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THE MECHANICAL BEHAVIOR OF A POROELASTIC MEDIUM SATURATED WITH A NEWTONIAN FLUID

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Abstract. In this paper we systematically derive, via the theory of homogenization, the macroscopic equations for the mechanical behavior of a deformable porous medium saturated with a Newtonian fluid. The derivation is first based on the equations of linear elasticity in the solid, the Stokes equations for the fluid, and suitable conditions at the fluid-solid interface. A detailed comparison between the equations derived here and those by Biot is given. The homogenization approach determines the form of the macroscopic constitutive relationships between variables and shows how to compute the coefficients in these relationships. The derivation is then extended to the nonlinear Navier-Stokes equations for the fluid in the deformable porous medium for the first time. A generalized Forchheimer law is obtained to take into account the nonlinear inertial effects on the flow of the Newtonian fluid through such a medium. Both quasi-static and transient flows are considered in this paper. The properties of the macroscopic coefficients are studied. The computational results show that the macroscopic equations predict well the behavior of the microscopic equations in certain reasonable test cases.

Key Words. deformable porous medium, Forchheimer law, homogenization, linear elasticity, high flow rate, Navier-Stokes equation, computational validation.

1. Introduction

We have recently employed the theory of homogenization to derive the Forchheimer law directly from the nonlinear Navier-Stokes equation in a rigid porous medium [11]. Unlike other studies based on the same approach that concluded the nonlinear correction to be cubic in velocity for an isotropic medium, our work has shown that the nonlinear correction is quadratic. In this paper we extend the techniques in [11] to a deformable porous medium.

The macroscopic mechanical behavior of a deformable porous medium has been studied by Biot [5, 6, 7] by means of an intuitive approach. Later studies have been based on mixture theory and constitutive assumptions [12]. Recent studies have utilized a group of averaging approaches [1, 9, 22] for treating Stokes flow through a periodic deformable medium. These averaging approaches have reproduced Biot's equations and shown how to calculate the coefficients in these equations. In this paper Stokes flow through a deformable medium is further examined by adapting the approach in [11]. This approach is simpler and is more direct than those in [1, 9, 22] since lower-order approximation terms are used, while higher-order terms were exploited in [1, 9, 22]. Also, the coefficients in the macroscopic equations are

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studied here in detail and a detailed comparison between the equations we obtain and those by Biot is given. Finally, we analyze the nonlinear Navier-Stokes flow in the framework of a deformable porous medium for the first time.

The approach in [11] is based on the theory of two-scale homogenization [4, 20]. The two-scale homogenization for a periodic porous medium averages the detailed microstructure of the pores and yields a set of simpler, macroscopic equations. This is achieved by a careful scaling of the microscopic equations by the ratio of two length scales associated with the microscopic and macroscopic phenomena in the periodic medium.

To compare the present theory with Biot's theory [5, 6, 7] for deformable porous medium flow, we carry out the analysis by starting with the equations of linear elasticity in the solid, the Stokes equations for the fluid, and suitable equations at the solid-fluid interface. The macroscopic equations derived here coincide with Biot's equations in the case where the scaled viscosity (see the definition in the next section) of the fluid is small. Moreover, the present theory determines the form of the macroscopic constitutive relationships between variables and shows how to compute the coefficients in these relationships. In the case where the scaled viscosity is large, we derive different differential equations and constitutive relationships. Two situations in this case are investigated. The first situation concerns the elastic single-phase behavior of the porous system, while the second concerns the viscoelastic single-phase behavior of this system.

We then extend the analysis to the nonlinear Navier-Stokes equations for the fluid. When these nonlinear equations are analyzed in the framework of a deformable medium, the situation is more complicated. It is well known that the simplest law for describing the flow of a fluid in a porous medium is the law obtained by Darcy (1856) [13]. Derived from empiricism, this law indicates a linear relationship between the fluid velocity relative to the solid and the pressure gradient. Subsequently, Dupuit (1863) [14] and Forchheimer (1901) [15] gave further empirical evidence that the linearity in Darcy's law does not hold for high rates of fluid flow and generalized this law in a nonlinear fashion (i.e., Forchheimer's law). In this paper we derive a generalized Forchheimer law for a deformable porous medium to take into account the nonlinear inertial effects on the fluid flow through such a medium.

The paper is organized as follows. In the next section we consider Stokes flow. Then, in the third section we analyze Navier-Stokes flow. The quasi-static case is considered in these two sections. Transient inertial effects are taken into account in the fourth section. In the fifth section we present a computational validation of some of the homogenized models derived. Concluding remarks are stated in the last section. The properties of the macroscopic coefficients are studied in the appendix. We end with two remarks. First, vectors and matrices will be represented by bold face variables, and the rectangular coordinates in $\Re^3$ are denoted by $x = (x_1, x_2, x_3)$ (not in bold face). Second, in this paper we focus on the systematical derivation of the macroscopic equations for the mechanical behavior of a deformable porous medium saturated with a Newtonian fluid via homogenization, the comparison of the present theory with Biot's, and the study of properties of macroscopic coefficients. A convergence proof of the homogenization approach in the present setting is beyond the scope of this paper.

2. Stokes Flow through a Deformable Medium

The development in this section is similar to that in the paper [9]. A quasi-static case is treated here, while a transient problem was considered in [9]. Also, the material in Section 2.2 appears new. Moreover, the properties of macroscopic coefficients are studied in this paper.

Let Ω be a bounded deformable porous medium in $\Re^3$. We assume that Ω is connected but not necessarily simply-connected. It is composed of a linear elastic material and its deformation is small. We study the quasi-static evolution of this medium, which is saturated with a viscous Newtonian fluid. Let $\Omega = \Omega_s \cup \Omega_f$, with Ω_s and Ω_f being the solid and fluid regions, respectively. The Navier equation in Ω_s, the Stokes equation in Ω_f, and the continuity equations of the normal stress and the displacement at the solid-fluid interface Γ_{fs} are stated as follows:

$$
\begin{aligned}
&\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_s + \rho_s \mathbf{g} = 0 && \text{in } \Omega_s,\\
&\boldsymbol{\sigma}_s = \mathbf{a}\mathbf{e}(\mathbf{u}_s) && \text{in } \Omega_s,\\
&\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f + \rho_f \mathbf{g} = 0 && \text{in } \Omega_f,\\
&\boldsymbol{\sigma}_f = -p\mathbf{I} + 2\mu\mathbf{e}(\mathbf{v}) && \text{in } \Omega_f,\\
&\boldsymbol{\nabla}\cdot\mathbf{v} = 0 && \text{in } \Omega_f,\\
&\boldsymbol{\sigma}_s\cdot\mathbf{n} = \boldsymbol{\sigma}_f\cdot\mathbf{n} && \text{on } \Gamma_{fs},\\
&\mathbf{u}_s = \mathbf{u}_f && \text{on } \Gamma_{fs},
\end{aligned}
\tag{2.1}
$$

where the solid has the stress tensor $\boldsymbol{\sigma}_s$, density ρ_s, displacement $\mathbf{u}_s$, and fourth rank elastic tensor $\mathbf{a}$, the fluid has the stress tensor $\boldsymbol{\sigma}_f$, density ρ_f, pressure p, viscosity μ, velocity $\mathbf{v}$, and displacement $\mathbf{u}_f$, $\mathbf{I}$ is the identity tensor, $\mathbf{n}$ is the unit normal to Γ_{fs} (pointing to the solid), $\rho_s\mathbf{g}$ (or $\rho_f\mathbf{g}$) is an external body force per volume, and $\boldsymbol{\nabla}\cdot$ indicates the divergence operator. The strain tensor $\mathbf{e}(\mathbf{u}_s) = (e_{ij})_{i,j=1,2,3}$ is defined by

$$
e_{ij} = \frac{1}{2}\left(\frac{\partial u_{si}}{\partial x_j} + \frac{\partial u_{sj}}{\partial x_i}\right), \qquad i,j = 1,2,3,\ \mathbf{u}_s = (u_{s1}, u_{s2}, u_{s3}); \tag{2.2}
$$

a similar definition can be given for $\mathbf{e}(\mathbf{v})$. The elastic tensor $\mathbf{a} = (a_{ijkh})$ is assumed constant (not essential for the later analysis and results) and satisfies the property of symmetry and positivity

$$
\begin{aligned}
&a_{ijkh} = a_{jikh} = a_{ijhk} = a_{khij}, && i,j,k,h = 1,2,3,\\
&a_{ijkh} z_{ij} z_{kh} \geq a_* z_{ij} z_{ij}, && a_* > 0,
\end{aligned}
\tag{2.3}
$$

where (z_{ij}) is a symmetric tensor and the convention that repeated indices indicate a summation is used.

We now consider the homogenization procedure for (2.1). The important characteristics of this procedure is existence of two vastly different length scales: the microscale l, which characterizes the typical layer thickness, and the macroscale L, which characterizes the global variation of external forces and boundary data. Let $\epsilon = l/L$, with $\epsilon \ll 1$.

Let the porous medium Ω have a periodic microstructure with period Y, where $Y = Y_f \cup Y_s$, with Y_f and Y_s being the fluid and solid parts, respectively. Define

$$
\Omega_{\epsilon f} = \Omega \cap \{x : x \in \epsilon Y_f\}. \tag{2.4}
$$

In this paper we only consider a formal expansion of the solution in (2.1), and the boundary of Ω does not play a role in this expansion. Consequently, let $\Omega = \Re^3$, and define

$$\Omega_{\epsilon f} = \{x : x \in \epsilon Y_f\}. \tag{2.5}$$

The domain $\Omega_{\epsilon s}$ can be defined in the same fashion.

We now consider the scaled problem

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_s^\epsilon + \rho_s \mathbf{g} = 0 && \text{in } \Omega_{\epsilon s},\\
&\boldsymbol{\sigma}_s^\epsilon = \mathbf{a}\mathbf{e}(\mathbf{u}_s^\epsilon) && \text{in } \Omega_{\epsilon s},\\
&\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_f^\epsilon + \rho_f \mathbf{g} = 0 && \text{in } \Omega_{\epsilon f},\\
&\boldsymbol{\sigma}_f^\epsilon = -p^\epsilon \mathbf{I} + 2\mu\epsilon^\beta \mathbf{e}(\mathbf{v}^\epsilon) && \text{in } \Omega_{\epsilon f},\\
&\boldsymbol{\nabla} \cdot \mathbf{v}^\epsilon = 0 && \text{in } \Omega_{\epsilon f},\\
&\boldsymbol{\sigma}_s^\epsilon \cdot \mathbf{n} = \boldsymbol{\sigma}_f^\epsilon \cdot \mathbf{n} && \text{on } \Gamma_{\epsilon f s},\\
&\mathbf{u}_s^\epsilon = \mathbf{u}_f^\epsilon && \text{on } \Gamma_{\epsilon f s},
\end{aligned} \tag{2.6}$$

where we have scaled the viscosity coefficient through ϵ^β (β is to be determined below). Note that a different scaling is used here from that in [2], where higher order approximations (e.g., $\boldsymbol{\sigma}^2$; see (2.9) below) have to be used due to the scaling used there. In particular, the continuity equations at the solid-fluid interface were scaled in [2].

