T}_{\tau_1}$.

Next, we move to the analysis for the solution of (4.3)-(4.4).

Lemma 5.2. *Assume that the regularity estimate (2.19) holds true with* $1 \le s \le \min(2, k_2+1)$ *and* $\Phi_{\tau_2} \subset H^{s-2}(\Omega)$. *Then there exists a constant* C *independent of* h *and* τ_2 *such that*

$$\begin{aligned} \|\Psi_{\tau_2}\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h\| &\le Ch^{\min(k_2+s-1,\beta_1)+\alpha_2\min(0,2-s)}(\|\boldsymbol{\phi}\|_{k_2+1} \\ &+ \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1}) \end{aligned} \tag{5.4}$$

where $\alpha_2 \in (0,1)$ *is as defined in (4.1).*

Proof. The definition of $\|\cdot\|$ and Ψ_{τ_2} implies that

$$\|\Psi_{\tau_2}\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h\| = \sup_{\gamma\in[L^2(\Omega)]^2, \|\gamma\|=1} |(\Psi_{\tau_2}\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h, \gamma)|$$

and

$$(\Psi_{\tau_2}\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h, \gamma) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \Psi_{\tau_2}\gamma).$$

Thus,

$$\|\Psi_{\tau_2}\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h\| = \sup_{\gamma\in[L^2(\Omega)]^2, \|\gamma\|=1} |(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \Psi_{\tau_2}\gamma)|.$$

Consider the following problem: find $(\chi, \lambda) \in [H_0^1(\Omega)]^2 \times H^1(\Omega)/R$ such that for all $(\psi, q) \in [H_0^1(\Omega)]^2 \times H^1(\Omega)/R$

$$(\nabla\psi, \nabla\chi) - (\psi, \nabla^\perp\lambda) = (\Psi_{\tau_2}\gamma, \psi) \tag{5.5}$$

$$-(\nabla^\perp q, \chi) - t^2(\nabla^\perp\lambda, \nabla^\perp q) = 0. \tag{5.6}$$

Let $\psi = \boldsymbol{\phi} - \boldsymbol{\phi}_h$ and $q = p - p_h$ in (5.5) and (5.6) respectively. Then adding (5.5) and (5.6) gives

$$\begin{aligned} (\Psi_{\tau_2}\gamma, \boldsymbol{\phi} - \boldsymbol{\phi}_h) &= (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla\chi) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \nabla^\perp\lambda) \\ &\quad -(\nabla^\perp(p - p_h), \chi) - t^2(\nabla^\perp(p - p_h), \nabla^\perp\lambda). \end{aligned} \tag{5.7}$$

Subtracting (4.3) from (2.3) and (4.4) from (2.4) and adding the resulting equations give that for all $\psi \in M_2$ and $q \in M_3$

$$\begin{aligned} &(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla\psi) - (\nabla^\perp(p - p_h), \psi) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \nabla^\perp q) \\ &\quad -t^2(\nabla^\perp(p - p_h), \nabla^\perp q) = (\nabla(r - \mathcal{R}_{\tau_1} r_h), \psi). \end{aligned} \tag{5.8}$$

Subtracting the above equation from (5.7) and using (2.19), we have

$$\begin{aligned} |(\Psi_{\tau_2}\gamma, \boldsymbol{\phi} - \boldsymbol{\phi}_h)| &\le \inf_{\psi\in M_2, q\in M_3} |(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\chi-\psi)) - (\boldsymbol{\phi}-\boldsymbol{\phi}_h, \nabla^{\perp}(\lambda - q)) \\ &- (\nabla^{\perp}(p-p_h), \chi-\psi) - t^2(\nabla^{\perp}(p-p_h), \nabla^{\perp}(\lambda-q)) \\ &- (\nabla(r-\mathcal{R}_{\tau_1} r_h), \psi)| \\ &\le Ch^{s-1}\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1\|\chi\|_s + h^{s-1}\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1\|\lambda\|_{s-1} \\ &+ h^{s-1}\|p-p_h\|\|\chi\|_s \\ &+ t^2h^{s-1}\|p-p_h\|_1\|\lambda\|_s + \|r-\mathcal{R}_{\tau_1}r_h\|\|\chi\|_s \\ &\le C(h^{k_2+s-1}\|\boldsymbol{\phi}\|_{k_2+1}\|\Psi_{\tau_2}\gamma\|_{s-2} + h^{k_2+s-1}\|p\|_{k_2+1}\|\Psi_{\tau_2}\gamma\|_{s-2} \\ &+ h^{\beta_1}(\|r\|_{n_1+1} + \|r\|_{k_1+1})\|\Psi_{\tau_2}\gamma\|_{s-2}. \end{aligned}$$

