INTERNATIONAL JOURNAL OF
INFORMATON AND SYSTEMS SCIENCES
Volume 6, Number 4, Pages 435–455

© 2010 Institute for Scientific
Computing and Information

MATHEMATICAL MODELING OF THE RECUPERATIVE HEAT EXCHANGERS - THE COMPARATIVE ANALYSIS AND GEOMETRIC OPTIMIZATION

DALIBOR Z. STEVIC

Abstract. The heat exchangers are frequently used as constructive elements in various plants and their dynamics are very important. Their operations are usually controlled by manipulating inlet fluid temperatures or mass flow rates. On the basis of the accepted and critically clarified assumptions, a linearized mathematical model of the cross-flow heat exchanger has been derived, taking into account the wall dynamics. The model is based on the fundamental law of energy conservation, covers all heat accumulation storages in the process, and leads to the set of partial differential equations (PDE), which solution is not possible in closed form. In order to overcome the solutions difficulties, in this paper, different methods are analyzed for modeling the heat exchanger: approach based on Laplace transformation, approximation of partial differential equations based on finite differences, the method of physical discretization and the transport approach. Specifying the input temperatures and output variables, under the constant initial conditions, the step transient responses have been simulated and presented in graphic form in order to compare these results for the four characteristic methods considered in this paper, and analyze its practical significance. The problem of the geometric optimization of the cross-flow heat exchanger based on its controllability characteristic is also considered in this paper.

Key Words. Heat exchanger, Laplace transformation, finite differences, physical discretization, transport method, geometric optimization.

1. Introduction

The heat exchangers are designed to archive certain requirements in the steady state which implies that transient response of heat exchanger must be known to define correct control strategy.

The mathematical model of the heat exchanger must be known in order to determine the transient response.

Solving of this problem in the mentioned papers is based on two approaches:

(1) numerical solving of PDE which pulls such drawback as convergence problems, stability, stiffness, etc.
(2) Laplace transformation which is complicated in this case and demands numerical to original time domain.

Received by the editors Received January 24, 2010, accepted in the revised form on June 23, 2010.

2000 *Mathematics Subject Classification.* 94A08; 68U10.

This problems and the similar ones have motivated the researches to invent new approaches for modeling of this processes in order to obtain larger practical use and not to lose their veracity as well.

In this paper the four characteristic methods for modeling of heat exchangers are considered:

(1) Analytical approach using Laplace transformation.
(2) Approximation of PDE based on finite differences.
(3) The method of physical discretization.
(4) Transport approach.

The comparative analysis of this methods and their practical signification is carried out on the model of cross flow heat exchanger.

Finally, the step responses of the derived mathematical models are presented in graphic form in order to compare this results for the different methods.

2. Analytical approach based on Laplace transformation

In this paper, the recuperative cross-flow heat exchanger is observed, shown in Fig.1, as a process with distributed parameters. Among the many kinds of water-to-air heat exchangers, the cross-flow geometry is very common. The geometry of cross-flow heat exchangers can be complicated, but in this paper we observe the case with the simplest geometry that can be easily computed. This heat exchanger consists of a single tube with fluid (water) flow inside and cross flow of hot air outside.

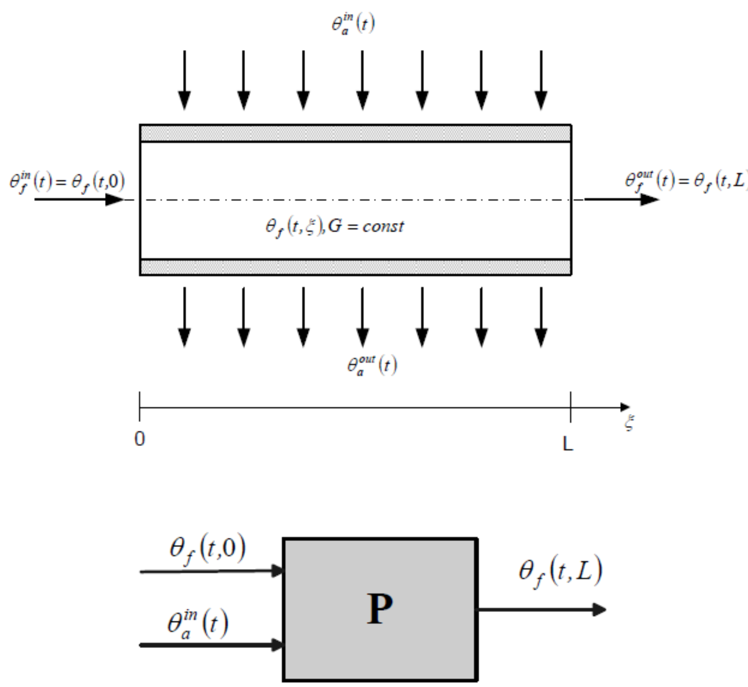

FIGURE 1

The mathematical model of the cross flow heat exchanger, shown in Fig. 1, is carried out on the basis of the following assumptions:

- Fluid in the tube is incompressible and viscous.
- Temperature field fluid in the tube is one dimensional.
- Temperature field of the tube wall is one dimensional.
- Fluid enthalpy can be expressed by means of corresponding temperature.
- Inlet air temperature is a function of time.
- Specific heat of fluid, wall and air has constant values.
- There is convective heat transfer between the cross flow air and the tube wall, conduction along the tube wall, and convection between the tube wall and the fluid in the tube wall.
- Newton's law of heat convection determines exactly enough the amount of exchange heat in the steady state and the transient state.
- Heat transfer coefficient on the both sides of the tubewall have constant values.
- The fluid flow in the tube is one phased.

On the basis of these assumptions, the mathematical model of the cross- flow heat exchanger after some simple mathematical transformations can be written in the following form.

- On the outside of the tube:

$$G_a c_a \left(\theta_a^{in}(t) - \theta_a^{out}(t)\right) = 2\alpha_{aw}\pi r_o L\left(\theta_a(t) - \theta_w(t,\xi)\right) \tag{1}$$

where L is the length of the tube, G_a is the mass-flow rate of air, c_a is the specific heat of air, θ_a^{in} and θ_a^{out} are incoming and outgoing air temperatures, $\alpha_{aw}$is the heat transfer coefficient between the hot air and the tube wall, $r_o$is the outer radius of the tube, θ_a is the air temperature surrounding the tube, and $\theta_w$is the tube wall temperature. For convenience, the air temperature can be assumed to be approximately:

$$\theta_a(t) = \frac{\theta_a^{in}(t) + \theta_a^{out}(t)}{2} \tag{2}$$

This can be substituted in Eq. (1).

- In the water:

$$\rho_f c_f r_i^2 \pi \frac{\partial\theta_f(t,\xi)}{\partial t} = \alpha_{wf} 2\pi r_i\left(\theta_w(t,\xi) - \theta_f(t,\xi)\right) - G_f c_f \frac{\partial\theta_f(t,\xi)}{\partial\xi} \tag{3}$$

where $\rho_f$is fluid (water) density, $c_f$is the specific heat of the fluid in the tube, r_i is the inner radius, θ_f is the fluid temperature, $\alpha_{wf}$is the heat transfer coefficient between the wall and fluid, and G_f is the mass-flow rate of fluid. - Finally, in the wall of the tube:

$$\begin{aligned} &\rho_w c_w \pi\left(r_o^2 - r_i^2\right)\tfrac{\partial\theta_w(t,\xi)}{\partial t} = \lambda_w \pi\left(r_o^2 - r_i^2\right)\tfrac{\partial^2\theta_w(t,\xi)}{\partial\xi^2} + \\ &+ 2\pi r_o \alpha_{aw}\left(\theta_a(t) - \theta_w(t,\xi)\right) - \\ &- 2\pi r_i \alpha_{wf}\left(\theta_w(t,\xi) - \theta_f(t,\xi)\right) \end{aligned} \tag{4}$$

where $\rho_w$is the wall density, c_w is the specific heat of the wall, λ_w is the wall thermal conductivity.