Following the custom of homogenization, we assume that any point is described by two coordinates: $x \in \Omega$ describing the general location of the point and $y \in Y$ giving the location of the point within the ϵ-cell ϵY. Obviously, x and y are related by the constant ϵ:

$$y \sim \epsilon^{-1} x, \tag{2.7}$$

(up to translation). Consequently, by the chain rule the relation holds

$$\boldsymbol{\nabla} \sim \boldsymbol{\nabla}_x + \epsilon^{-1} \boldsymbol{\nabla}_y, \tag{2.8}$$

where $\boldsymbol{\nabla}_x$ and $\boldsymbol{\nabla}_y$ represent the gradient operators with respect to x and y, respectively. Assuming that the solution to (2.6) behaves as if it was a function of these two coordinates and that it can be expanded in a power series in terms of ϵ, the stress, displacement, velocity, and pressure are then expanded in the asymptotic form

$$\begin{aligned}
&\boldsymbol{\sigma}_\alpha^\epsilon(x,t) = \boldsymbol{\sigma}_\alpha^0(x,y,t) + \epsilon^1 \boldsymbol{\sigma}_\alpha^1(x,y,t) + \dots, && \alpha = s,\ f,\\
&\mathbf{u}_\alpha^\epsilon(x,t) = \mathbf{u}_\alpha^0(x,y,t) + \epsilon^1 \mathbf{u}_\alpha^1(x,y,t) + \dots, && \alpha = s,\ f,\\
&\mathbf{v}^\epsilon(x,t) = \mathbf{v}^0(x,y,t) + \epsilon^1 \mathbf{v}^1(x,y,t) + \dots, &&\\
&p^\epsilon(x,t) = p^0(x,y,t) + \epsilon^1 p^1(x,y,t) + \dots, &&
\end{aligned} \tag{2.9}$$

where $\boldsymbol{\sigma}_\alpha^i$, $\mathbf{u}_\alpha^i$, $\mathbf{v}^i$, and p^i are Y-periodic in y, $x \in \Omega$, $y \in Y_f$. We shall substitute (2.9) into (2.6), apply (2.8), and analyze the resulting equations. Before this, we need to determine the value of β in (2.6).

2.1. Fluid-solid macroscopic behavior. Different choices for the value of β lead to different macroscopic equations for the deformable medium considered. Note that, with Δ being the Laplacian operator, it follows from (2.8) that

$$\Delta \sim \Delta_{xx} + 2\epsilon^{-1} \boldsymbol{\nabla}_y \cdot \boldsymbol{\nabla}_x + \epsilon^{-2} \Delta_{yy}, \tag{2.10}$$

where Δ_{xx} and Δ_{yy} denote the Laplacian operators with respect to x and y, respectively. Also, by the third, fourth, and fifth equations of (2.6), we see that

$$-\boldsymbol{\nabla} p^\epsilon + \mu\epsilon^\beta \Delta \mathbf{v}^\epsilon + \rho_f \mathbf{g} = 0. \tag{2.11}$$

From (2.10) and (2.11) we see that a suitable choice for β is two in order for the leading term $\mathbf{v}^0$ to be significant. In this subsection we take $\beta = 2$; other choices will be considered in later subsections. The choice of $\beta = 2$ can be also seen as follows. We assume that the quantity μ/l^2, appropriate to the microscale, is of order unity. Thus this quantity, appropriate to the macroscale μ/L^2, is of order ϵ^2. To accomplish this, we scale μ in (2.6) by ϵ^2.

We now substitute (2.9) into (2.6) with $\beta = 2$, apply (2.8), and collect terms with like powers of ϵ. The ϵ^{-1} term of the first, second, third, and fifth equations and the ϵ^0 term of the fourth, sixth, and seventh equations in (2.6) yield

$$
\begin{aligned}
&\boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_s^0 = 0 && \text{in } Y_s,\\
&\mathbf{a}\mathbf{e}_y(\mathbf{u}_s^0) = 0 && \text{in } Y_s,\\
&\boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_f^0 = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_f^0 = -p^0\mathbf{I} && \text{in } Y_f,\\
&\boldsymbol{\nabla}_y \cdot \mathbf{v}^0 = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_s^0 \cdot \mathbf{n} = \boldsymbol{\sigma}_f^0 \cdot \mathbf{n} && \text{on } Y_{fs},\\
&\mathbf{u}_s^0 = \mathbf{u}_f^0 && \text{on } Y_{fs},
\end{aligned}
\tag{2.12}
$$

where Y_{fs} is the interface between Y_f and Y_s, and the ϵ^0 term of the first, second, third, and fifth equations and the ϵ^1 term of the fourth, sixth, and seventh equations in (2.6) lead to

$$
\begin{aligned}
&\boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_s^0 + \boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_s^1 + \rho_s\mathbf{g} = 0 && \text{in } Y_s,\\
&\boldsymbol{\sigma}_s^0 = \mathbf{a}\mathbf{e}_x(\mathbf{u}_s^0) + \mathbf{a}\mathbf{e}_y(\mathbf{u}_s^1) && \text{in } Y_s,\\
&\boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_f^0 + \boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_f^1 + \rho_f\mathbf{g} = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_f^1 = -p^1\mathbf{I} + 2\mu\mathbf{e}_y(\mathbf{v^0}) && \text{in } Y_f,\\
&\boldsymbol{\nabla}_x \cdot \mathbf{v}^0 + \boldsymbol{\nabla}_y \cdot \mathbf{v}^1 = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_s^1 \cdot \mathbf{n} = \boldsymbol{\sigma}_f^1 \cdot \mathbf{n} && \text{on } Y_{fs},\\
&\mathbf{u}_s^1 = \mathbf{u}_f^1 && \text{on } Y_{fs}.
\end{aligned}
\tag{2.13}
$$

We now analyze these equations.

First, by the third and fourth equations of (2.12), we have

$$-\boldsymbol{\nabla}_y p^0(x, y, t) = 0. \tag{2.14}$$

This implies that

$$p^0 = p^0(x, t), \tag{2.15}$$

and, consequently,

$$\boldsymbol{\sigma}_f^0 = \boldsymbol{\sigma}_f^0(x, t). \tag{2.16}$$

That is, p^0 and $\boldsymbol{\sigma}_f^0$ are independent of y. This corresponds to the intuition that the local average of p and $\boldsymbol{\sigma}_f$ does not oscillate. Also, using the second equation of (2.12) and assumption (2.3) on $\mathbf{a}$, we find that

$$\mathbf{u}_s^0 = \mathbf{u}_s^0(x, t). \tag{2.17}$$

Second, it follows from the fourth and seventh equations of (2.12) and the third and fourth equations of (2.13) that

$$
\begin{aligned}
&\mu\Delta_y\mathbf{v}^0 = \boldsymbol{\nabla}_y p^1 - \rho_f\mathbf{g} + \boldsymbol{\nabla}_x p^0 && \text{in } Y_f,\\
&\boldsymbol{\nabla}_y \cdot \mathbf{v}^0 = 0 && \text{in } Y_f,\\
&\mathbf{v}^0 = \dot{\mathbf{u}}_s^0 && \text{on } Y_{fs},
\end{aligned}
\tag{2.18}
$$

where $\dot{\mathbf{u}}_s^0$ denotes the differentiation of $\mathbf{u}_s^0$ with respect to time. This is a Stokes problem with an inhomogeneous boundary condition, which can be solved as follows. The usual Sobolev spaces $W^{m,\pi}(\Omega)$ with the norm $\|\cdot\|_{W^{m,\pi}(\Omega)}$ will be used, where m is a nonnegative integer and $0 \le \pi \le \infty$. When $\pi = 2$, we simply write $H^m(\Omega) = W^{m,2}(\Omega)$. When $m = 0$, we have $L^2(\Omega) = H^0(\Omega)$. Below $(\cdot,\cdot)_Q$ denotes the $L^2(Q)$ inner product (or sometimes the duality pairing). For notational convenience, $(H^m(\Omega))^3$ will be simply indicated by $H^m(\Omega)$. Also, we introduce the space of Y-periodic functions

$$V_Y = \{\mathbf{w} \in H^1(Y_f) : \mathbf{w}|_{Y_{fs}} = 0, \boldsymbol{\nabla}_y \cdot \mathbf{w} = 0, \text{ and } Y\text{-periodic}\}, \tag{2.19}$$

equipped with the inner product

$$(\boldsymbol{\nabla}_y \mathbf{w}_1, \boldsymbol{\nabla}_y \mathbf{w}_2)_{Y_f} \equiv \int_{Y_f} \boldsymbol{\nabla}_y \mathbf{w}_1 \cdot \boldsymbol{\nabla}_y \mathbf{w}_2 \, \mathrm{d}y, \qquad \mathbf{w}_1, \mathbf{w}_2 \in V_Y. \tag{2.20}$$

Note that V_Y is a Hilbert space and the associated norm is equivalent to the usual $H^1(Y_f)$-norm.

For $i = 1, 2, 3$, define $\mathbf{K}^i \in V_Y$ to be the solution of

$$(\boldsymbol{\nabla}_y \mathbf{K}^i, \boldsymbol{\nabla}_y \mathbf{w})_{Y_f} = (1, w_i)_{Y_f} \qquad \forall \mathbf{w} = (w_1, w_2, w_3) \in V_Y. \tag{2.21}$$

The tensor $\mathbf{K}$ is defined by

$$\mathbf{K} = (\mathbf{K}_j^i)_{i,j=1,2,3}. \tag{2.22}$$

Now, the velocity $\mathbf{v}^0$ in (2.18) is expressed by

$$\mathbf{v}^0 - \dot{\mathbf{u}}_s^0 = \frac{\mathbf{K}}{\mu}(\rho_f \mathbf{g} - \boldsymbol{\nabla}_x p^0). \tag{2.23}$$

For any generic function ϕ defined on Y (respectively, on Y_f and Y_s), we introduce its volume average over Y (respectively, on Y_f and Y_s) by

$$\langle \phi \rangle^\alpha = \frac{1}{|Y|} \int_{Y_\alpha} \phi(y) \mathrm{d}y, \tag{2.24}$$

where $|Y|$ indicates the volume of Y and α is empty, f, or s. We now apply the average operator $\langle \cdot \rangle^f$ to (2.23):

$$\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 = \frac{\langle \mathbf{K} \rangle^f}{\mu}(\rho_f \mathbf{g} - \boldsymbol{\nabla}_x p^0), \tag{2.25}$$

where $\phi = |Y_f|/|Y|$ is the porosity. This is the generalized Darcy law in the setting of a deformable porous medium. The matrix $\langle \mathbf{K} \rangle^f$ is the permeability tensor.