Hence, the desired estimate (5.4) follows from the above with an application of the inverse inequality. □

Theorem 5.2. *Assume that the regularity estimate (2.19) holds true with* $1 \le s \le \min(2, k_2+1)$ *and* $\Phi_{\tau_2} \subset H^{s-2}(\Omega)$. *Then, we have*

$$\begin{aligned} \|\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h\| &+ h^{\alpha_2}\|\nabla_{\tau_2}(\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h)\| \le Ch^{\beta_2}(\|\boldsymbol{\phi}\|_{n_2+1} + \|\boldsymbol{\phi}\|_{k_2+1} + \|p\|_{k_2+1} \\ &+ \|r\|_{n_1+1} + \|r\|_{k_1+1}) \end{aligned} \tag{5.9}$$

where $\alpha_2 = \frac{\min\{k_2+s-1,\beta_1\}}{n_2+1-\min(0,2-s)}$ *and* $\beta_2 = \alpha_2(n_2+1)$.

Proof. By the definition of Ψ_{τ_2}, we have

$$\|\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h\| \le C\tau_2^{n_2+1}\|\boldsymbol{\phi}\|_{n_2+1} = Ch^{\alpha_2(n_2+1)}\|\boldsymbol{\phi}\|_{n_2+1}. \tag{5.10}$$

Combining (5.4) and (5.10) gives

$$\begin{aligned} \|\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h\| &\le \|\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}\| + \|\Psi_{\tau_2}\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h\| \le Ch^{\alpha_2(n_2+1)}\|\boldsymbol{\phi}\|_{n_2+1} \\ &+ Ch^{\min(k_2+s-1,\beta_1)+\alpha_2\min(0,2-s)}(\|\boldsymbol{\phi}\|_{k_2+1} \\ &+ \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1}). \end{aligned}$$

The above error estimate can be optimized by choosing α_2 such that

$$\alpha_2(n_2+1) = \min(k_2+s-1, \beta_1) + \alpha_2\min(0, 2-s).$$

Solving the above equation gives

$$\alpha_2 = \frac{\min(k_2+s-1,\beta_1)}{n_2+1-\min(0,2-s)}. \tag{5.11}$$

Thus,

$$\|\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h\| \le Ch^{\beta_2}(\|\boldsymbol{\phi}\|_{n_2+1} + \|\boldsymbol{\phi}\|_{k_2+1} + \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1})$$

with $\beta_2 = \frac{\min(k_2+s-1,\beta_1)(n_2+1)}{n_2+1-min(0,2-s)}$.

The gradient term $\|\nabla_{\tau_2}(\boldsymbol{\phi}-\Psi_{\tau_2}\boldsymbol{\phi}_h)\|$ can be estimated in a similar manner, and is thus omitted. □