The boundary and initial conditions are:

$$\theta_f(t,0) = \theta_f^{in}(t),\ \theta_f(t,L) = \theta_f^{out}(t),\ \theta_f(0,\xi) = 0,\ \theta_w(0,\xi) = 0 \tag{5}$$

Introducing relative deviations in this form:

$$\begin{gathered} \overline{\Delta\theta}_f(t,\xi) = \tfrac{\theta_f(t,\xi) - \theta_{fN}(\xi)}{\theta_{fN}(\xi_N)}, \quad \overline{\Delta\theta}_w(t,\xi) = \tfrac{\theta_w(t,\xi) - \theta_{wN}(\xi)}{\theta_{wN}(\xi_N)}, \\ \overline{\Delta\theta}_a^{in}(t) = \tfrac{\theta_a^{in}(t) - \theta_{aN}^{in}}{\theta_{aN}^{in}} \end{gathered} \tag{6}$$

where $\theta_{fN}, \theta_{wN}, \theta_{aN}^{in}$ are nominal values.

¿From Eq. (2), (3), (4), (5) mathematical model of cross-flow heat exchanger is obtained in following form:

$$\frac{\partial\overline{\Delta\theta}_f(t,\xi)}{\partial t}=a_1\overline{\Delta\theta}_w(t,\xi)-a_2\overline{\Delta\theta}_f(t,\xi)\ -a_3\frac{\partial\overline{\Delta\theta}_f(t,\xi)}{\partial\xi}\tag{7}$$

$$\frac{\partial\overline{\Delta\theta}_w(t,\xi)}{\partial t}=b_1\frac{\partial^2\overline{\Delta\theta}_w(t,\xi)}{\partial\xi^2}-b_2\overline{\Delta\theta}_w(t,\xi)+b_3\overline{\Delta\theta}_f(t,\xi)+b_4\overline{\Delta\theta}_a^{in}(t)\tag{8}$$

where

$$a_1=\frac{2\alpha_{wf}}{\rho_f c_f r_i}\frac{\theta_{wN}}{\theta_{fN}},\quad a_2=\frac{2\alpha_{wf}}{\rho_f c_f r_i},\quad a_3=\frac{G_f}{\rho_f r_i^2\pi},$$

$$b_1=\frac{\lambda_w}{\rho_w c_w}\ ,\quad b_2=\frac{2G_a c_a\left(r_o\alpha_{aw}+r_i\alpha_{wf}\right)+2\alpha_{aw}\alpha_{wf}r_o r_i\pi L}{\rho_w c_w\left(r_o^2-r_i^2\right)\left(G_a c_a+\alpha_{wa}\pi r_o L\right)},$$

$$b_3=\frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2-r_i^2\right)}\frac{\theta_{fN}}{\theta_{wN}},b_4=\frac{2r_o\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_o^2-r_i^2\right)\left(G_a c_a+\alpha_{aw}\pi r_o L\right)}\frac{\theta_{aN}^{in}}{\theta_{wN}}$$

Taking the Laplace transform of (6), (7) first with respect to t, and then with respect to spatial coordinate ξ,$L_{\xi\to\sigma}$, with initial conditions (5) we find:

$$\left(s+a_2+a_3\sigma\right)\overline{\Delta\theta}_f(s,\sigma)=a_1\overline{\Delta\theta}_w(s,\sigma)+a_3\overline{\Delta\theta}_f(s,0)\tag{9}$$

$$\left(s-b_1\sigma^2+b_2\right)\overline{\Delta\theta}_w(s,\sigma)=b_3\overline{\Delta\theta}_f(s,\sigma)-b_1\sigma\overline{\Delta\theta}_w(s,0)+b_4\overline{\Delta\theta}_a^{in}(s)\tag{10}$$

Solution of the system (9), (10) for $\theta_f(s,\sigma)$ has the following form:

$$\begin{aligned}\theta_f(s,\sigma)=&\tfrac{a_3b_1\sigma^2-M(s)}{a_3b_1\sigma^3+P(s)\sigma^2-Q(s)\sigma-R(s)}\overline{\Delta\theta}_f(s,0)-\\ &-\tfrac{a_1b_4}{a_3b_1\sigma^3+P(s)\sigma^2-Q(s)\sigma-R(s)}\overline{\Delta\theta}_a^{in}(s)+\\ &+\tfrac{a_1b_1\sigma}{a_3b_1\sigma^3+P(s)\sigma^2-Q(s)\sigma-R(s)}\overline{\Delta\theta}_w(s,0)\end{aligned}\tag{11}$$

where

$$M(s)=a_3s+a_3b_2,\quad P(s)=b_1s+a_2b_1,\quad Q(s)=a_3s+a_3b_2,$$

$$Q(s)=a_3s+a_3b_2,\quad R(s)=s^2+(a_2+b_2)s+a_2b_2-a_1b_3.$$

Taking inverse Laplace transform of (11) with respect to spatial coordinate, $L^{-1}_{\sigma\to\xi}$, and switching $\xi=L$, appropriate transfer functions is obtained in the following form:

$$W_1(s,L)=\frac{\overline{\Delta\theta}_f(s,L)}{\overline{\Delta\theta}_f(s,0)}=F(\sigma_1,s,L)\ +F(\sigma_2,s,L)+F(\sigma_3,s,L),\tag{12}$$

$$W_2(s,L)=\frac{\overline{\Delta\theta}_f(s,L)}{\overline{\Delta\theta}_a^{in}(s)}=G(\sigma_1,s,L)+G(\sigma_2,s,L)+G(\sigma_3,s,L)\tag{13}$$

where

$$F(\sigma_n,s,L)=\frac{a_3b_1\sigma_n^2-U(s)}{3a_3b_1\sigma_n^2+V(s)\sigma_n-U(s)}e^{-\sigma_n L},$$

$$G(\sigma_n,s,L)=\frac{-b_4a_1}{3a_3b_1\sigma_n^2+V(s)\sigma_n-U(s)}e^{-\sigma_n L},$$

$$U(s)=a_3s+a_3b_2,\quad V(s)=2\left(b_1s+a_2b_1\right).$$

and $\sigma_n, n=1,2,3$ are the roots of equation:

$$a_3b_1\sigma^3+(b_1s+a_2b_1)\sigma^2-(a_3s+a_3b_2)\sigma-s^2-(a_2+b_2)s+a_1b_3-a_2b_2=0\tag{14}$$

In order to obtain the step response for mathematical model of the cross-flow heat exchanger it is necessary to take another inverse Laplace transform with respect to time, $L^{-1}_{s\to t}$. Considering the complexity of the right side of Eq. (12), (13) this can be done using the numerical methods for inverse Laplace transform, [10].

Applying the *Gaver-Stehfest* algorithm for inverse Laplace transform of (12),(13) we find:

$$\frac{\overline{\Delta\theta}_f(t,L)}{\overline{\Delta\theta}_f(t,0)} = f(t,M) = \frac{\ln(2)}{t}\sum_{k=1}^{2M}\varsigma_k F\left(\frac{k\ln(2)}{t}\right) \tag{15}$$

$$\frac{\overline{\Delta\theta}_f(t,L)}{\overline{\Delta\theta}_a^{in}(t)} = g(t,M) = \frac{\ln(2)}{t}\sum_{k=1}^{2M}\varsigma_k G\left(\frac{k\ln(2)}{t}\right) \tag{16}$$

where

$$\varsigma_k = (-1)^{M+k}\sum_{j=[(k+1)/2]}^{k\wedge M}\frac{j^{M+1}}{M!}\begin{pmatrix} M \\ j \end{pmatrix}\begin{pmatrix} 2j \\ j \end{pmatrix}\begin{pmatrix} j \\ k-j \end{pmatrix}$$

with $[(k+1)/2]$ being the greatest integer less then or equal to $(k+1)/2$, $k\wedge M \equiv \min\{k, M\}$ and positive integer M.