Third, apply the first and sixth equations of (2.12), the second equation of (2.13), and (2.17) to see that

$$\begin{aligned} &\boldsymbol{\nabla}_y \cdot \left[\mathbf{a}\mathbf{e}_y(\mathbf{u}_s^1)\right] = 0 && \text{in } Y_s, \\ &\mathbf{a}\mathbf{e}_y(\mathbf{u}_s^1) \cdot \mathbf{n} = -\left[\mathbf{a}\mathbf{e}_x(\mathbf{u}_s^0) + p^0 \mathbf{I}\right] \cdot \mathbf{n} && \text{on } Y_{fs}. \end{aligned} \tag{2.26}$$

This system forms an elliptic problem in y for $\mathbf{u}_s^1$ that can be solved in terms of $\mathbf{e}_x(\mathbf{u}_s^0)$ and p^0. To this end, for each k and h $(k, h = 1, 2, 3)$ we define the vector $\boldsymbol{\xi}^{kh}(y)$ to be the solution of

$$\begin{aligned} &\boldsymbol{\nabla}_y \cdot \left[\mathbf{a}\mathbf{e}_y(\boldsymbol{\xi}^{kh})\right] = 0 && \text{in } Y_s, \\ &\mathbf{a}\mathbf{e}_y(\boldsymbol{\xi}^{kh}) \cdot \mathbf{n} = -\mathbf{a}\tilde{\mathbf{e}} \cdot \mathbf{n} && \text{on } Y_{fs}, \end{aligned} \tag{2.27}$$

where the tensor $\tilde{\mathbf{e}} = (\tilde{e}_{ij})$ is given by

$$\tilde{e}_{ij} = \delta_{ik}\delta_{jh}, \qquad i, j = 1, 2, 3, \tag{2.28}$$

with δ_{ik} being the Kronecker symbol. Also, let the vector $\boldsymbol{\zeta}(y)$ satisfy

$$\begin{aligned} &\boldsymbol{\nabla}_y \cdot [\mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta})] = 0 && \text{in } Y_s, \\ &\mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta}) \cdot \mathbf{n} = -\mathbf{I} \cdot \mathbf{n} && \text{on } Y_{fs}. \end{aligned} \tag{2.29}$$

Now, $\mathbf{u}_s^1$ can be represented in terms of $\boldsymbol{\xi}^{kh}$ and $\boldsymbol{\zeta}$:

$$\mathbf{u}_s^1 = \boldsymbol{\xi}(y) \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \boldsymbol{\zeta}(y)p^0, \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}^{kh}), \tag{2.30}$$

up to an additive function of x and t; since only $\boldsymbol{\nabla}_y \cdot \mathbf{u}_s^1$ is used below, the ambiguity in this additive function is irrelevant. Utilizing the second equation of (2.13) and (2.30), we obtain

$$\boldsymbol{\sigma}_s^0 = \mathbf{a}\mathbf{e}_x(\mathbf{u}_s^0) + \mathbf{a}\mathbf{e}_y(\boldsymbol{\xi})\mathbf{e}_x(\mathbf{u}_s^0) + \mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta})p^0. \tag{2.31}$$

The total stress is defined as

$$\boldsymbol{\sigma}_T^0 = \begin{cases} \boldsymbol{\sigma}_f^0 & \text{in } Y_f, \\ \boldsymbol{\sigma}_s^0 & \text{in } Y_s. \end{cases} \tag{2.32}$$

Averaging $\boldsymbol{\sigma}_T^0$ over Y and using the fourth equation of (2.12) and (2.31), we see that

$$\left\langle \boldsymbol{\sigma}_T^0 \right\rangle = \langle \mathbf{a}\{\mathbf{I} + \mathbf{e}_y(\boldsymbol{\xi})\}\rangle^s \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \left(\langle \mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta})\rangle^s - \phi\mathbf{I}\right) p^0. \tag{2.33}$$

Fourth, integration of the first equation of (2.13) over Y_s leads to

$$\int_{Y_s} \boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_s^0 \mathrm{d}y + \int_{Y_s} \boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_s^1 \mathrm{d}y + \int_{Y_s} \rho_s \mathbf{g} \mathrm{d}y = 0. \tag{2.34}$$

Applying the divergence theorem, the periodicity condition, and the sixth equation of (2.13) to the second term in the left-hand side of this equation, we obtain

$$\int_{Y_s} \boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_s^0 \mathrm{d}y + \int_{Y_{fs}} \boldsymbol{\sigma}_f^1 \cdot \mathbf{n} \mathrm{d}\tau + \int_{Y_s} \rho_s \mathbf{g} \mathrm{d}y = 0. \tag{2.35}$$

Again, applying the divergence theorem, the periodicity condition, and the third equation of (2.13) to the second term in the left-hand side of this equation, we see that

$$\int_{Y} \boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_T^0 \mathrm{d}y + \int_{Y_s} \rho_s \mathbf{g} \mathrm{d}y + \int_{Y_f} \rho_f \mathbf{g} \mathrm{d}y = 0. \tag{2.36}$$

The mass density of the bulk material is defined as

$$\rho = \begin{cases} \rho_f & \text{in } Y_f, \\ \rho_s & \text{in } Y_s. \end{cases} \tag{2.37}$$

Then, by dividing by $|Y|$, (2.36) implies that

$$\boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle + \langle \rho \rangle \, \mathbf{g} = 0. \tag{2.38}$$

In the derivation of (2.38), we have used a so-called volume averaging theorem [10, 22], which, together with the periodicity condition, implies that $\left\langle \boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_T^0 \right\rangle = \boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle$.

Finally, integrating the fifth equation of (2.13) over Y_f, we find

$$\int_{Y_f} \boldsymbol{\nabla}_x \cdot \mathbf{v}^0 \mathrm{d}y + \int_{Y_f} \boldsymbol{\nabla}_y \cdot \mathbf{v}^1 \mathrm{d}y = 0. \tag{2.39}$$

Applying the divergence theorem, the periodicity condition, and the seventh equation of (2.13) to the second term in the left-hand side of this equation, we get

$$\int_{Y_f} \boldsymbol{\nabla}_x \cdot \mathbf{v}^0 \mathrm{d}y = -\int_{Y_{fs}} \dot{\mathbf{u}}_s^1 \cdot \mathbf{n} \mathrm{d}\tau. \tag{2.40}$$

Then it follows from a further application of the divergence theorem and periodicity condition and use of (2.30) that

$$\int_{Y_f} \boldsymbol{\nabla}_x \cdot \mathbf{v}^0 \mathrm{d}y = \int_{Y_s} (\boldsymbol{\nabla}_y \cdot \boldsymbol{\xi}) \cdot \mathbf{e}_x(\dot{\mathbf{u}}_s^0) \mathrm{d}y + \int_{Y_s} \boldsymbol{\nabla}_y \cdot \boldsymbol{\zeta} \dot{p}^0 \mathrm{d}y, \tag{2.41}$$

i.e.,

$$\boldsymbol{\nabla}_x \cdot \left(\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 \right) = \left(\left\langle \boldsymbol{\nabla}_y \cdot \boldsymbol{\xi} \right\rangle^s - \phi \mathbf{I} \right) \cdot \mathbf{e}_x(\dot{\mathbf{u}}_s^0) + \left\langle \boldsymbol{\nabla}_y \cdot \boldsymbol{\zeta} \right\rangle^s \dot{p}^0. \tag{2.42}$$

2.1.1. The macroscopic equations. The macroscopic equations are given by (2.25), (2.33), (2.38), and (2.42). It follows from (2.27) and (2.29) that (see Appendix)

$$\left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s = \left\langle \boldsymbol{\nabla}_y \cdot \boldsymbol{\xi} \right\rangle^s. \tag{2.43}$$

Introducing the notation

$$\mathbf{c} = \left\langle \mathbf{a}\{\mathbf{I} + \mathbf{e}_y(\boldsymbol{\xi})\} \right\rangle^s, \quad \boldsymbol{\alpha} = \left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s - \phi \mathbf{I}, \quad \gamma = \left\langle \boldsymbol{\nabla}_y \cdot \boldsymbol{\zeta} \right\rangle^s, \tag{2.44}$$

we can rewrite the macroscopic equations as follows:

$$\begin{aligned} &\left\langle \boldsymbol{\sigma}_T^0 \right\rangle = \mathbf{c} \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \boldsymbol{\alpha} p^0, \\ &\boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle + \left\langle \rho \right\rangle \mathbf{g} = 0, \\ &\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 = \left\langle \mathbf{K} \right\rangle^f (\rho_f \mathbf{g} - \boldsymbol{\nabla}_x p^0)/\mu, \\ &\boldsymbol{\nabla}_x \cdot \left(\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 \right) = \boldsymbol{\alpha} \cdot \mathbf{e}_x(\dot{\mathbf{u}}_s^0) + \gamma \dot{p}^0. \end{aligned} \tag{2.45}$$

We summarize the properties of the coefficients here. First, it can be shown that $\left\langle \mathbf{K} \right\rangle^f$ is symmetric and positive definite [20]. Also, note that $\boldsymbol{\alpha} = (\alpha_{ij})$ satisfies

$$\alpha_{ij} = \left\langle a_{ijkh} e_{ykh}(\boldsymbol{\zeta}) \right\rangle^s - \phi I_{ij} = \alpha_{ji}, \tag{2.46}$$

i.e., $\boldsymbol{\alpha}$ is symmetric. Next, with $\mathbf{c} = (c_{ijkh})$, it is proven in the appendix that

$$\begin{aligned} &c_{ijkh} = c_{jikh} = c_{ijhk} = c_{khij}, && i, j, k, h = 1, 2, 3, \\ &c_{ijkh} z_{ij} z_{kh} \geq c_* z_{ij} z_{ij}, && a_* > 0, \end{aligned} \tag{2.47}$$

where (z_{ij}) is a symmetric tensor. That is, $\mathbf{c}$ is symmetric and positive-definite. Finally, in the appendix we show that $\gamma < 0$. $\mathbf{c}$, $\boldsymbol{\alpha}$, and γ are the elastic coefficients.