Lemma 5.3. *Assume that (2.19) holds true with* $1 \le s \le \min(2, k_2+1)$ *and* $P_{\tau_3} \subset H^{s-2}(\Omega)$. *Then there exists a constant* C *independent of* h *and* τ_3 *such that*

$$\begin{aligned} \|\mathcal{P}_{\tau_3}p - \mathcal{P}_{\tau_3}p_h\| &\le Ch^{\min(k_2+s-1,\beta_1)+\alpha_3\min(0,2-s)}(\|\boldsymbol{\phi}\|_{k_2+1} \\ &+ \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1}) \end{aligned} \tag{5.12}$$

where $\alpha_3 \in (0,1)$ *is as defined in (4.1).*

Proof. The definition of $\|\cdot\|$ and $\mathcal{P}_{\tau_3}$ gives

$$\|\mathcal{P}_{\tau_3}p - \mathcal{P}_{\tau_3}p_h\| = \sup_{\gamma\in L^2(\Omega),\|\gamma\|=1} |(\mathcal{P}_{\tau_3}p - \mathcal{P}_{\tau_3}p_h, \gamma)|$$

and

$$(\mathcal{P}_{\tau_3}p - \mathcal{P}_{\tau_3}p_h, \gamma) = (p - p_h, \mathcal{P}_{\tau_3}\gamma).$$

Then

$$\|\mathcal{P}_{\tau_3}p - \mathcal{P}_{\tau_3}p_h\| = \sup_{\gamma\in L^2(\Omega),\|\gamma\|=1} |(p - p_h, \mathcal{P}_{\tau_3}\gamma)|.$$

Consider the following problem: find $(\xi,\mu) \in [H_0^1(\Omega)]^2 \times H^1(\Omega)/R$ such that for all $(\psi, q) \in [H_0^1(\Omega)]^2 \times H^1(\Omega)/R$

$$(\nabla\psi,\nabla\xi) - (\psi,\nabla^\perp\mu) = 0, \tag{5.13}$$

$$-(\nabla^\perp q,\xi) - t^2(\nabla^\perp\mu,\nabla^\perp q) = (\mathcal{P}_{\tau_3}\gamma, q). \tag{5.14}$$

Let $\psi = \boldsymbol{\phi} - \boldsymbol{\phi}_h$ and $q = p - p_h$ in (5.13) and (5.14) respectively. Then adding (5.13) and (5.14) gives

$$\begin{aligned}(\mathcal{P}_{\tau_3}\gamma, p - p_h) &= (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h),\nabla\xi) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h,\nabla^\perp\mu)\\ &- (\nabla^\perp(p-p_h),\xi) - t^2(\nabla^\perp(p-p_h),\nabla^\perp\mu).\end{aligned} \tag{5.15}$$

Subtracting (5.8) from (5.15) and using (2.19), we have that for all $\psi \in M_2$ and $q \in M_3$

$$\begin{aligned}|(\mathcal{P}_{\tau_3}\gamma, p - p_h)| &\le \inf_{\psi\in M_2, q\in M_3} |(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h),\nabla(\xi-\psi)) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h,\nabla^\perp(\mu-q))\\ &- (\nabla^\perp(p-p_h),\xi-\psi) - t^2(\nabla^\perp(p-p_h),\nabla^\perp(\mu-q))\\ &- (\nabla(r - \mathcal{R}_{\tau_1}r_h),\psi)|\\ &\le Ch^{s-1}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1\|\xi\|_s + h^{s-1}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1\|\mu\|_{s-1}\\ &+ h^{s-1}\|p-p_h\|\|\xi\|_s\\ &+ t^2h^{s-1}\|p-p_h\|_1\|\mu\|_s + \|r - \mathcal{R}_{\tau_1}r_h\|\|\chi\|_s\\ &\le C(h^{k_2+s-1}\|\boldsymbol{\phi}\|_{k_2+1}\|\mathcal{P}_{\tau_3}\gamma\|_{s-2} + h^{k_2+s-1}\|p\|_{k_2+1}\|\mathcal{P}_{\tau_3}\gamma\|_{s-2}\\ &+ h^{\beta_1}(\|r\|_{n_1+1} + \|r\|_{k_1+1}\|\mathcal{P}_{\tau_3}\gamma\|_{s-2}.\end{aligned} \tag{5.16}$$

Hence, the desired estimate (5.12) follows from the inverse inequality. □

The following is a superconvergence result for the auxiliary variable p which was introduced by using the Helmholtz decomposition (2.1).