3. Differential-discrete mathematical model of the cross-flow heat exchanger

The system of PDE is very complicated to be solved analytically as it was consider in analytical approach for modeling the cross-flow heat exchanger. A numerical method based on the finite differences is developed to approximate infinite-dimensional equations by finite-dimensional ones. The spatial coordinate of the cross-flow heat exchanger is discretized by means of the finite difference method. In this manner the system of PDE is transformed to the system of ODE. Discretization of spatial variable is carried out by means of substituting partial derivation with respect to spatial coordinate in the certain numbers of point. In the physical meaning, the observed heat exchanger is divided into the equal p-cells with the spatial coordinate discretization.

The mathematical model of the cross-flow heat exchanger is derived in previous section, described with Eq. (1), (3), (4). The finite difference method has a different form for the first, k-th (k= 2,3,..., p-1) and the last cell. This method has a lot of different forms. Observing one dimensional problem the finite difference method has the following form:

$$\begin{gathered} \Delta\xi = \xi_{k+1} - \xi_k = l, \quad \theta_{f,k}(t) = \theta_f(t,k), \quad k = 1,2,...,p \\ \frac{\partial\theta_f(t,\xi)}{\partial\xi}|_k \cong \frac{1}{2l}\left(\theta_{f,k+1}(t) - \theta_{f,k-1}(t)\right), \\ \frac{\partial^2\theta_f(t,\xi)}{\partial\xi^2}|_k \cong \frac{1}{l^2}\left(\theta_{f,k+1}(t) - 2\theta_{f,k}(t) + \theta_{f,k-1}(t)\right) \end{gathered} \tag{17}$$

Substituting (22) into (3), (4) the differential-discrete model of the cross-flow heat exchanger is obtained in the following form:

- for the *first* cell:

$$\frac{d\theta_{f,1}(t)}{dt} = \frac{2\alpha_{wf}}{\rho_f c_f r_u}\theta_{w,1}(t) - \frac{2\alpha_{wf}}{\rho_f c_f r_u}\theta_{f,1}(t) - \frac{G_f}{\rho_f r_u^2\pi}\left(\frac{\theta_{f,2}(t) - \theta_f^{in}(t)}{2l}\right) \tag{18}$$

(19)
$$\frac{d\theta_{w,1}(t)}{dt} = \frac{\lambda_w}{\rho_w c_w}\left(\frac{\theta_{w,2}(t)-2\theta_{w,1}(t)}{l^2}\right) + \frac{2r_s\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\theta_a^{in}(t) + $$
$$+ \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2-r_i^2\right)}\theta_{f,1}(t) - \frac{2G_a c_a(r_o\alpha_{aw}+r_i\alpha_{wf})+2\alpha_{aw}\alpha_{wf}r_o r_i\pi L}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\theta_{w,1}(t)$$

-k-th cell:

(20)
$$\frac{d\theta_{f,k}(t)}{dt} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,k}(t) - \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{f,k}(t) - \frac{G_f}{\rho_f r_i^2\pi}\left(\frac{\theta_{f,k+1}(t)-\theta_{f,k-1}(t)}{2l}\right)$$

(21)
$$\frac{d\theta_{w,k}(t)}{dt} = \frac{\lambda_w}{\rho_w c_w}\left(\frac{\theta_{w,k+1}(t)-2\theta_{w,k}(t)+\theta_{w,k-1}(t)}{l^2}\right) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2-r_i^2\right)}\theta_{f,k}(t)$$
$$+ \frac{2r_o\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\theta_a^{in}(t) +$$
$$- \frac{2G_a c_a(r_o\alpha_{aw}+r_i\alpha_{wf})+2\alpha_{aw}\alpha_{wf}r_o r_i\pi L}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\theta_{w,k}(t)$$

-last cell:

(22)
$$\frac{d\theta_{f,p}(t)}{dt} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,p}(t) - \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{f,p}(t) + \frac{G_f}{\rho_f r_i^2\pi}\frac{\theta_{f,p-1}(t)}{2l}$$

(23)
$$\frac{d\theta_{w,p}(t)}{dt} = -\frac{\lambda_w}{\rho_w c_w}\frac{2\theta_{w,p}(t)-\theta_{w,p-1}(t)}{l^2} + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2-r_i^2\right)}\theta_{f,p}(t)$$
$$+ \frac{2r_o\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\theta_a^{in}(t)$$
$$- \frac{2G_a c_a(r_o\alpha_{aw}+r_i\alpha_{wf})+2\alpha_{aw}\alpha_{wf}r_o r_i\pi L}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\theta_{w,p}(t)$$

where

$$\theta_f(t,L) = \theta_{f,p}(t), \theta_{w,0}(t) = 0, \quad \theta_{f,p+1}(t) = 0, \theta_{f,0}(t) = \theta_f^{in}(t).$$

Introduce relative deviations and define the state variables in this form:

(24)
$$\overline{\Delta\theta}_{f,k}(t) = \frac{\theta_{f,k}(t)-\theta_{fN,k}}{\theta_{fN,k}} = x_k(t), \quad \overline{\Delta\theta}_{w,k}(t) = \frac{\theta_{w,k}(t)-\theta_{wN,k}}{\theta_{wN,k}} = x_k^*(t),$$
$$\overline{\Delta\theta}_f^{in}(t) = \frac{\theta_f^{in}(t)-\theta_{fN}^{in}}{\theta_{fN}^{in}} = u_1(t), \quad \overline{\Delta\theta}_a^{in}(t) = \frac{\theta_a^{in}(t)-\theta_{aN}^{in}}{\theta_{aN}^{in}} = u_2(t),$$
$$\mathbf{x}_k(t) = \begin{bmatrix} x_k(t) \\ x_k^*(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The mathematical model of the cross-flow heat exchanger is obtained in the state space:

-for the k-th cell:

(25)
$$\dot{\mathbf{x}}_k(t) = A_k^1\mathbf{x}_{k-1}(t) + A_k^2\mathbf{x}_k(t) + A_k^3\mathbf{x}_{k+1}(t) + B_k\mathbf{u}(t)$$
$$A_k^1 = \begin{pmatrix} a_{k3} & 0 \\ 0 & b_{k3} \end{pmatrix}, \quad A_k^2 = \begin{pmatrix} -a_{k1} & a_{k4} \\ b_{k4} & -b_{k1} \end{pmatrix}, \quad A_k^3 = \begin{pmatrix} -a_{k2} & 0 \\ 0 & b_{k2} \end{pmatrix}$$
$$B_k = \begin{pmatrix} 0 & 0 \\ 0 & b_{k5} \end{pmatrix}, \quad k = 2,3,\ldots,p-1$$
$$a_{k1} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}, \quad a_{k2} = \frac{G_f}{\rho_f r_i^2\pi 2l}\frac{\theta_{fN,k+1}}{\theta_{fN,k}}, \quad a_{k3} = \frac{G_f}{\rho_f r_i^2\pi 2l}\frac{\theta_{fN,k-1}}{\theta_{fN,k}}, \quad a_{k4} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\frac{\theta_{wN,k}}{\theta_{fN,k}}$$
$$b_{k1} = \left(\frac{2\lambda_w}{\rho_w c_w l^2} + \frac{2G_a c_a(r_o\alpha_{aw}+r_i\alpha_{wf})+2\alpha_{aw}\alpha_{wf}r_o r_i\pi L}{\rho_w c_w\left(r_o^2-r_i^2\right)(G_a c_a+\alpha_{aw}\pi r_o L)}\right), \quad b_{k2} = \frac{\lambda_w}{\rho_w c_w l^2}\frac{\theta_{wN,k+1}}{\theta_{wN,k}}$$
$$b_{k3} = \frac{\lambda_w}{\rho_w c_w l^2}\frac{\theta_{wN,k-1}}{\theta_{wN,k}}, \quad b_{k4} = \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2-r_i^2\right)}\frac{\theta_{fN,k}}{\theta_{wN,k}}$$
$$b_{k5} = \frac{2r_s\alpha_{gz}G_g c_g}{\rho_z c_z(r_s^2-r_u^2)(G_g c_g+\alpha_{gz}\pi r_s L)}\frac{\theta_{gN}^{ul}}{\theta_{zN,k}} \quad ,k = 1,2,...,p$$