2.1.2. Comparison with Biot's model. Let us recall the Biot's macroscopic equations for an anisotropic medium [6]. First, using Biot's notation, equation (2.16) in [6] reads as follows:

$$\begin{bmatrix} -\partial p/\partial x + \rho_f X \\ -\partial p/\partial y + \rho_f Y \\ -\partial p/\partial z + \rho_f Z \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}^{-1} \begin{bmatrix} \dot{U}_x - \dot{u}_x \\ \dot{U}_y - \dot{u}_y \\ \dot{U}_z - \dot{u}_z \end{bmatrix}. \tag{2.48}$$

This equation coincides with the third equation of (2.45) since it can be written, with $\left\langle \mathbf{v}^0 \right\rangle^f = \phi \dot{\mathbf{u}}_f^0$, as

$$\dot{\mathbf{u}}_f^0 - \dot{\mathbf{u}}_s^0 = \frac{\left\langle \mathbf{K} \right\rangle^f}{\phi \mu} (\rho_f \mathbf{g} - \boldsymbol{\nabla}_x p^0). \tag{2.49}$$

Notice the correspondence between our notation and Biot's:

$$\mathbf{U} = \mathbf{u}_f^0, \quad \mathbf{u} = \mathbf{u}_s^0, \quad \mathbf{g} = (X, Y, Z), \quad p = p^0, \quad k_{ij} = \frac{1}{|Y_f|\mu} \int_{Y_f} K_j^i \mathrm{d}y. \tag{2.50}$$

Next, equation (2.15) in [6] is stated as follows:

$$\begin{aligned} &\frac{\partial}{\partial x}(\sigma_{xx} + \sigma) + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho X = 0, \\ &\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} + \sigma) + \frac{\partial \sigma_{yz}}{\partial z} + \rho Y = 0, \\ &\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial}{\partial z}(\sigma_{zz} + \sigma) + \rho Z = 0. \end{aligned} \tag{2.51}$$

This is exactly the second equation of (2.45) since $\sigma = -fp$ in Biot's notation (f is ϕ in the present notation). Finally, equation (2.12) in [6] is written as

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} \\ & & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} \\ & & & c_{44} & c_{45} & c_{46} & c_{47} \\ & & & & c_{55} & c_{56} & c_{57} \\ & & & & & c_{66} & c_{67} \\ & & & & & & c_{77} \end{pmatrix} \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ \epsilon \end{pmatrix}, \tag{2.52}$$

where $c_{ij} = c_{ji}$ (i.e., the coefficient matrix is symmetric). Again, (2.52) is stated in Biot's notation, which may have a different meaning than the notation in this paper. In [6], $\epsilon = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$, for example, where ϵ_{ij} represents the strain component for the fluid. We now recover (2.52) from our macroscopic equations in (2.45).

It follows from the fourth equation of (2.45) that

$$\dot{p}^0 = \gamma^{-1} \phi \boldsymbol{\nabla}_x \cdot \left(\dot{\mathbf{u}}_f^0 - \dot{\mathbf{u}}_s^0 \right) - \gamma^{-1} \boldsymbol{\alpha} \cdot \mathbf{e}_x(\dot{\mathbf{u}}_s^0). \tag{2.53}$$

Integrating this equation over time and using the definition of $\boldsymbol{\alpha}$, we see that

$$p^0 = -\gamma^{-1} \left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \gamma^{-1} \phi \boldsymbol{\nabla}_x \cdot \mathbf{u}_f^0, \tag{2.54}$$

since the integration constant is zero by the fact that p^0, $\mathbf{u}_s^0$, and $\mathbf{u}_f^0$ vanish at the same time (see (2.12) and (2.31)). Also, by (2.31), we have

$$\left\langle \boldsymbol{\sigma}_s^0 \right\rangle^s = \mathbf{c} \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s p^0. \tag{2.55}$$

Substituting (2.54) into this equation, we see that

$$\left\langle \boldsymbol{\sigma}_s^0 \right\rangle^s = \left(\mathbf{c} - \gamma^{-1} \left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s \left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s \right) \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \gamma^{-1} \phi \left\langle \mathbf{a} \mathbf{e}_y(\boldsymbol{\zeta}) \right\rangle^s \boldsymbol{\nabla}_x \cdot \mathbf{u}_f^0. \tag{2.56}$$

Then (2.52) follows from (2.54) and (2.56) with the coefficients calculated by (2.27) and (2.29) (see Section 4.1 in the identification of the present variables and Biot's).

We remark that the equations derived here via homogenization coincide with Biot's equations via an intuitive approach. The present approach allows for the derivation of the constitutive relationships and the calculation of the coefficients in them. In particular, the coefficients can be determined by (2.21), (2.27), and (2.29). When the porous medium is isotropic, the present equations reduce to those derived in [5] for such a medium.

2.2. Elastic single-phase behavior. As mentioned in the last subsection, different choices for the value of β are possible, depending upon the magnitude of the quantity μ/l^2. In this subsection, we consider the case where this quantity is larger than that considered in the last subsection; accordingly, we take $\beta = 1$. That is, the fourth equation of (2.6) takes the form

$$\boldsymbol{\sigma}_f^\epsilon = -p^\epsilon \mathbf{I} + 2\mu\epsilon \mathbf{e}(\mathbf{v}^\epsilon) \qquad \text{in } \Omega_{\epsilon f}. \tag{2.57}$$

All other equations of (2.6) remain the same and so does their analysis. With the scaling in (2.57), system (2.18) becomes

$$\begin{aligned} &\mu\Delta_y \mathbf{v}^0 = \boldsymbol{\nabla}_y p^0 && \text{in } Y_f, \\ &\boldsymbol{\nabla}_y \cdot \mathbf{v}^0 = 0 && \text{in } Y_f, \\ &\mathbf{v}^0 = \dot{\mathbf{u}}_s^0 && \text{on } Y_{fs}. \end{aligned} \tag{2.58}$$

It has the trivial solution

$$\mathbf{v}^0 = \dot{\mathbf{u}}_s^0(x,t), \quad p^0 = p^0(x,t). \tag{2.59}$$

This means that no relative displacement occurs between the fluid and solid at the zero-th order. Thus the macroscopic behavior of the saturated porous medium is that of a single phase.

By (2.59), the fourth equation of (2.45) reduces to

$$\boldsymbol{\alpha} \cdot \mathbf{e}_x(\dot{\mathbf{u}}_s^0) + \gamma \dot{p}^0 = 0. \tag{2.60}$$

Integrating this equation over time, as before, we obtain

$$p^0 = -\gamma^{-1} \boldsymbol{\alpha} \cdot \mathbf{e}_x(\mathbf{u}_s^0). \tag{2.61}$$

Substituting this relation into the first equation of (2.45), we obtain the macroscopic elastic single-phase behavior

$$\begin{aligned} &\left\langle \boldsymbol{\sigma}_T^0 \right\rangle = (\mathbf{c} - \gamma^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}) \cdot \mathbf{e}_x(\mathbf{u}_s^0), \\ &\boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle + \langle \rho \rangle \, \mathbf{g} = 0. \end{aligned} \tag{2.62}$$

2.3. Viscoelasticity. We now consider the final case where $\beta = 0$. This choice corresponds to the case where the quantity μ/L^2, appropriate to the macroscale, is of order unity. Accordingly, the fourth equation of (2.6) becomes

$$\boldsymbol{\sigma}_f^\epsilon = -p^\epsilon \mathbf{I} + 2\mu \mathbf{e}(\mathbf{v}^\epsilon) \qquad \text{in } \Omega_{\epsilon f}. \tag{2.63}$$

Using (2.63), the fourth equations of (2.12) and (2.13) take the form, respectively,

$$\begin{aligned} &2\mu \mathbf{e}_y(\mathbf{v}^0) = 0 && \text{in } Y_f, \\ &\boldsymbol{\sigma}_f^0 = -p^0\mathbf{I} + 2\mu\mathbf{e}_x(\mathbf{v}^0) + 2\mu\mathbf{e}_y(\mathbf{v}^1) && \text{in } Y_f. \end{aligned} \tag{2.64}$$

The first equation of this system yields that

$$\mathbf{v}^0 = \mathbf{v}^0(x,t), \tag{2.65}$$

while substituting the second equation of this system into the third equation of (2.12) and using the fifth equation of (2.13) leads to

$$\mu\Delta_y \mathbf{v}^1 = \boldsymbol{\nabla}_y p^0, \qquad \text{in } Y_f. \tag{2.66}$$

To analyze (2.66), we introduce the displacement

$$\mathbf{u} = \begin{cases} \mathbf{u}_f & \text{in } Y_f, \\ \mathbf{u}_s & \text{in } Y_s, \end{cases} \tag{2.67}$$

and consider time harmonic motions with angular frequency ω:

$$\mathbf{v} = i\omega\mathbf{u}, \tag{2.68}$$

where i is the complex unit (i.e., $i^2 = -1$). Now, applying the first equation of (2.12), the second and fifth equations of (2.13), and (2.66), we obtain the system

$$\begin{aligned} &\boldsymbol{\nabla}_y \cdot \left[\mathbf{a}\mathbf{e}_y(\mathbf{u}^1)\right] = 0 && \text{in } Y_s, \\ &i\omega\mu\Delta_y \mathbf{u}^1 = \boldsymbol{\nabla}_y p^0 && \text{in } Y_f, \\ &\boldsymbol{\nabla}_x \cdot \mathbf{u}^0 + \boldsymbol{\nabla}_y \cdot \mathbf{u}^1 = 0 && \text{in } Y_f, \end{aligned} \tag{2.69}$$

together with the boundary conditions given by the last two equations of (2.13). This system can be solved in the same manner as for (2.26):

$$\mathbf{u}^1 = \boldsymbol{\xi}(y, i\omega) \cdot \mathbf{e}_x(\mathbf{u}^0), \quad p^0 = \boldsymbol{\zeta}(y, i\omega) \cdot \mathbf{e}_x(\mathbf{u}^0), \tag{2.70}$$

where we ignore an additive function of x with the same reasoning as before. Note that p^0 has the same expression as in (2.61). Consequently, with the same analysis as in the last subsection, we derive the macroscopic equations

$$\begin{aligned} &\left\langle \boldsymbol{\sigma}_T^0 \right\rangle = \tilde{\mathbf{c}} \cdot \mathbf{e}_x(\mathbf{u}^0), \\ &\boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle + \langle \rho \rangle \, \mathbf{g} = 0. \end{aligned} \tag{2.71}$$

These equations again correspond to single phase behavior. Now $\tilde{\mathbf{c}}$ is complex-valued and depends on ω. These equations describe the macroscopic behavior of a linear viscoelastic system.

Stokes flow through a deformable porous medium saturated with a Newtonian fluid has been analyzed in this section. In the case where the scaled viscosity of the fluid is small, the macroscopic equations for the mechanical behavior of such a medium coincide with Biot's equations. In the case where the scaled viscosity is large, we have derived different differential equations and constitutive relationships. Navier-Stokes flow is considered in the next section.