Theorem 5.3. *Assume that (2.19) holds true with* $1 \le s \le \min(2, k_2+1)$ *and* $P_{\tau_3} \subset H^{s-2}(\Omega)$*. Then, we have*

$$\begin{aligned}\|p - \mathcal{P}_{\tau_3}p_h\| &+ h^{\alpha_3}\|\nabla_{\tau_3}(p - \mathcal{P}_{\tau_3}p_h)\| \le Ch^{\beta_3}(\|p\|_{n_3+1} + \|\boldsymbol{\phi}\|_{k_2+1}\\ &+ \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1})\end{aligned} \tag{5.17}$$

where $\alpha_3 = \frac{\min(k_2+s-1,\beta_1)}{n_3+1-\min(0,2-s)}$ *and* $\beta_3 = \alpha_3(n_3+1)$.

Proof. By the definition of $\mathcal{P}_{\tau_3}$, we have

$$\|p - \mathcal{P}_{\tau_3}p\| \le C\tau_3^{n_3+1}\|p\|_{n_3+1} = Ch^{\alpha_3(n_3+1)}\|p\|_{n_3+1}. \tag{5.18}$$

Combining (5.12) and (5.18) gives

$$\begin{aligned}\|p - \mathcal{P}_{\tau_3}p_h\| &\le \|p - \mathcal{P}_{\tau_3}p\| + \|\mathcal{P}_{\tau_3}p - \mathcal{P}_{\tau_3}p_h\| \le Ch^{\alpha_3(n_3+1)}\|p\|_{n_3+1}\\ &+ Ch^{\min(k_2+s-1,\beta_1)+\alpha_3\min(0,2-s)}(\|\boldsymbol{\phi}\|_{k_2+1} + \|p\|_{k_2+1}\\ &+ \|r\|_{n_1+1} + \|r\|_{k_1+1}).\end{aligned}$$

The above error estimate can be optimized by choosing α_3 such that

$$\alpha_3(n_3+1) = \min(k_2+s-1,\beta_1) + \alpha_3 \min(0, 2-s).$$

Solving the above equation gives

$$\alpha_3 = \frac{\min(k_2+s-1,\beta_1)}{n_3+1-\min(0,2-s)}. \tag{5.19}$$

Thus

$$\|p - \mathcal{P}_{\tau_3} p_h\| \le Ch^{\beta_3}(\|p\|_{n_3+1} + \|\boldsymbol{\phi}\|_{k_2+1} + \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1})$$

where $\beta_3 = \frac{\min(k_2+s-1,\beta_1)(n_3+1)}{n_3+1-min(0,2-s)}$.

The gradient term $\|\nabla_{\tau_3}(p - \mathcal{P}_{\tau_3} p_h)\|$ can be estimated in a similar manner, and is thus omitted. □

The rest of this section is devoted to a superconvergence for the approximation of the transverse deflection of the plate.

Lemma 5.4. *Assume that (2.19) holds true with $1 \le s \le \min(2, k_3+1)$ and $W_{\tau_4} \subset H^{s-2}(\Omega)$. Then there exists a constant C independent of h and τ_4 such that*

$$\begin{aligned} \|\mathcal{W}_{\tau_4} w - \mathcal{W}_{\tau_4} w_h\| &\le Ch^{\min(k_3+s-1,\beta_2)+\alpha_4 \min(0,2-s)}(\|\boldsymbol{\phi}\|_{k_2+1} + \|p\|_{k_2+1} \\ &+ \|w\|_{k_3+1} + \|\boldsymbol{\phi}\|_{n_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1}) \end{aligned} \tag{5.20}$$

where $\alpha_4 \in (0,1)$ is as defined in (4.1).