The mathematical model of the first and the last cell in the state space will be slightly modified, because this two cells, through control boundary, have a contact with environment.

The mathematical model of the first cell has the following form:

$$\dot{\mathbf{x}}_1(t) = A_1^2\mathbf{x}_1(t) + A_1^3\mathbf{x}_2(t) + B_1\mathbf{u}(t) \tag{26}$$

where

$$A_1^2 = \begin{pmatrix} -a_{11} & a_{14} \\ b_{14} & -b_{11} \end{pmatrix}, \quad A_1^3 = \begin{pmatrix} -a_{12} & 0 \\ 0 & b_{12} \end{pmatrix}, \quad B_1 = \begin{pmatrix} a_{15} & 0 \\ 0 & b_{15} \end{pmatrix},$$

while this model in the state space for the last cell can be written as follows:

$$\begin{gathered} \dot{\mathbf{x}}_p(t) = A_p^1\mathbf{x}_{p-1}(t) + A_p^2\mathbf{x}_p(t) + B_p\mathbf{u}(t) \\ A_p^1 = \begin{pmatrix} a_{p3} & 0 \\ 0 & b_{p3} \end{pmatrix}, A_p^2 = \begin{pmatrix} -a_{p1} & a_{p4} \\ b_{p4} & -b_{p1} \end{pmatrix}, B_p = \begin{pmatrix} 0 & 0 \\ 0 & b_{p5} \end{pmatrix} \end{gathered} \tag{27}$$

The state equation and the output of the cross-flow heat exchanger are given with:

$$\begin{gathered} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \\ x_i(t) = \mathbf{c}^T\mathbf{x}(t) \\ A = \begin{pmatrix} A_1^2 & A_1^3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ A_2^1 & A_2^2 & A_2^3 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & A_3^1 & A_3^2 & A_3^3 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & A_{p-1}^1 & A_{p-1}^2 & A_{p-1}^3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & A_p^1 & A_p^2 \end{pmatrix}, \\ B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{p-1} \\ B_p \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ \dots \\ \vdots \\ \dots \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_p(t) \end{bmatrix}, \quad x_i(t) = \overline{\Delta\theta}_{f,p}(t) \end{gathered} \tag{28}$$

4. The method of physical discretization

To avoid partial differential equations, the method of physical discretization is used [3]. By this method the observed heat exchanger is divided into the same p cells with the spatial coordinate discretization. This cells assumes to have homogenous fields of specific physical values. The qualitative description of the process with distributed parameters can be obtained taking into consideration a big enough number of cells.

The advantage of such an approach is avoiding complex PDE to be set and solved. Balance differential equations are set for an arbitrary cell, taking especially into consideration the first an the last cell which through control boundary have contact with environment. The flowing index indicates the position of the cell in the heat exchanger.

In the physical meaning, the observed cross-flow heat exchanger is divided into the equal p cells with the spatial coordinate discretization. Assuming that number of cells is big enough, each of this cells can be considered as a process with certain parameters. The fundamental low of energy conservation is derived for every cell, so the mathematical model of the cross-flow heat exchanger can be represented as a system of ODE.

Considering k-th cell, the balance equation can be written in the following form:

$$\frac{d\theta_{f,k}(t)}{dt} = \frac{G_f}{\rho_f r_i^2 \pi l}\left(\theta_{f,k-1}(t) - \theta_{f,k+1}(t)\right) + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\left(\theta_{w,k}(t) - \theta_{f,k}(t)\right) \tag{29}$$

$$\begin{aligned}\frac{d\theta_{w,k}(t)}{dt} &= \frac{\lambda_w}{\rho_w c_w l}\left(\theta_{w,k-1}(t) - \theta_{w,k}(t)\right) + \frac{2r_o\alpha_{aw}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\left(\theta_a(t) - \theta_{w,k}(t)\right) - \\ &- \frac{2r_i\alpha_{wf}}{\rho_w c_w (r_o^2 - r_u^2)}\left(\theta_{w,k}(t) - \theta_{f,k}(t)\right)\end{aligned} \tag{30}$$

$$\left(\theta_a^{in}(t) - \theta_a^{out}(t)\right) = \frac{2\alpha_{aw}\pi r_o l}{G_a c_a}\left(\theta_a(t) - \theta_{w,k}(t)\right) \tag{31}$$

where

$$\theta_a(t) = \frac{\theta_a^{in}(t) + \theta_a^{out}(t)}{2} \tag{32}$$

Using Eq. (29)-(32) the mathematical model of cross flow heat exchanger can be obtained in the following form:

- k-th cell:

$$\begin{aligned}\frac{d\theta_{f,k}(t)}{dt} &= -\frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{f,k}(t) + \frac{G_f}{\rho_f r_i^2\pi l}\theta_{f,k-1}(t) - \frac{G_f}{\rho_f r_i^2 \pi l}\theta_{f,k+1}(t) \\ &+ \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,k}(t)\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{d\theta_{w,k}(t)}{dt} &= \frac{4\alpha_{aw}^2 r_o^2 \pi l}{\rho_w c_w\left(r_o^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_o l\right)}\theta_{w,k}(t) - \frac{\lambda_w}{\rho_w c_w l}\theta_{w,k}(t) - \\ &- \frac{2r_o\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\theta_{w,k}(t) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\theta_{f,k}(t) + \\ &+ \frac{\lambda_w}{\rho_w c_w l}\theta_{w,k-1}(t) + \frac{2r_o\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_o l\right)}{\rho_w c_w\left(r_o^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_o l\right)}\theta_a^{in}(t)\end{aligned} \tag{34}$$

- for the first cell:

$$\frac{d\theta_{f,1}(t)}{dt} = -\frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{f,1}(t) + \frac{G_f}{\rho_f r_i^2 \pi l}\theta_f^{in}(t) - \frac{G_f}{\rho_f r_i^2 \pi l}\theta_{f,2}(t) + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,1}(t) \tag{35}$$

$$\begin{aligned}\frac{d\theta_{w,1}(t)}{dt} &= \frac{4\alpha_{aw}^2 r_o^2 \pi l}{\rho_w c_w\left(r_o^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_o l\right)}\theta_{w,1}(t) - \frac{\lambda_w}{\rho_w c_w l}\theta_{w,1}(t) - \\ &- \frac{2r_o\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\theta_{w,1}(t) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\theta_{f,1}(t) + \\ &+ \frac{2r_o\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_o l\right)}{\rho_w c_w\left(r_o^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_o l\right)}\theta_a^{in}(t)\end{aligned} \tag{36}$$