3. Navier-Stokes Flow through a Deformable Medium

In this section we extend the techniques developed in the previous section to the Navier-Stokes flow through a deformable medium. Namely, we use the Navier equation in Ω_s, the Navier-Stokes equation in Ω_f, and the continuity equations of the normal stress and the displacement at the solid-fluid interface Γ_{fs}:

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_s + \rho_s \mathbf{g} = 0 && \text{in } \Omega_s, \\ &\boldsymbol{\sigma}_s = \mathbf{a}\mathbf{e}(\mathbf{u}_s) && \text{in } \Omega_s, \\ &\rho_f (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_f + \rho_f \mathbf{g} && \text{in } \Omega_f, \\ &\boldsymbol{\sigma}_f = -p\mathbf{I} + 2\mu\mathbf{e}(\mathbf{v}) && \text{in } \Omega_f, \\ &\boldsymbol{\nabla} \cdot \mathbf{v} = 0 && \text{in } \Omega_f, \\ &\boldsymbol{\sigma}_s \cdot \mathbf{n} = \boldsymbol{\sigma}_f \cdot \mathbf{n} && \text{on } \Gamma_{fs}, \\ &\mathbf{u}_s = \mathbf{u}_f && \text{on } \Gamma_{fs}. \end{aligned} \tag{3.1}$$

With the same notation as in the last section, the scaled problem is stated as follows:

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_s^\epsilon + \rho_s \mathbf{g} = 0 && \text{in } \Omega_{\epsilon s}, \\ &\boldsymbol{\sigma}_s^\epsilon = \mathbf{a}\mathbf{e}(\mathbf{u}_s^\epsilon) && \text{in } \Omega_{\epsilon s}, \\ &\rho_f (\mathbf{v}^\epsilon \cdot \boldsymbol{\nabla})\mathbf{v}^\epsilon = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_f^\epsilon + \rho_f \mathbf{g} && \text{in } \Omega_{\epsilon f}, \\ &\boldsymbol{\sigma}_f^\epsilon = -p^\epsilon \mathbf{I} + 2\mu\epsilon^\beta \mathbf{e}(\mathbf{v}^\epsilon) && \text{in } \Omega_{\epsilon f}, \\ &\boldsymbol{\nabla} \cdot \mathbf{v}^\epsilon = 0 && \text{in } \Omega_{\epsilon f}, \\ &\boldsymbol{\sigma}_s^\epsilon \cdot \mathbf{n} = \boldsymbol{\sigma}_f^\epsilon \cdot \mathbf{n} && \text{on } \Gamma_{\epsilon f s}, \\ &\epsilon^\eta \mathbf{u}_s^\epsilon = \mathbf{u}_f^\epsilon && \text{on } \Gamma_{\epsilon f s}. \end{aligned} \tag{3.2}$$

Note that we have not just scaled the viscosity coefficient but also the displacement continuity equation at the solid-fluid interface based on the consideration below. The solutions of (3.2) are expanded as in (2.9) except for $\mathbf{v}^\epsilon$ and $\mathbf{u}_f^\epsilon$. They are now expanded in the asymptotic form

$$\begin{aligned} \mathbf{v}^\epsilon(x,t) &= \epsilon^\alpha \mathbf{v}^0(x,y,t) + \epsilon^{\alpha+1}\mathbf{v}^1(x,y,t) + \dots, \\ \mathbf{u}_f^\epsilon(x,t) &= \epsilon^\alpha \mathbf{u}_f^0(x,y,t) + \epsilon^{\alpha+1}\mathbf{u}_f^1(x,y,t) + \dots, \end{aligned} \tag{3.3}$$

where α (together with β and η) is to be determined below. As demonstrated in the last section, β needs to be larger than one to have a fluid-solid two-phase macroscopic behavior. Also, keep in mind our attempt to model nonlinear inertial effects in the present case. Then, with the same analysis as in [11], a reasonable choice for α and β is

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}. \tag{3.4}$$

To balance the displacement continuity at Γ_{fs}, we take $\eta = 1/2$.

With the above choice, all the equations in (2.12) and (2.13) remain the same except the third equation of (2.13), which now becomes

$$\rho_f(\mathbf{v}^0 \cdot \boldsymbol{\nabla}_y)\mathbf{v}^0 = \boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma}_f^0 + \boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_f^1 + \rho_f \mathbf{g} \qquad \text{in } Y_f. \tag{3.5}$$

Consequently, we focus on the analysis of this equation.

It follows from the fourth and seventh equations of (2.12), the fourth equation of (2.13), and (3.5) that

$$\begin{aligned} &\rho_f(\mathbf{v}^0 \cdot \boldsymbol{\nabla}_y)\mathbf{v}^0 = \mu\Delta_y \mathbf{v}^0 - \boldsymbol{\nabla}_y p^1 + \rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0 && \text{in } Y_f, \\ &\boldsymbol{\nabla}_y \cdot \mathbf{v}^0 = 0 && \text{in } Y_f, \\ &\mathbf{v}^0 = \dot{\mathbf{u}}_s^0 && \text{on } Y_{fs}. \end{aligned} \tag{3.6}$$

With V_Y defined as before, (3.6) can be written in a variational formulation. First, observe that, by the definition of V_Y and the divergence theorem,

$$(\boldsymbol{\nabla}_y p^1, \mathbf{w})_{Y_f} = (p^1, \mathbf{w}\cdot\mathbf{n})_{Y_{fs}} = 0 \qquad \forall \mathbf{w} \in V_Y. \tag{3.7}$$

Then a further application of the definition of V_Y and the divergence theorem to the first equation of (3.6) implies that

$$\rho_f\left((\mathbf{v}^0 \cdot \boldsymbol{\nabla}_y)\mathbf{v}^0, \mathbf{w}\right)_{Y_f} + \mu(\boldsymbol{\nabla}_y\mathbf{v}^0, \boldsymbol{\nabla}_y\mathbf{w})_{Y_f} = (\rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0, \mathbf{w})_{Y_f} \qquad \forall \mathbf{w} \in V_Y. \tag{3.8}$$

Note that (3.8) always has a solution, and if $\|\rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0\|_{V_Y^*}$ is not very large, the solution is unique [16], where $\|\cdot\|_{V_Y^*}$ is the dual norm to V_Y. For large values of $\|\rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0\|_{V_Y^*}$, the uniqueness fails and bifurcations may arise. In this situation, there is no Darcy's law.

For a fixed $\mathbf{z}_0 \in H^1(Y_f)$, we introduce an operator $\mathbf{J} = \mathbf{J}(\mathbf{z}_0) \in \mathcal{L}(V_Y, V_Y)$ in the following way: for each $\mathbf{z} \in V_Y$, $\mathbf{J}\mathbf{z} \in V_Y$ is the solution of the Stokes problem

$$(\boldsymbol{\nabla}_y(\mathbf{J}\mathbf{z}), \boldsymbol{\nabla}_y\mathbf{w})_{Y_f} = ((\mathbf{z}_0 \cdot \boldsymbol{\nabla}_y)\mathbf{z}, \mathbf{w})_{Y_f} \qquad \forall \mathbf{w} \in V_Y. \tag{3.9}$$

The operator $\mathbf{J}$ depends on $\mathbf{z}_0$ as a parameter. Note that, by the Sobolev imbedding,

$$\begin{aligned} |\left((\mathbf{z}_0 \cdot \boldsymbol{\nabla}_y)\mathbf{z}, \mathbf{w}\right)_{Y_f}| &\le C\|\mathbf{z}_0\|_{L^4(Y_f)}\|\mathbf{w}\|_{L^4(Y_f)}|\mathbf{z}|_{H^1(Y_f)} \\ &\le C|\mathbf{z}_0|_{H^1(Y_f)}|\mathbf{w}|_{H^1(Y_f)}|\mathbf{z}|_{H^1(Y_f)}. \end{aligned} \tag{3.10}$$

Thus (3.9) is well-defined. With the definition of $\mathbf{K}$ (see (2.21)) and $\mathbf{J}$, (3.8) can be written as

$$\left\{\rho_f\mathbf{J}(\mathbf{v}^0) + \mu\mathbf{I}\right\}(\mathbf{v}^0 - \dot{\mathbf{u}}_s^0) = \mathbf{K}(\rho\mathbf{g} - \boldsymbol{\nabla}_x p^0). \tag{3.11}$$

Applying the average operator $\langle\cdot\rangle^f$ to (3.11), we obtain

$$\rho_f \left\langle \mathbf{J}(\mathbf{v}^0)(\mathbf{v}^0 - \dot{\mathbf{u}}_s^0)\right\rangle^f + \mu\left(\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right) = \langle\mathbf{K}\rangle^f\,(\rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0). \tag{3.12}$$

When $\mathbf{J}(\mathbf{v}^0)$ is zero, (3.12) represents the generalized Darcy's law in (2.25). When it is not zero, (3.12) is the generalized Forchheimer's law for a deformable porous medium.

Other macroscopic equations can be obtained in the exactly same manner as in the last section; we just summarize them, together with (3.12), as follows:

$$\begin{aligned}
&\left\langle \boldsymbol{\sigma}_T^0\right\rangle = \mathbf{c}\cdot\mathbf{e}_x(\mathbf{u}_s^0) + \boldsymbol{\alpha}p^0,\\
&\boldsymbol{\nabla}_x\cdot\left\langle \boldsymbol{\sigma}_T^0\right\rangle + \langle\rho\rangle\,\mathbf{g} = 0,\\
&\rho_f \left\langle \mathbf{J}(\mathbf{v}^0)(\mathbf{v}^0 - \dot{\mathbf{u}}_s^0)\right\rangle^f + \mu\left(\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right) = \langle\mathbf{K}\rangle^f\,(\rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0),\\
&\boldsymbol{\nabla}_x\cdot\left(\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right) = \boldsymbol{\alpha}\cdot\mathbf{e}_x(\dot{\mathbf{u}}_s^0) + \gamma\dot{p}^0.
\end{aligned} \tag{3.13}$$

Again, the coefficients $\mathbf{c}$, $\boldsymbol{\alpha}$, and γ are determined by (2.44) and calculated from (2.26) and (2.27). They have the same properties as in the last section. Furthermore, $\mathbf{J}$ is symmetric as shown in [11].