Proof. The definition of $\|\cdot\|$ and $\mathcal{W}_{\tau_4}$ gives

$$\|\mathcal{W}_{\tau_4} w - \mathcal{W}_{\tau_4} w_h\| = \sup_{\gamma \in L^2(\Omega), \|\gamma\|=1} |(\mathcal{W}_{\tau_4} w - \mathcal{W}_{\tau_4} w_h, \gamma)|$$

and

$$(\mathcal{W}_{\tau_4} w - \mathcal{W}_{\tau_4} w_h, \gamma) = (w - w_h, \mathcal{W}_{\tau_4}\gamma).$$

Then

$$\|\mathcal{W}_{\tau_4} w - \mathcal{W}_{\tau_4} w_h\| = \sup_{\gamma \in L^2(\Omega), \|\gamma\|=1} |(w - w_h, \mathcal{W}_{\tau_4}\gamma)|.$$

Consider the following problem: find $\omega \in H_0^1(\Omega)$ such that

$$(\nabla\psi, \nabla\omega) = (\mathcal{W}_{\tau_4}\gamma, \psi) \quad \forall\, \psi \in H_0^1(\Omega). \tag{5.21}$$

Subtracting (4.5) from (2.6) gives

$$(\nabla(w - w_h), \nabla u) = (\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h, \nabla u) \quad \forall u \in M_4. \tag{5.22}$$

Let $\psi = w - w_h$ in (5.21). Using (5.22), we have

$$(\mathcal{W}_{\tau_4}\gamma, w - w_h) = (\nabla(\omega - u), \nabla(w - w_h)) + (\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h, \nabla u).$$

Thus,

$$\begin{aligned} |(\mathcal{W}_{\tau_4}\gamma, w - w_h)| &\le \sup_{u \in M_4} |(\nabla(\omega - u), \nabla(w - w_h)) + (\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h, \nabla u)| \\ &\le C\left(h^{k_3+s-1}\|w\|_{k_3+1}\|\omega\|_s + \|\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h\|\|\omega\|_s\right) \\ &\le Ch^{\min(k_3+s-1,\beta_2)}(\|w\|_{k_3+1} + \|\boldsymbol{\phi}\|_{n_2+1} + \|\boldsymbol{\phi}\|_{k_2+1} \\ &+ \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1})\,\|\mathcal{W}_{\tau_4}\gamma\|_{s-2}, \end{aligned}$$

which, together with an use of the standard inverse inequality, gives rise to (5.20). □

A similar analysis can be applied to yield the following result.

Theorem 5.4. *Assume that (2.19) holds true with* $1 \le s \le \min(2, k_3+1)$ *and* $W_{\tau_4} \subset H^{s-2}(\Omega)$. *Then, we have*

$$\begin{aligned}\|w - \mathcal{W}_{\tau_4} w_h\| &+ h^{\alpha_4}\|\nabla_{\tau_4}(w - \mathcal{W}_{\tau_4} w_h)\| \le Ch^{\beta_4}(\|w\|_{n_4+1} + \|w\|_{K_3+1} \\ &+ \|\boldsymbol{\phi}\|_{k_2+1} + \|\boldsymbol{\phi}\|_{n_2+1} + \|p\|_{k_2+1} + \|r\|_{n_1+1} + \|r\|_{k_1+1})\end{aligned}$$

where $\alpha_4 = \frac{\min(k_3+s-1, \beta_2)}{n_4+1-\min(0,2-s)}$ *and* $\beta_4 = \alpha_4(n_4+1)$.

6. An Application

In this section, we will apply the results derived in the previous section to an element introduce by Brezzi and Fortin [3]. In this finite element method, the finite element spaces M_1, M_2, M_3 and M_4 are given as follows:

$$\begin{aligned} M_1 &= \{u: \ u \in H_0^1(\Omega), \ u|_T \in P_1, \ \forall\, T \in \mathcal{T}^h\}, \\ M_2 &= [M_1 \oplus B_3]^2, \\ M_3 &= \{q: \ q \in H^1(\Omega), \ q|_T \in P_1, \ \forall T \in \mathcal{T}^h\}, \\ M_4 &= M_1, \end{aligned}$$

where B_3 is the cubic bubble function and $\mathcal{T}^h$ is a regular triangulation of the domain Ω. Let $(r, \boldsymbol{\phi}, p, w)$ and $(\bar{r}_h, \bar{\boldsymbol{\phi}}_h, \bar{p}_h, \bar{w}_h)$ be the solutions of (2.2)-(2.5) and (2.8)-(2.11) respectively. The following estimate was obtained in [3].