- for the last cell:

$$\frac{d\theta_{f,p}(t)}{dt} = -\frac{2\alpha_{wf}}{\rho_f c_f r_u}\theta_{f,p}(t) + \frac{G_f}{\rho_f r_i^2 \pi l}\theta_{f,p-1}(t) + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,p}(t) \tag{37}$$

$$\begin{aligned}\frac{d\theta_{w,p}(t)}{dt} &= \frac{4\alpha_{aw}^2 r_o^2 \pi l}{\rho_w c_w\left(r_o^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_o l\right)}\theta_{w,p}(t) - \frac{\lambda_w}{\rho_w c_w l}\theta_{w,p}(t) - \\ &- \frac{2r_o\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\theta_{w,p}(t) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_o^2 - r_i^2\right)}\theta_{f,p}(t) + \\ &+ \frac{\lambda_w}{\rho_w c_w l}\theta_{w,p-1}(t) + \frac{2r_o\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_o l\right)}{\rho_w c_w\left(r_o^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_o l\right)}\theta_a^{in}(t)\end{aligned} \tag{38}$$

Introducing relative deviations and define the state variables in this form:

$$\begin{array}{ll}\overline{\Delta\theta}_{f,k}(t)=\frac{\theta_{f,k}(t)-\theta_{fN,k}}{\theta_{fN,k}}=x_k(t), & \overline{\Delta\theta}_{w,k}(t)=\frac{\theta_{w,k}(t)-\theta_{wN,k}}{\theta_{wN,k}}=x_k^*(t)\\ \overline{\Delta\theta}_f^{in}(t)=\frac{\theta_f^{in}(t)-\theta_{fN}^{in}}{\theta_{fN}^{in}}=u_1(t), & \overline{\Delta\theta}_a^{in}(t)=\frac{\theta_a^{in}(t)-\theta_{aN}^{in}}{\theta_{aN}^{in}}=u_2(t)\end{array} \tag{39}$$

The mathematical model for k-th cell is obtained in the following form:

$$\frac{dx_k(t)}{dt}=-a_{k1}x_k(t)+a_{k2}x_{k-1}(t)-a_{k3}x_{k+1}(t)+a_{k4}x_k^*(t) \tag{40}$$

$$\frac{dx_k^*(t)}{dt}=b_{k1}x_k^*(t)+b_{k2}x_{k-1}^*(t)+b_{k3}x_k(t)+b_{k4}u_2(t) \tag{41}$$

where $k=1,2,\ldots,p$ and

$$a_{k1}=\frac{2\alpha_{zf}}{\rho_f c_f r_u}, a_{k2}=\frac{G_f}{\rho_f r_u^2\pi l}\frac{\theta_{fN,k-1}}{\theta_{fN,k}},$$

$$a_{k3}=\frac{G_f}{\rho_f r_u^2\pi l}\frac{\theta_{fN,k+1}}{\theta_{fN,k}}, a_{k4}=\frac{2\alpha_{zf}}{\rho_f c_f r_u}\frac{\theta_{zN,k}}{\theta_{fN,k}},$$

$$b_{k1}=\frac{4\alpha_{gz}^2 r_s^2\pi l}{\rho_z c_z\left(r_s^2-r_u^2\right)\left(G_g c_g+\alpha_{gz}\pi r_s l\right)}-\frac{\lambda_z}{\rho_z c_z l}-\frac{2r_s\alpha_{gz}+2r_u\alpha_{zf}}{\rho_z c_z\left(r_s^2-r_u^2\right)},$$

$$b_{k2}=\frac{\lambda_z}{\rho_z c_z\ l}\frac{\theta_{zN,k-1}}{\theta_{zN,k}}, b_{k3}=\frac{2r_u\alpha_{zf}}{\rho_z c_z\left(r_s^2-r_u^2\right)}\frac{\theta_{fN,k}}{\theta_{zN,k}},$$

$$b_{k4}=\frac{2r_s\alpha_{gz}\left(G_g c_g-\alpha_{gz}\pi r_s l\right)}{\rho_z c_z\left(r_s^2-r_u^2\right)\left(G_g c_g+\alpha_{gz}\pi r_s l\right)}\frac{\theta_{gN}^u}{\theta_{zN,k}}k=1,2,\ldots,p.$$

The mathematical model for the k-th cell of the cross-flow heat exchanger is obtained in the state space:

$$\begin{array}{c}\frac{d\mathbf{x}_k(t)}{dt}=A_k^1\mathbf{x}_{k-1}(t)+A_k^2\mathbf{x}_k(t)+A_k^3\mathbf{x}_{k+1}(t)+B_k\mathbf{u}(t)\\ A_k^1=\begin{pmatrix}a_{k2}&0\\0&b_{k2}\end{pmatrix},\quad A_k^2=\begin{pmatrix}-a_{k1}&a_{k4}\\b_{k3}&b_{k1}\end{pmatrix},\quad A_k^3=\begin{pmatrix}-a_{k3}&0\\0&0\end{pmatrix},\\ B_k=\begin{pmatrix}0&0\\0&b_{k4}\end{pmatrix},\quad \mathbf{x}_k(t)=\begin{bmatrix}x_k(t)&x_k^*(t)\end{bmatrix}^T,\quad \mathbf{u}(t)=\begin{bmatrix}u_1(t)&u_2(t)\end{bmatrix}^T\end{array} \tag{42}$$

For the first cell of the cross-flow heat exchanger the state space equation has the following form:

$$\begin{array}{c}\dot{\mathbf{x}}_1(t)=A_1^2\mathbf{x}_1(t)+A_1^3\mathbf{x}_2(t)+B_1\mathbf{u}(t)\\ A_1^2=\begin{pmatrix}-a_{11}&a_{14}\\b_{13}&b_{11}\end{pmatrix},\quad A_1^3=\begin{pmatrix}-a_{13}&0\\0&0\end{pmatrix},\quad B_1=\begin{pmatrix}a_{12}&0\\0&b_{14}\end{pmatrix}\end{array} \tag{43}$$

For the last cell the state space equation can be written as follows:

$$\begin{array}{c}\dot{\mathbf{x}}_p(t)=A_p^1\mathbf{x}_{p-1}(t)+A_p^2\mathbf{x}_p(t)+B_p\mathbf{u}(t)\\ A_p^1=\begin{pmatrix}a_{p2}&0\\0&-b_{p2}\end{pmatrix},A_p^2=\begin{pmatrix}-a_{p1}&a_{p4}\\b_{p3}&b_{p1}\end{pmatrix},\ B_p=\begin{pmatrix}0&0\\0&b_{p4}\end{pmatrix}\end{array} \tag{44}$$

The state equation and the output of the cross-flow heat exchanger are given with:

(45)
$$\dot{\mathbf{x}}\left(t\right)=A\mathbf{x}\left(t\right)+B\mathbf{u}\left(t\right),\quad \mathbf{x}_i=\mathbf{c}^T\mathbf{x}\left(t\right),$$
$$A=\begin{pmatrix} A_1^2 & A_1^3 & 0 & 0 & \dots & 0 & 0\\ A_2^1 & A_2^2 & A_2^3 & 0 & \dots & 0 & 0\\ 0 & A_3^1 & A_3^2 & A_3^3 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & A_p^1 & A_p^2 \end{pmatrix},$$
$$B=\begin{bmatrix} B_1\\ B_2\\ B_3\\ \vdots\\ B_p \end{bmatrix},\quad \mathbf{c}=\begin{bmatrix} 0\\ 0\\ \dots\\ 0\\ 0\\ \dots\\ \vdots\\ \dots\\ 1\\ 0 \end{bmatrix},\quad \mathbf{x}\left(t\right)=\begin{bmatrix} \mathbf{x}_1\left(t\right)\\ \mathbf{x}_2\left(t\right)\\ \vdots\\ \mathbf{x}_p\left(t\right) \end{bmatrix}$$

5. The transport approach

The transport approach represents the special method for mathematical modeling of heat exchangers [11]. This approach is based on dividing observed heat exchanger into the same p cells.