3.1. The isotropic case. We now examine the nonlinear correction term in the Forchheimer law (3.12) for an isotropic medium, following the treatment presented in [3, 11, 17]. We first review some classical results on second-order, tensor-valued isotropic functions [8, 21]. A second-order, symmetric, tensor-valued, isotropic function of a vector $\mathbf{w} = (w_1, w_2, w_3)$, $\mathbf{L} = (L_{ij})_{i,j=1,2,3}$, can be expressed as follows:

$$L_{ij}(\mathbf{w}) = a_0(|\mathbf{w}|)\delta_{ij} + a_1(|\mathbf{w}|)w_iw_j, \tag{3.14}$$

where a_0 and a_1 are scalar-valued isotropic functions of $|\mathbf{w}|$ and $|\mathbf{w}| = (w_1^2 + w_2^2 + w_3^2)^{1/2}$. On the other hand, a second-order, skew-symmetric, tensor-valued, isotropic function of a vector is identically zero. Applying a series expansion to (3.14), we see that

$$\begin{aligned}
L_{ij}(\mathbf{w}) =& (a_0^0 + a_0^1|\mathbf{w}| + a_0^2|\mathbf{w}|^2 + \ldots)\delta_{ij}\\
&+(a_1^0 + a_1^1|\mathbf{w}| + a_1^2|\mathbf{w}|^2 + \ldots)w_iw_j.
\end{aligned} \tag{3.15}$$

With $\mathbf{v}^T$ being the transpose of the vector $\mathbf{v}$, set

$$\begin{aligned}
\mathbf{H} =& \mathbf{H}\left(\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right)\\
\equiv& \frac{\rho_f}{\left|\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right|^2}\left\langle \mathbf{J}(\mathbf{v}^0)\left(\mathbf{v}^0 - \dot{\mathbf{u}}_s^0\right)\right\rangle^f \left(\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right)^T\mathbf{I} + \mu\mathbf{I}.
\end{aligned} \tag{3.16}$$

Note that

$$\mathbf{H} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) + \frac{1}{2}(\mathbf{H} - \mathbf{H}^T), \tag{3.17}$$

where $\mathbf{H}^T$ indicates the transpose of the operator $\mathbf{H}$ (or the adjoint of $\mathbf{H}$ with respect to the inner product (2.20)). With the application of (3.15) to $(\mathbf{H}+\mathbf{H}^T)/2$ and the skew-symmetry property of $(\mathbf{H} - \mathbf{H}^T)/2$, we see that

$$\mathbf{H} = \left(H_0^0 + H_0^1\left|\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right|\right)\mathbf{I} + O\left(\left|\left\langle \mathbf{v}^0\right\rangle^f - \phi\dot{\mathbf{u}}_s^0\right|^2\right), \tag{3.18}$$

for some constants H_0^0 and H_0^1. Substituting (3.18) into (3.12) and retaining the first two terms in (3.18), we obtain

$$\left(H_0^0 + H_0^1 \left|\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 \right|\right) \left(\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0\right) = \langle \mathbf{K} \rangle^f (\rho \mathbf{g} - \boldsymbol{\nabla}_x p^0). \tag{3.19}$$

This is the Forchheimer law for an isotropic deformable porous medium, and we see a quadratic correction here. If desired, it is possible to retain higher order terms (e.g., the third order term in $\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0$) in (3.19).

We have derived the macroscopic equations for Navier-Stokes flow in the framework of a deformable medium in this section. A generalized Forchheimer law has been obtained to take into account the nonlinear inertial effects on the fluid flow through such a medium. The nonlinear correction term in this law has been shown to be quadratic in velocity for an isotropic medium for the present choice of scaling. Qausi-static flows have been considered in this and last sections. Transient flow is analyzed in the next section.

4. Transient Flow through a Deformable Medium

In this section we take into account transient inertial effects at the pore scale. As an example, we only consider the transient Stokes flow; the transient Navier-Stokes flow can be analyzed in a similar fashion using the techniques in the previous section. The microscopic equations are given by

$$\begin{aligned}
&\rho_s \ddot{\mathbf{u}}_s = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_s + \rho_s \mathbf{g} && \text{in } \Omega_s, \\
&\boldsymbol{\sigma}_s = \mathbf{a} \mathbf{e}(\mathbf{u}_s) && \text{in } \Omega_s, \\
&\rho_f \dot{\mathbf{v}} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_f + \rho_f \mathbf{g} && \text{in } \Omega_f, \\
&\boldsymbol{\sigma}_f = -p\mathbf{I} + 2\mu \mathbf{e}(\mathbf{v}) && \text{in } \Omega_f, \\
&\boldsymbol{\nabla} \cdot \mathbf{v} = 0 && \text{in } \Omega_f, \\
&\boldsymbol{\sigma}_s \cdot \mathbf{n} = \boldsymbol{\sigma}_f \cdot \mathbf{n} && \text{on } \Gamma_{fs}, \\
&\mathbf{u}_s = \mathbf{u}_f && \text{on } \Gamma_{fs}.
\end{aligned} \tag{4.1}$$

For simplicity we seek solutions of the form $\mathbf{u}e^{i\omega t}$ with angular frequency ω. Then the microscopic equations become (with modified body force terms, which are denoted with the same notation for convenience)

$$\begin{aligned}
&-\omega^2 \rho_s \mathbf{u}_s = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_s + \rho_s \mathbf{g} && \text{in } \Omega_s, \\
&\boldsymbol{\sigma}_s = \mathbf{a} \mathbf{e}(\mathbf{u}_s) && \text{in } \Omega_s, \\
&i\omega \rho_f \mathbf{v} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_f + \rho_f \mathbf{g} && \text{in } \Omega_f, \\
&\boldsymbol{\sigma}_f = -p\mathbf{I} + 2\mu \mathbf{e}(\mathbf{v}) && \text{in } \Omega_f, \\
&\boldsymbol{\nabla} \cdot \mathbf{v} = 0 && \text{in } \Omega_f, \\
&\boldsymbol{\sigma}_s \cdot \mathbf{n} = \boldsymbol{\sigma}_f \cdot \mathbf{n} && \text{on } \Gamma_{fs}, \\
&\mathbf{u}_s = \mathbf{u}_f && \text{on } \Gamma_{fs}.
\end{aligned} \tag{4.2}$$

The scaled problem is then given by

$$
\begin{aligned}
&-\omega^2\rho_s\mathbf{u}_s^\epsilon = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_s^\epsilon + \rho_s\mathbf{g} && \text{in } \Omega_{\epsilon s},\\
&\boldsymbol{\sigma}_s^\epsilon = \mathbf{ae}(\mathbf{u}_s^\epsilon) && \text{in } \Omega_{\epsilon s},\\
&i\omega\rho_f\mathbf{v}^\epsilon = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f^\epsilon + \rho_f\mathbf{g} && \text{in } \Omega_{\epsilon f},\\
&\boldsymbol{\sigma}_f^\epsilon = -p^\epsilon\mathbf{I} + 2\mu\epsilon^2\mathbf{e}(\mathbf{v}^\epsilon) && \text{in } \Omega_{\epsilon f},\\
&\boldsymbol{\nabla}\cdot\mathbf{v}^\epsilon = 0 && \text{in } \Omega_{\epsilon f},\\
&\boldsymbol{\sigma}_s^\epsilon\cdot\mathbf{n} = \boldsymbol{\sigma}_f^\epsilon\cdot\mathbf{n} && \text{on } \Gamma_{\epsilon fs},\\
&\mathbf{u}_s^\epsilon = \mathbf{u}_f^\epsilon && \text{on } \Gamma_{\epsilon fs}.
\end{aligned} \tag{4.3}
$$

Note that we have in mind the two-phase fluid-solid macroscopic behavior, so we have taken $\beta = 2$ above.

As in the second section, we substitute (2.9) into (4.3), apply (2.8), and collect terms with like powers of ϵ. The following equations are analogous to those in (2.12) and (2.13):

$$
\begin{aligned}
&\boldsymbol{\nabla}_y\cdot\boldsymbol{\sigma}_s^0 = 0 && \text{in } Y_s,\\
&\mathbf{ae}_y(\mathbf{u}_s^0) = 0 && \text{in } Y_s,\\
&\boldsymbol{\nabla}_y\cdot\boldsymbol{\sigma}_f^0 = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_f^0 = -p^0\mathbf{I} && \text{in } Y_f,\\
&\boldsymbol{\nabla}_y\cdot\mathbf{v}^0 = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_s^0\cdot\mathbf{n} = \boldsymbol{\sigma}_f^0\cdot\mathbf{n} && \text{on } Y_{fs},\\
&\mathbf{u}_s^0 = \mathbf{u}_f^0 && \text{on } Y_{fs},
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
&-\omega^2\rho_s\mathbf{u}_s^0 = \boldsymbol{\nabla}_x\cdot\boldsymbol{\sigma}_s^0 + \boldsymbol{\nabla}_y\cdot\boldsymbol{\sigma}_s^1 + \rho_s\mathbf{g} && \text{in } Y_s,\\
&\boldsymbol{\sigma}_s^0 = \mathbf{ae}_x(\mathbf{u}_s^0) + \mathbf{ae}_y(\mathbf{u}_s^1) && \text{in } Y_s,\\
&i\omega\rho_f\mathbf{v}^0 = \boldsymbol{\nabla}_x\cdot\boldsymbol{\sigma}_f^0 + \boldsymbol{\nabla}_y\cdot\boldsymbol{\sigma}_f^1 + \rho_f\mathbf{g} && \text{in } Y_f,\\
&\boldsymbol{\sigma}_f^1 = -p^1 I + 2\mu\mathbf{e}_y(\mathbf{v^0}) && \text{in } Y_f,\\
&\boldsymbol{\nabla}_x\cdot\mathbf{v}^0 + \boldsymbol{\nabla}_y\cdot\mathbf{v}^1 = 0 && \text{in } Y_f,\\
&\boldsymbol{\sigma}_s^1\cdot\mathbf{n} = \boldsymbol{\sigma}_f^1\cdot\mathbf{n} && \text{on } Y_{fs},\\
&\mathbf{u}_s^1 = \mathbf{u}_f^1 && \text{on } Y_{fs}.
\end{aligned} \tag{4.5}
$$

As for (2.15)–(2.17), we have

$$
p^0 = p^0(x,t), \quad \boldsymbol{\sigma}_f^0 = \boldsymbol{\sigma}_f^0(x,t), \quad \mathbf{u}_s^0 = \mathbf{u}_s^0(x,t). \tag{4.6}
$$

Next, it follows from the fourth and seventh equations of (4.4) and the third and fourth equations of (4.5) that

$$
\begin{aligned}
&i\omega\rho_f\mathbf{v}^0 = \mu\Delta_y\mathbf{v}^0 - \boldsymbol{\nabla}_y p^1 + \rho_f\mathbf{g} - \boldsymbol{\nabla}_x p^0 && \text{in } Y_f,\\
&\boldsymbol{\nabla}_y\cdot\mathbf{v}^0 = 0 && \text{in } Y_f,\\
&\mathbf{v}^0 = i\omega\mathbf{u}_s^0 && \text{on } Y_{fs}.
\end{aligned} \tag{4.7}
$$

To analyze (4.7), let

$$
\mathbf{w} = \mathbf{v}^0 - i\omega\mathbf{u}_s^0. \tag{4.8}
$$

Then (4.7) becomes

$$
\begin{aligned}
&i\omega\rho_f \mathbf{w} - \mu\Delta_y \mathbf{w} = -\boldsymbol{\nabla}_y p^1 + \rho_f \mathbf{g} - \boldsymbol{\nabla}_x p^0 + \omega^2 \rho_f \mathbf{u}_s^0 && \text{in } Y_f,\\
&\boldsymbol{\nabla}_y \cdot \mathbf{w} = 0 && \text{in } Y_f,\\
&\mathbf{w} = 0 && \text{on } Y_{fs}.
\end{aligned} \tag{4.9}
$$