$$\|r - \bar{r}_h\|_1 + \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}_h\|_1 + \|p - \bar{p}_h\| + t\|p - \bar{p}_h\|_1 + \|w - \bar{w}_h\|_1 \le Ch\|g\|.$$

Recall that the results established in Section 5 are based on H^s-regularity for the problems (2.2)-(2.5) with $1 \le s \le k_i + 1$ with $i = 1, 2, 3$. For this element, we have $k_1 = k_2 = k_3 = 1$. Therefore, the problem (2.2)-(2.5) must have H^s-regularity ($1 \le s \le 2$) in order to make use of those estimates. For simplicity, assume that the problem has H^2-regularity.

The fitting finite element space R_{τ_1}, Φ_{τ_2}, P_{τ_3} and W_{τ_4} must be selected to be subsets of $H^{s-2}(\Omega)$. Since $s = 2$, we see that R_{τ_1}, Φ_{τ_2}, P_{τ_3} and W_{τ_4} could be chosen as finite element spaces consisting of discontinuous piecewise polynomials of degree n_1, n_2, n_3 and n_4 respectively. Let $(r_h, \boldsymbol{\phi}_h, p_h, w_h)$ be the solution of (4.2)-(4.5). Using Theorems 5.1–5.4 we obtain the following estimates:

$$\|r - \mathcal{R}_{\tau_1} r_h\| \le Ch^2(\|r\|_{n_1+1} + \|r\|_2)$$

and

$$\|\nabla_{\tau_1}(r - \mathcal{R}_{\tau_1} r_h)\| \le Ch^{\frac{2n_1}{n_1+1}}(\|r\|_{n_1+1} + \|r\|_2).$$

Since $\beta_1 = 2$, we have

$$\|\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h\| \le Ch^2(\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\phi}\|_{n_2+1} + \|p\|_2 + \|r\|_{n_1+1} + \|r\|_2),$$

$$\|\nabla_{\tau_2}(\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h)\| \le Ch^{\frac{2n_2}{n_2+1}}(\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\phi}\|_{n_2+1} + \|p\|_2 + \|r\|_{n_1+1} + \|r\|_2),$$

$$\|p - \mathcal{P}_{\tau_3} p_h\| \le Ch^2(\|\boldsymbol{\phi}\|_2 + \|p\|_{n_3+1} + \|p\|_2 + \|r\|_{n_1+1} + \|r\|_2),$$

and

$$\|\nabla_{\tau_3}(p - \mathcal{P}_{\tau_3} p_h)\| \le Ch^{\frac{2n_3}{n_3+1}}(\|\boldsymbol{\phi}\|_2 + \|p\|_{n_3+1} + \|p\|_2 + \|r\|_{n_1+1} + \|r\|_2).$$

Since $\beta_1 = \beta_2 = 2$, it follows that

$$\|w - \mathcal{W}_{\tau_4} w_h\| \le Ch^2(\|w\|_{n_4+1} + \|w\|_2 + \|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\phi}\|_{n_2+1} + \|p\|_2 + \|r\|_{n_1+1} + \|r\|_2),$$

and

$$\begin{aligned}\|\nabla_{\tau_4}(w - \mathcal{W}_{\tau_4} w_h)\| &\le Ch^{\frac{2n_4}{n_4+1}}(\|w\|_2 + \|w\|_{n_4+1} + \|\boldsymbol{\phi}\|_2 \\ &+ \|\boldsymbol{\phi}\|_{n_2+1} + \|p\|_2 + \|r\|_{n_1+1} + \|r\|_2).\end{aligned}$$