The fundamental low of energy conservation is derived by means of the finite differences for every cell with respect to finite time interval Δt. In this manner the system of partial differential equations is transformed to the system of algebraic equations which are easy to solve by numeric, iterative methods.

Lets observe the k-th cell, k=1,2,...,p, of the cross-flow heat exchanger. Considering heat balance within observed cell, it can be identified the heat input and output by fluid flow through boundary of the cell and heat exchange between the fluid in the cell and the wall.

Considering the changes of the specific physical values in the certain time interval defined with t and $t+\Delta t$. The amount of heat exchange into the k-th cell for the time interval Δt can be written in the following form:

(46)
$$\overline{Q}\left(\Delta t\right)=\tfrac{Q(t)+Q(t+\Delta t)}{2}\Delta t\quad \overline{Q}_{in}\left(\Delta t\right)=\tfrac{Q_{in}(t)+Q_{in}(t+\Delta t)}{2}\Delta t$$
$$\overline{Q}_{out}\left(\Delta t\right)=\tfrac{Q_{out}(t)+Q_{out}(t+\Delta t)}{2}\Delta t,\quad \overline{Q}_{wf}\left(\Delta t\right)=\tfrac{Q_{wf}(t)+Q_{wf}(t+\Delta t)}{2}\Delta t$$

where $\overline{Q}_{in}\left(\Delta t\right)$ is the heat change at the cell inlet for the time interval Δt, $\overline{Q}_{out}\left(\Delta t\right)$ is the heat change at cell outlet and $\overline{Q}_{wf}\left(\Delta t\right)$ the heat exchange between the wall and the fluid in the cell.

The heat flow can be presented as:

(47)
$$Q_{in}\left(t\right)=c_fG_f\theta_f^{in}\left(t\right),\quad Q_{out}\left(t\right)=c_fG_f\theta_f^{out}\left(t\right),\quad Q_{wf}\left(t\right)=\alpha_{wf}A_{wf}\left(\theta_w\left(t\right)-\theta_f\left(t\right)\right)$$

Assuming that fluid temperature in the cell can be expressed in the following form:

(48)
$$\theta_{fsr}\left(t\right)=\frac{\theta_f^{in}\left(t\right)+\theta_f^{out}\left(t\right)}{2}.$$

The heat content in the fluid for the observed cell can be expressed in the following form:

$$\overline{Q}\left(t\right)=m_{f}c_{f}\theta_{fsr}\left(t\right)=m_{f}c_{f}\frac{\theta_{f}^{in}\left(t\right)+\theta_{f}^{out}\left(t\right)}{2}. \tag{49}$$

The total amount of heat accumulated in the observed cell for the time interval $((t+\Delta t)-t)=\Delta t$can be written as follows:

$$\Delta\overline{Q}\left(\Delta t\right)=\overline{Q}\left(t+\Delta t\right)-\overline{Q}\left(t\right)=\overline{Q}_{in}\left(\Delta t\right)-\overline{Q}_{out}\left(\Delta t\right)+\overline{Q}_{wf}\left(\Delta t\right) \tag{50}$$

where

$$\overline{Q}_{in}\left(\Delta t\right)=c_{f}G_{f}\frac{\theta_{f}^{in}\left(t\right)+\theta_{f}^{in}\left(t+\Delta t\right)}{2}\Delta t,$$

$$\overline{Q}_{out}\left(\Delta t\right)=c_{f}G_{f}\frac{\theta_{f}^{out}\left(t\right)+\theta_{f}^{out}\left(t+\Delta t\right)}{2}\Delta t,$$

$$\begin{array}{c}\overline{Q}_{wf}\left(\Delta t\right)=\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w}\left(t\right)-\frac{\theta_{f}^{in}(t)+\theta_{f}^{out}(t)}{2}\right)+\\+\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w}\left(t+\Delta t\right)-\frac{\theta_{f}^{in}(t+\Delta t)+\theta_{f}^{out}(t+\Delta t)}{2}\right)\end{array}$$

The balance equation for the observed cell is obtained in the following form:

-for the fluid in the tube:

$$\begin{array}{c}m_{f}c_{f}\theta_{fsr,k}\left(t+\Delta t\right)-m_{f}c_{f}\theta_{fsr,k}\left(t\right)=\frac{c_{f}G_{f}\Delta t}{2}\left(\theta_{f,k}^{in}\left(t\right)-\theta_{f,k}^{in}\left(t+\Delta t\right)\right)-\\-\frac{c_{f}G_{f}\Delta t}{2}\left(\theta_{f,k}^{out}\left(t\right)-\theta_{f,k}^{out}\left(t+\Delta t\right)\right)\\+\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}\left(t\right)-\frac{\theta_{f,k}^{in}(t)+\theta_{f,k}^{out}(t)}{2}\right)+\\+\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}\left(t+\Delta t\right)-\frac{\theta_{f,k}^{in}(t+\Delta t)+\theta_{f,k}^{out}(t+\Delta t)}{2}\right)\end{array} \tag{51}$$

- for the wall:

$$\begin{array}{c}m_{w}c_{w}\left(\theta_{w,k}\left(t+\Delta t\right)-\theta_{w,k}\left(t\right)\right)=\frac{\alpha_{aw}A_{aw}\Delta t}{2}\left(\theta_{a}\left(t\right)-\theta_{w,k}\left(t\right)\right)\\+\frac{\alpha_{aw}A_{aw}\Delta t}{2}\left(\theta_{a}\left(t+\Delta t\right)-\theta_{w,k}\left(t+\Delta t\right)\right)-\\-\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}\left(t\right)-\frac{\theta_{f,k}^{in}(t)+\theta_{f,k}^{out}(t)}{2}\right)\\-\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}\left(t+\Delta t\right)-\frac{\theta_{f,k}^{in}(t+\Delta t)+\theta_{f,k}^{out}(t+\Delta t)}{2}\right)\end{array} \tag{52}$$

- for the air:

$$G_{a}c_{a}\left(\theta_{a}^{in}\left(t\right)-\theta_{a}^{out}\left(t\right)\right)=\alpha_{aw}A_{aw}\left(\theta_{a}\left(t\right)-\theta_{w,k}\left(t\right)\right) \tag{53}$$