Using (3.7), (4.9) is written in the variational formulation

$$
\begin{aligned}
&(i\omega\rho_f \mathbf{w}, \mathbf{z})_{Y_f} + (\mu \boldsymbol{\nabla}_y \mathbf{w}, \boldsymbol{\nabla}_y \mathbf{z})_{Y_f}\\
&\quad = -(\boldsymbol{\nabla}_x p^0 - \rho_f \mathbf{g} - \omega^2 \rho_f \mathbf{u}_s^0, \mathbf{z})_{Y_f}, \qquad \forall \mathbf{z} \in V_Y.
\end{aligned} \tag{4.10}
$$

For $i = 1, 2, 3$, the components of the permeability tensor $\mathbf{K}^i \in V_Y$ are defined to be the solution of

$$
(i\omega\rho_f \mathbf{K}^i, \mathbf{z})_{Y_f} + \mu(\boldsymbol{\nabla}_y \mathbf{K}^i, \boldsymbol{\nabla}_y \mathbf{z})_{Y_f} = (1, z_i)_{Y_f} \qquad \forall \mathbf{z} = (z_1, z_2, z_3) \in V_Y. \tag{4.11}
$$

Now, the solution to (4.10) is expressed by

$$
\mathbf{w} = \mathbf{v}^0 - i\omega \mathbf{u}_s^0 = -\mathbf{K}\left(\boldsymbol{\nabla}_x p^0 - \rho_f \mathbf{g} - \omega^2 \rho_f \mathbf{u}_s^0\right). \tag{4.12}
$$

Applying the average operator $\langle \cdot \rangle^f$ to this equation, we see that

$$
\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 = -\langle \mathbf{K} \rangle^f \left(\boldsymbol{\nabla}_x p^0 - \rho_f \mathbf{g} + \rho_f \ddot{\mathbf{u}}_s^0\right). \tag{4.13}
$$

This is the generalized Darcy law for transient flow through a deformable porous medium. Note that $\langle \mathbf{K} \rangle^f$ is complex-valued and depends on ω.

In the exactly same manner as for (2.33), it follows from the first and sixth equations of (4.4) and the second equation of (4.5) that

$$
\left\langle \boldsymbol{\sigma}_T^0 \right\rangle = \mathbf{c} \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \boldsymbol{\alpha} p^0, \tag{4.14}
$$

where $\mathbf{c}$ and $\boldsymbol{\alpha}$ are given by (2.44). Also, as for (2.38), we apply the first, third, and sixth equations of (4.5) to obtain

$$
\rho_f \left\langle \dot{\mathbf{v}}^0 \right\rangle^f + \rho_s \left\langle \ddot{\mathbf{u}}_s^0 \right\rangle^s = \boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle + \langle \rho \rangle \mathbf{g}, \tag{4.15}
$$

and, as for (2.42), employing the fifth and seven equations of (4.5), we see that

$$
\boldsymbol{\nabla}_x \cdot \left(\left\langle \mathbf{v}^0 \right\rangle^f - \phi \dot{\mathbf{u}}_s^0 \right) = \boldsymbol{\alpha} \cdot \mathbf{e}_x(\dot{\mathbf{u}}_s^0) + \gamma \dot{p}^0. \tag{4.16}
$$

The macroscopic equations are given by (4.13)–(4.16).

4.1. Comparison with Biot's model. For transient flow through an anisotropic poroelastic medium, equations (2.3), (5.1), and (5.2) in [7] are

$$
\begin{aligned}
&\frac{\partial \tau_{ij}}{\partial x_j} = \rho \ddot{u}_i + \rho_f \ddot{w}_i,\\
&\tau_{ij} = A_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij}\zeta,\\
&p_f = M_{ij} e_{ij} + M\zeta,\\
&-\frac{\partial p_f}{\partial x_i} - \rho_f \ddot{u}_i = \overline{Y}_{ij}(p) \dot{w}_j.
\end{aligned} \tag{4.17}
$$

The displacement of the fluid relative to the solid is introduced

$$
\mathbf{w} = \phi\left(\mathbf{u}_f^0 - \mathbf{u}_s^0\right). \tag{4.18}
$$

Then, using the definition of ρ in (2.37), (4.15) becomes

$$
\boldsymbol{\nabla}_x \cdot \left\langle \boldsymbol{\sigma}_T^0 \right\rangle = \langle \rho \rangle \ddot{\mathbf{u}}_s^0 + \rho_f \ddot{\mathbf{w}} - \langle \rho \rangle \mathbf{g}. \tag{4.19}
$$

This equation agrees with the first equation of (4.17) that omits the gravity term. Also, it follows from (2.54) and the definition of $\boldsymbol{\alpha}$ in (2.44) that

$$p^0 = -\gamma^{-1}\boldsymbol{\alpha} \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \gamma^{-1}\boldsymbol{\nabla}_x \cdot \mathbf{w}. \tag{4.20}$$

Substituting (4.20) into (4.14), we see that

$$\langle \boldsymbol{\sigma}_T^0 \rangle = \left(\mathbf{c} - \gamma^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}\right) \cdot \mathbf{e}_x(\mathbf{u}_s^0) + \gamma^{-1}\boldsymbol{\alpha}\boldsymbol{\nabla}_x \cdot \mathbf{w}. \tag{4.21}$$

This is the second equation of (4.17), where ζ is $-\boldsymbol{\nabla}_x \cdot \mathbf{w}$. Next, (4.20) is the third equation of (4.17). Finally, with the definition of $\mathbf{w}$ in (4.18), (4.13) becomes

$$-\boldsymbol{\nabla}_x p^0 - \rho_f \ddot{\mathbf{u}}_s^0 = \phi \left(\langle \mathbf{K} \rangle^f\right)^{-1} \dot{\mathbf{w}} - \rho_f \mathbf{g}. \tag{4.22}$$

This agrees with the fourth equation of (4.17) that again omits the gravity term. The following table identifies the present variables and the corresponding Biot variables:

(4.23)

Present variables	Biot's in [7]
$\langle \boldsymbol{\sigma}_T^0 \rangle$	$\boldsymbol{\tau}$
$\langle \rho \rangle$	ρ
$\mathbf{u}_s^0$	$\mathbf{u}$
ρ_f	ρ_f
$\mathbf{w}$	$\mathbf{w}$
$\left(\mathbf{c} - \gamma^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}\right)$	$\mathbf{A} = (A_{ij}^{\mu\nu})$
$-\gamma^{-1}\boldsymbol{\alpha}$	$\mathbf{M} = (M_{ij})$
$-\boldsymbol{\nabla}_x \cdot \mathbf{w}$	ζ
$\mathbf{e}_x(\mathbf{u}_s^0)$	$\mathbf{e} = (e_{ij})$
p^0	p_f
$-\gamma^{-1}\phi$	M
$\phi \left(\langle \mathbf{K} \rangle^f\right)^{-1}$	$\overline{\mathbf{Y}}(p)$

5. Numerical Experiments

In order to obtain a solution to the microscopic model, we consider a relatively simple problem where fluid flows in connected pores and small channels of a periodic porous medium. The geometry of the pore structure is shown in Fig. 1 [18]. The initial channel width variation is given by $\theta(x) = 0.1\sin(x_1)$, and the initial channel width is defined by $W(x) = 2\theta(x)$.

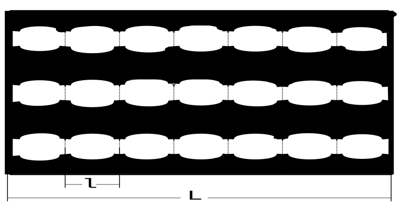

Fig. 1. A periodic porous medium.

The microscopic equations governing the motion of the fluid and solid are determined by (4.1). For the present problem, we solve for the vertical displacement

$w(x_1, w_2, t)$ of the solid, and the elastic tensor $\mathbf{a}$ is chosen such that the Navier equation in Ω_s in (4.1) is of the form

$$\rho_s \ddot{w} = G \frac{\partial^2 w}{\partial x_1^2} + (2G + \lambda) \frac{\partial^2 w}{\partial x_2^2} \quad \text{in } \Omega_s, \tag{5.1}$$

where G and λ are the solid rigidity modulus and Lamé parameter, respectively, and the gravity term is ignored. In the numerical examples of this section, these constants are given by $\rho_s = 4.907$ g/cm^3 and $G = \lambda = 10^{10}$ Pa. Also, we assume that the pressure profile over the cross section of the channel is flat; i.e., the pressure is of the form $p(x_1, t)$. The microscopic equations of the fluid are given as in (4.1); that is,

$$\begin{aligned} &\rho_f \dot{\mathbf{v}} = \boldsymbol{\nabla} \cdot (-p\mathbf{I} + 2\mu \mathbf{e}(\mathbf{v})) && \text{in } \Omega_f, \\ &\boldsymbol{\nabla} \cdot \mathbf{v} = 0 && \text{in } \Omega_f, \end{aligned} \tag{5.2}$$

where $\mu = 10^{-3}$ Pa $\cdot$ s and $\rho_f = 0.9814$ g/cm^3. At the solid-fluid interface, the continuity equations of the normal stress and displacement are imposed as in (4.1). Periodic boundary conditions are used at the outer boundary of a periodic cell, and the initial conditions are given by

$$w = \dot{w} = 0, \quad \mathbf{v} = \mathbf{0} \qquad \text{at } t = 0. \tag{5.3}$$

The microscale l and macroscale L are taken to be $l = 10^{-3}$ m and $L = 10^2$ m. The corresponding macroscopic equations can be derived as in the fourth section.

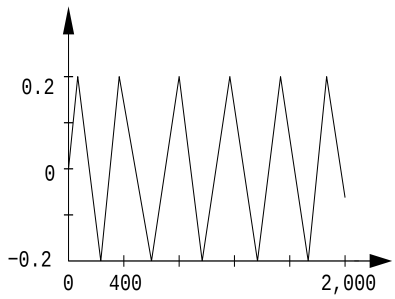

Fig. 2. Perturbation signal.

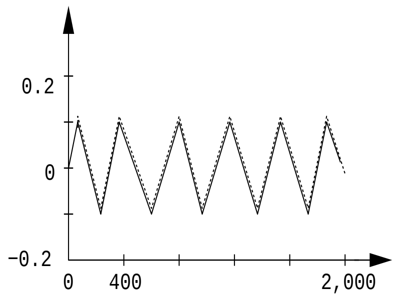

Fig. 3. Averaged fluid velocities of the first example: $\cdots$ pore scale, — homogenized.

The microscopic and macroscopic equations are numerically solved for the pressure p, fluid velocity v, and solid displacement w. The mixed finite element method, based on the lowest-order Raviart-Thomas mixed space on triangles [19], is utilized to solve these equations. The solution procedure is based on a sequential scheme;

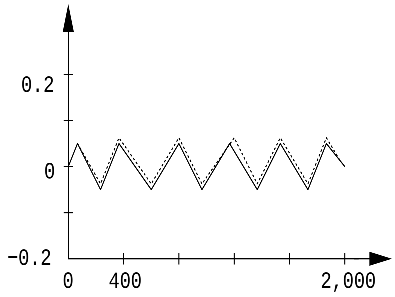

Fig. 4. Averaged solid velocities of the first example: $\cdots$ pore scale, — homogenized.

namely, the fluid and solid equations are decoupled and solved in an alternating manner.