We view $\Psi_{\tau_2}\boldsymbol{\phi}_h$ and $\mathcal{W}_{\tau_4}w_h$ as new approximate solutions to the unknown functions $\boldsymbol{\phi}$ and w. The above estimates certainly do not indicate any sort of superconvergence for this new set of approximations in the L^2 norm. However, superconvergence for the gradient of the variables can be achieved. For example, with $n_1 = n_2 = n_3 = n_4 = 2$ (piecewise quadratic elements), the post-processed approximation $\mathcal{R}_{\tau_1}r_h$, $\Psi_{\tau_2}\boldsymbol{\phi}_h$, $\mathcal{P}_{\tau_3}p_h$ and $\mathcal{W}_{\tau_4}w_h$ have the following superconvergence:

$$\|\nabla_{\tau_1}(r - \mathcal{R}_{\tau_1}r_h)\| \le Ch^{\frac{4}{3}}(\|r\|_3 + \|r\|_2),$$

$$\|\nabla_{\tau_2}(\boldsymbol{\phi} - \Psi_{\tau_2}\boldsymbol{\phi}_h)\| \le Ch^{\frac{4}{3}}(\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\phi}\|_3 + \|p\|_2 + \|r\|_3 + \|r\|_2),$$

$$\|\nabla_{\tau_3}(p - \mathcal{P}_{\tau_3}p_h)\| \le Ch^{\frac{4}{3}}(\|\boldsymbol{\phi}\|_2 + \|p\|_3 + \|p\|_2 + \|r\|_3 + \|r\|_2),$$

and

$$\|\nabla_{\tau_4}(w - \mathcal{W}_{\tau_4}w_h)\| \le Ch^{\frac{4}{3}}(\|w\|_2 + \|w\|_3 + \|\boldsymbol{\phi}\|_2 + \|p\|_3 + \|p\|_2 + \|r\|_3 + \|r\|_2).$$

Assume that the exact solution is sufficiently smooth. Then it can be shown that

$$\|\nabla_{\tau_1}(r - \mathcal{R}_{\tau_1}r_h)\| \approx O(h^2), \quad \text{as } n_1 \to \infty$$

Similar results can be established for the approximate solution $\Psi_{\tau_2}\boldsymbol{\phi}_h$ and $\mathcal{W}_{\tau_4}w_h$.

References

[1] D. N. Arnold and R. S. Falk, A Uniformly accurate finite element method for the Reissner-Mindlin plate, SIAM J. Numer. Anal. 26 (1989) 1276-1290.

[2] D. N. Arnold and X. Liu, Interior estimates for a low order finite element method for the Reissner-Mindlin plate model, Adv. in Comp. Math., 7 (1997) 337-360.

[3] F. Brezzi and M. Fortin, Numerical approximation of Mindlin-Reissner plates, Math. Comp., 47 (1986) 151-158.

[4] F. Brezzi, K. Bathe, and M. Fortin, Mixed interpolated elements for Reissner-Mindlin plates, Int. J. Numer. Methods Eng., 28 (1989) 1787-1801.

[5] F. Brezzi, M. Fortin and R. Stenberg, Quasi-optimal error bounds for approximation of shear-stresses in Mindlin-Reissner plate models, Math. Models Meth. Appl. Sci., 1 (1991).

[6] F. Brezzi, L. Franca, T. Hughes and A. Russo, Stabilization techniques and subgrid scales capturing, The state of the art in numerical analysis (York, 1996), pp. 391-406, Inst. Math. Appl. Conf. Ser. New Ser., 63, Oxford University Press, New York, 1997.

[7] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, New York, 1978.

[8] V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Springer-Verlag, New York, 1986.

[9] R. Pierre, Convergence properties and numerical approximation of the solution of the Mindlin plate bending problem, Math Comp., 51 (1988) 15-25.

[10] R. Rannacher and S. Turek, Simple nonconforming quadrilateral Stokes element, Numerical Methods for Partial Differential Equations, 8 (1992) 97-111.

[11] J. Wang, A superconvergence analysis for finite element solutions by the least-squares surface fitting on irregular meshes for smooth problems, Journal of Mathematical Study, Vol. 33, No. 3 (2000) 229-243.

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