The mathematical model for the observed cell after some simple mathematical transformation can be obtained in the following form:

$$\begin{array}{c}\theta_{f,k}^{out}\left(t+\Delta t\right)=a_{1}\theta_{f,k}^{out}\left(t\right)+a_{2}\theta_{f,k}^{in}\left(t\right)+a_{3}\theta_{w,k}\left(t\right)\\+a_{4}\theta_{f,k}^{in}\left(t+\Delta t\right)+a_{3}\theta_{w,k}\left(t+\Delta t\right)\end{array} \tag{54}$$

$$\begin{array}{l}\theta_{w,k}\left(t+\Delta t\right)=b_{1}\theta_{w,k}\left(t\right)+b_{2}\theta_{a}^{in}\left(t\right)+b_{2}\theta_{a}^{in}\left(t+\Delta t\right)+b_{3}\theta_{f,k}^{in}\left(t\right)+\\\quad+b_{3}\theta_{f,k}^{in}\left(t+\Delta t\right)+b_{3}\theta_{f,k}^{out}\left(t\right)+b_{3}\theta_{f,k}^{out}\left(t+\Delta t\right)\end{array} \tag{55}$$

where

$$a_{1}=\frac{2m_{f}c_{f}-2c_{f}G_{f}\Delta t-\alpha_{wf}A_{wf}\Delta t}{2m_{f}c_{f}+2c_{f}G_{f}\Delta t+\alpha_{wf}A_{wf}\Delta t},a_{2}=\frac{2m_{f}c_{f}+2c_{f}G_{f}\Delta t-\alpha_{wf}A_{wf}\Delta t}{2m_{f}c_{f}+2c_{f}G_{f}\Delta t+\alpha_{wf}A_{wf}\Delta t},$$

$$a_{3}=\frac{2\alpha_{wf}A_{wf}\Delta t}{2m_{f}c_{f}+2c_{f}G_{f}\Delta t+\alpha_{wf}A_{wf}\Delta t},a_{4}=\frac{2c_{f}G_{f}\Delta t-2m_{f}c_{f}-\alpha_{wf}A_{wf}\Delta t}{2m_{f}c_{f}+2c_{f}G_{f}\Delta t+\alpha_{wf}A_{wf}\Delta t},$$

$$b_1 = \frac{(2m_z c_z - \alpha_{gz} A_{gz} \Delta t - \alpha_{zf} A_{zf} \Delta t)(2G_g c_g + \alpha_{gz} A_{gz}) + \alpha_{gz}^2 A_{gz}^2 \Delta t}{(2m_z c_z + \alpha_{gz} A_{gz} \Delta t + \alpha_{zf} A_{zf} \Delta t)(2G_g c_g + \alpha_{gz} A_{gz}) - \alpha_{gz}^2 A_{gz}^2 \Delta t},$$

$$b_2 = \frac{2\alpha_{gz} A_{gz} G_g c_g \Delta t}{(2m_z c_z + \alpha_{gz} A_{gz} \Delta t + \alpha_{zf} A_{zf} \Delta t)(2G_g c_g + \alpha_{gz} A_{gz}) - \alpha_{gz}^2 A_{gz}^2 \Delta t},$$

$$b_3 = \frac{\alpha_{zf} A_{zf} \Delta t (2G_g c_g + \alpha_{gz} A_{gz})}{2(2m_z c_z + \alpha_{gz} A_{gz} \Delta t + \alpha_{zf} A_{zf} \Delta t)(2G_g c_g + \alpha_{gz} A_{gz}) - 2\alpha_{gz}^2 A_{gz}^2 \Delta t}$$

Eliminating $\theta_{w,k}(t + \Delta t)$ from Eq. (54), (55) the outlet temperature from the k-th cell can be expressed in the following form:

$$\begin{aligned} \theta_{f,k}^{out}(t+\Delta t) = c_1 \theta_{f,k}^{out}(t) + c_2 \theta_{f,k}^{in}(t) + c_3 \theta_{w,k}(t) + c_4 \theta_a^{in}(t) + \\ + c_5 \theta_{f,k}^{in}(t+\Delta t) + c_4 \theta_a^{in}(t+\Delta t) \end{aligned} \tag{56}$$

where

$$c_1 = \frac{a_1 + a_3 b_3}{1 - a_3 b_3}, c_2 = \frac{a_2 + a_3 b_3}{1 - a_3 b_3}, c_3 = \frac{a_3 + a_3 b_1}{1 - a_3 b_3}, c_4 = \frac{a_3 b_2}{1 - a_3 b_3}, c_5 = \frac{a_4 + a_3 b_3}{1 - a_3 b_3}.$$

Equation (56) is used for the simulation of process in cross-flow heat exchanger.

Assuming, in the first step, $\theta_{f,k}^{in}(t+\Delta t) = \theta_{f,k}^{in}(t)$ and $\theta_a^{in}(t+\Delta t) = \theta_a^{in}(t)$ where Δt is small enough, the outlet temperature is determined for the k-th cell, k=1,2,...,p. Repeating this procedure for different values of Δt the fluid temperature of cross-flow heat exchanger at the cells outlet is obtained as a function of time.

6. Simulation and comparative analysis

This section presents the analysis of mathematical model of the cross-flow heat exchanger with respect to different methods considered in the previous sections. The characteristic of real cross-flow heat exchanger are given in the Table 1.

The observed heat exchanger is divided into five cells. If the results are unsatisfactory the number of cells must be increased.

Table 1.

	Parameter	Dimension	Value		Parameter	Dimension	Value
1.	c_w	kJ/kgK	0,53	10.	c_f	kJ/kgK	4,233
2.	ρ_w	kg/m^3	7850	11.	α_{aw}	$W/\left(m^2K\right)$	220
3.	λ_w	W/mK	40	12.	ρ_a	kg/m^3	0,748
4.	r_o	m	0,016	13.	α_{wf}	$W/\left(m^2K\right)$	3000
5.	r_i	m	0,012	14.	c_a	kJ/kgK	1,097
6.	w_f	m/s	0,5	15.	L	m	10
7.	G_f	kg/s	0,12	16.	θ_{fN}^{in}	$^\circ C$	110
8.	G_a	kg/s	1,8	17.	θ_{aN}^{in}	$^\circ C$	220
9.	ρ_f	kg/m^3	952,38				

6.1. Results of the simulation for the analytical approach based on Laplace transformation. In the Figure 2 and 3 are shown the step response of the outlet temperature of fluid in the tube for the step change of the inlet fluid temperature and inlet air temperature. Since the temperature function obtained using the approach explained in the section 2 is complex valued, the amplitude of this function is used for the simulation.

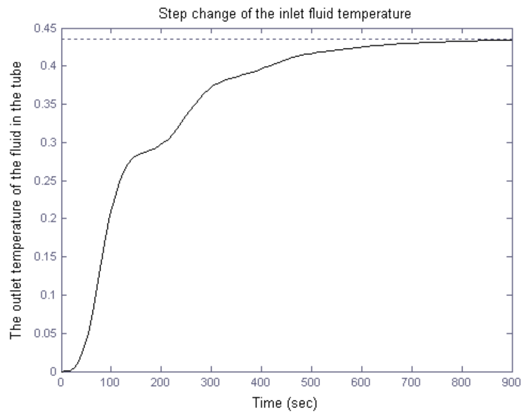

Figure 2

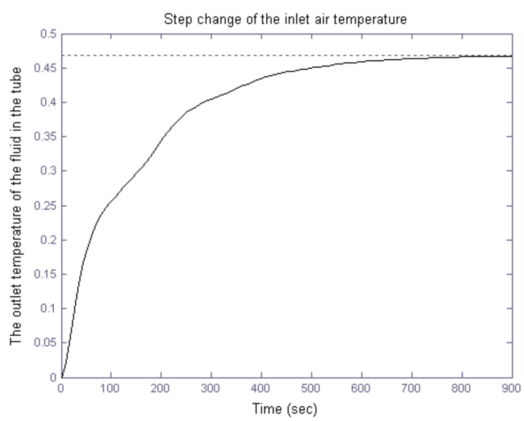

Figure 3

6.2. Results of the simulation for the differential discrete mathematical model based on the finite differences method. In the Figure 4,5,6 and 7 are shown the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change in the fluid inlet temperature and the step change in the air inlet temperature.

6.3. Results of the simulation for the mathematical model based on the physical discretization. In the Figure 8,9,10 and 11 are shown the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change in the fluid inlet temperature and the step change in the air inlet temperature.

6.4. Results of the simulation for the mathematical model based on the transport approach. In the Figure 12 and 13 are shown the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change (from 220C to 257 C) in the air inlet temperature.