The numerical examples in this section are employed to investigate the response of a fluid in an elastic channel to sinusoidal pressure perturbations and to present a computational validation of the homogenized model in the fourth section. The pressure perturbation is given by

$$p(x_1, t) = p_s \sin(\omega t) \qquad \text{at } x_1 = 0, \tag{5.4}$$

where ω is the perturbation signal frequency and p_s is the magnitude constant of pressure signal. In the first example, this frequency is chosen by $\omega = 10^2$ Hz. The following dimensionless pressure and velocities are displayed:

$$p^\star = \frac{p}{\rho_f v_0^2}, \quad v_1^\star = \frac{v_1}{v_0}, \tag{5.5}$$

where v_0 is a characteristic fluid velocity and $\mathbf{v} = (v_1, v_2, v_3)$. Furthermore, the fluid and solid velocities are averaged over the pore and solid space, respectively, in the displays. The perturbation pressure signal $-\mathrm{d}p^\star/\mathrm{d}x_1$ (also volume-averaged), fluid velocity, and solid velocity verse the dimensionless time ωt are shown in Figs.2–4, respectively, where the microscopic and macroscopic solutions are displayed. The volume-averaged results obtained by solving the microscopic equations match well with those obtained from the macroscopic equations. The averaged solid velocity is much smaller than the averaged fluid velocity. Thus this problem shows fluid motion dominated behavior at the macroscopic scale.

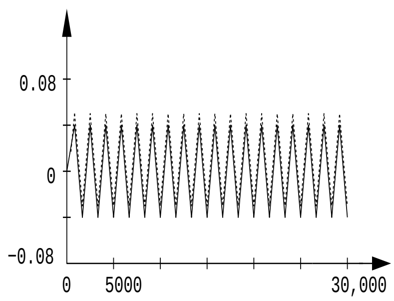

Fig. 5. Averaged fluid velocities of the second example: $\cdots$ pore scale, — homogenized.

The second example uses the same set of data as the first example except that the perturbation pressure signal of frequency is $\omega = 10^3$ Hz, which is ten times higher than the signal frequency in the first example. The responses are considerably different from those of the first example; see Figs. 5 and 6. Now, the magnitude of the solid velocity is of the same order as that of the fluid velocity. Also, it takes

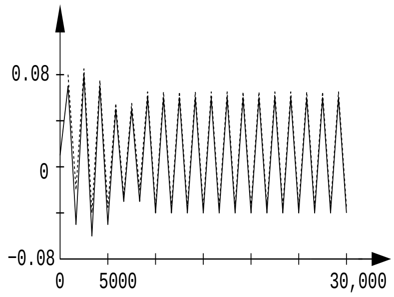

Fig. 6. Averaged solid velocities of the second example: $\cdots$ pore scale, — homogenized.

longer time for the porous system to stabilize in the second example. Again, the microscopic and macroscopic results match very well.

Finally, the responses of the fluid velocity to the perturbation signal frequency are shown in Figs. 7, where the dimensionless quantity $\omega^\star = \omega l / v_0$ is exploited. At low frequency, the magnitude of this velocity is almost fixed, and at high frequency, it seems proportional to $(\omega^\star)^{-1}$.

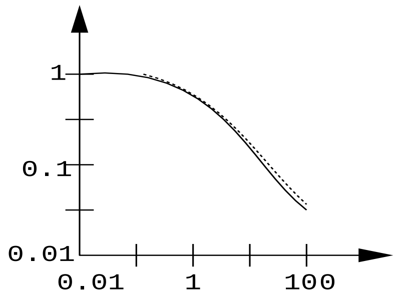

Fig. 7. Responses of the fluid velocity to $\omega^\star$: $\cdots$ pore scale, — homogenized.

6. Conclusions

We have systematically derived the macroscopic equations for the mechanical behavior of a deformable porous medium saturated with a Newtonian fluid via the theory of homogenization. In the case of Stokes flow, these macroscopic equations coincide with Biot's equations provided that the scaled viscosity of the fluid is small. In the case where the scaled viscosity is large, we have derived different differential equations and constitutive relationships.

We have also derived the macroscopic equations for Navier-Stokes flow in the framework of a deformable medium. A generalized Forchheimer law has been obtained to take into account the nonlinear inertial effects on the fluid flow through such a medium. The nonlinear correction term in this law has been shown to be quadratic in velocity for an isotropic medium. Next, transient inertial effects at the pore scale have been analyzed. In the transient case, the coefficients are complex-valued and depend on the angular frequency for time harmonic motions. The present homogenization approach determines the form of the macroscopic constitutive relationships between variables and shows how to compute the coefficients in these relationships. Finally, our computational results show that the macroscopic equations predict well the behavior of the microscopic equations in certain reasonable test cases.

Appendix. Properties of the Macroscopic Coefficients. In this appendix we establish the properties of the macroscopic coefficients. To that end, we introduce the space of Y-periodic functions

$$U_Y = \{\mathbf{w}_1 \in H^1(Y_s) : \int_{Y_s} \mathbf{w}_1 \mathrm{d}y = 0 \text{ and } Y\text{-periodic}\}, \tag{A.1}$$

equipped with the inner product

$$(\mathbf{w}_1, \mathbf{w}_2)_{Y_s} \equiv \int_{Y_s} \mathbf{e}_y(\mathbf{w}_1) \cdot \mathbf{e}_y(\mathbf{w}_2)\, \mathrm{d}y, \qquad \mathbf{w}_1,\, \mathbf{w}_2 \in U_Y. \tag{A.2}$$

With $\mathbf{w}_1 \in U_Y$, it follows from (2.27) and the first equation of (2.3) that

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\xi}^{kh})\mathbf{e}_y(\mathbf{w}_1)\mathrm{d}y = -\int_{Y_{fs}} \mathbf{a}\tilde{\mathbf{e}} \cdot \mathbf{n}\mathbf{w}_1 \mathrm{d}\tau. \tag{A.3}$$

Also, it follows from (2.29) that

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta})\mathbf{e}_y(\mathbf{w}_2)\mathrm{d}y = -\int_{Y_{fs}} \mathbf{n} \cdot \mathbf{w}_2 \mathrm{d}\tau, \qquad \mathbf{w}_2 \in U_Y. \tag{A.4}$$

We take $\mathbf{w}_1 = \boldsymbol{\zeta}$ in (A.3) and use the divergence theorem to see that

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\xi}^{kh})\mathbf{e}_y(\boldsymbol{\zeta})\mathrm{d}y = -\int_{Y_s} a_{ijkh} e_{yij}(\boldsymbol{\zeta})\mathrm{d}y, \tag{A.5}$$

and $\mathbf{w}_2 = \boldsymbol{\xi}^{kh}$ in (A.4)

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta})\mathbf{e}_y(\boldsymbol{\xi}^{kh})\mathrm{d}y = -\int_{Y_s} \boldsymbol{\nabla}_y \cdot \boldsymbol{\xi}^{kh}\mathrm{d}y. \tag{A.6}$$

These two equations imply that (2.43) holds.

Now, we choose $\mathbf{w}_2 = \boldsymbol{\zeta}$ in (A.4) to have

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\zeta})\mathbf{e}_y(\boldsymbol{\zeta})\mathrm{d}y = -\int_{Y_s} \boldsymbol{\nabla}_y \cdot \boldsymbol{\zeta}\mathrm{d}y. \tag{A.7}$$

By the first equation of (2.3), we see that $\gamma < 0$.

Note that

$$c_{ijkh} = \left\langle a_{ijkh} + a_{ijmn} e_{ymn}(\boldsymbol{\xi}^{kh}) \right\rangle^s. \tag{A.8}$$

By the first equation of (2.3) again, it is obvious that

$$c_{ijkh} = c_{jikh}. \tag{A.9}$$

We take $\mathbf{w}_1 = \boldsymbol{\xi}^{ij}$ in (A.3) and use the divergence theorem to see that

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\xi}^{kh})\mathbf{e}_y(\boldsymbol{\xi}^{ij})\mathrm{d}y = -\int_{Y_s} a_{khmn} e_{ymn}(\boldsymbol{\xi}^{ij})\mathrm{d}y. \tag{A.10}$$

Similarly,

$$\int_{Y_s} \mathbf{a}\mathbf{e}_y(\boldsymbol{\xi}^{ij})\mathbf{e}_y(\boldsymbol{\xi}^{kh})\mathrm{d}y = -\int_{Y_s} a_{ijmn} e_{ymn}(\boldsymbol{\xi}^{kh})\mathrm{d}y. \tag{A.11}$$

These two equations, together with (2.3), yield

$$c_{khij} = \left\langle a_{khij} + a_{khmn} e_{ymn}(\boldsymbol{\xi}^{ij}) \right\rangle^s = \left\langle a_{ijkh} + a_{ijmn} e_{ymn}(\boldsymbol{\xi}^{kh}) \right\rangle^s = c_{ijkh}. \tag{A.12}$$

Analogously, we have

$$c_{ijkh} = c_{ijhk}. \tag{A.13}$$

Hence we see that $\mathbf{c}$ is symmetric.

Finally, for a symmetric tensor (z_{ij}), it follows from the definition of $\mathbf{c}$ that

$$c_{ijkh} z_{ij} z_{kh} = \frac{1}{|Y|} \int_{Y_s} a_{ijkh} z_{ij} \left(z_{kh} + e_{ykh}(\boldsymbol{\xi}^{mn}) z_{mn} \right) \mathrm{d}y. \tag{A.14}$$

We apply (A.3), the divergence theorem, and the first equation of (2.3) to see that

$$\int_{Y_s} a_{ijkh} e_{yij}(\boldsymbol{\xi}^{qr}) e_{ykh}(\boldsymbol{\xi}^{mn}) \mathrm{d}y + \int_{Y_s} a_{uvmn} e_{yuv}(\boldsymbol{\xi}^{qr}) \mathrm{d}y = 0. \tag{A.15}$$

These two equations lead to

$$c_{ijkh} z_{ij} z_{kh} = \frac{1}{|Y|} \int_{Y_s} a_{ijkh} \left(z_{ij} + e_{yij}(\boldsymbol{\xi}^{qr}) z_{qr} \right) \left(z_{kh} + e_{ykh}(\boldsymbol{\xi}^{mn}) z_{mn} \right) \mathrm{d}y. \tag{A.16}$$

Therefore, the positive-definiteness of $\mathbf{c}$ follows from the corresponding property of $\mathbf{a}$ (i.e., the second equation of (2.3)).

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