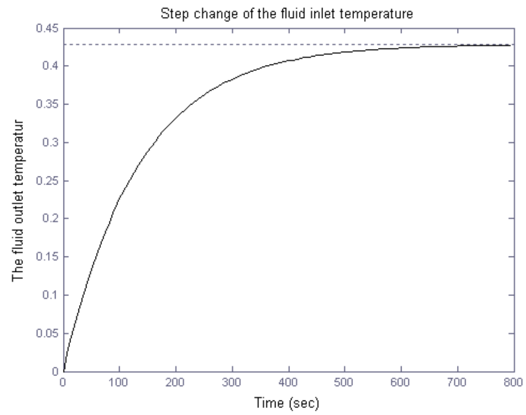

FIGURE 4

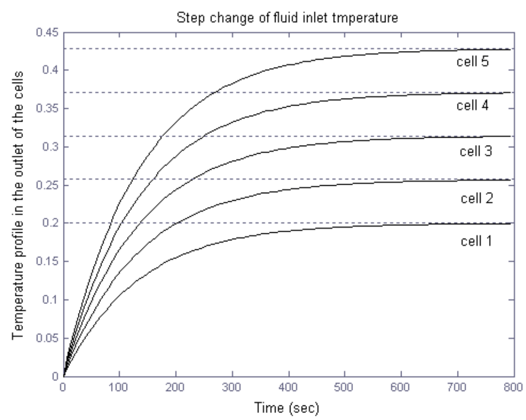

FIGURE 5

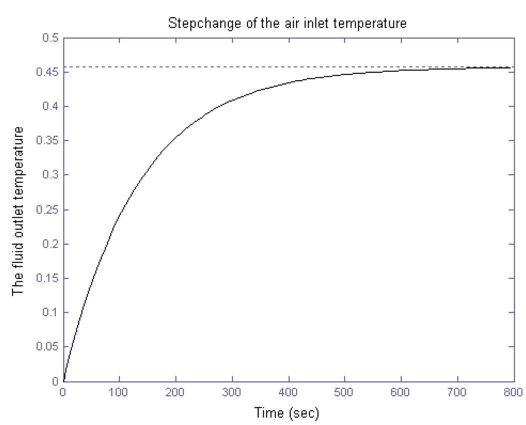

FIGURE 6

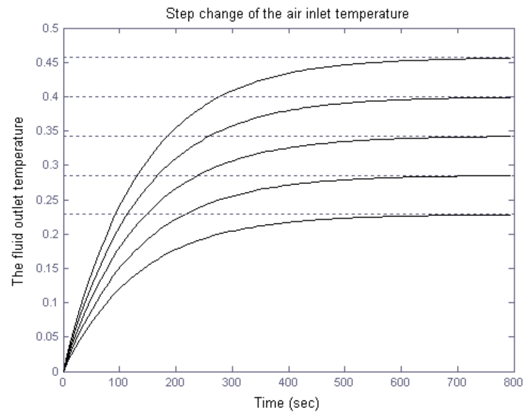

FIGURE 7

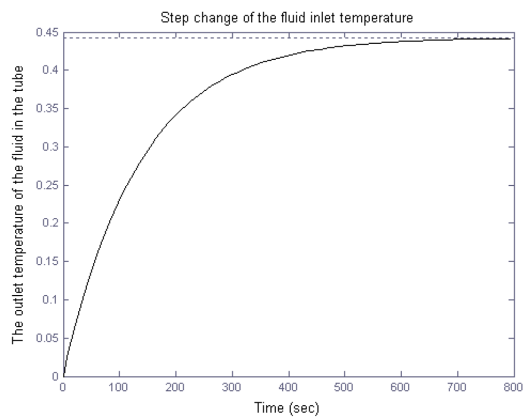

FIGURE 8

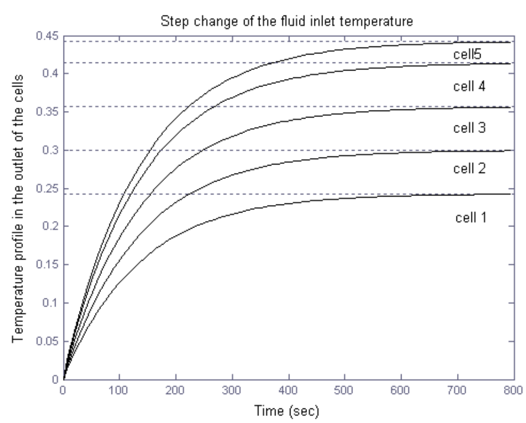

FIGURE 9

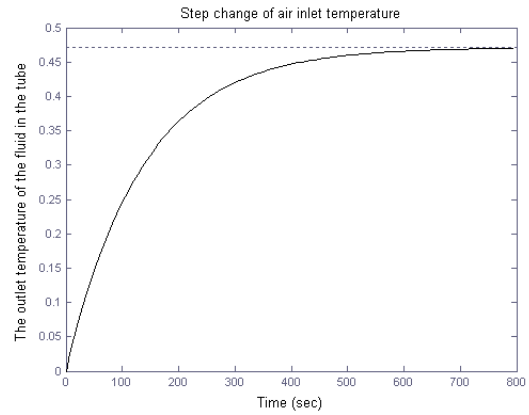

FIGURE 10

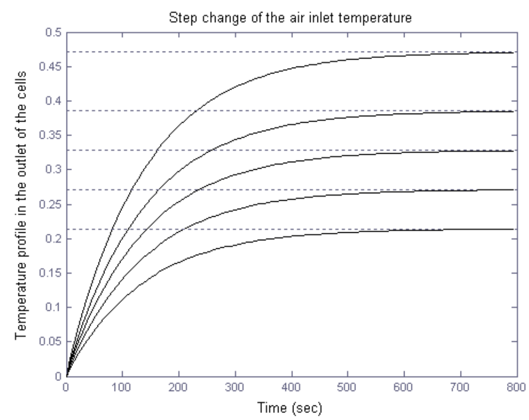

FIGURE 11

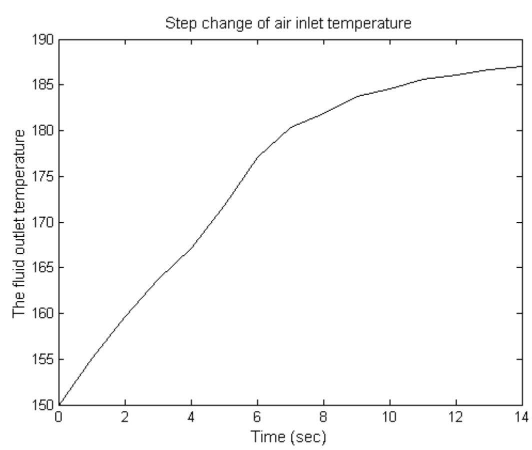

FIGURE 12

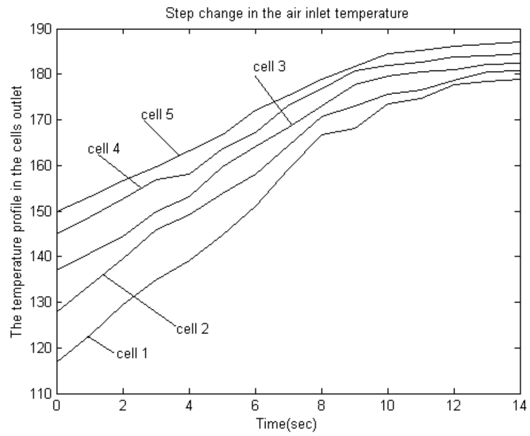

FIGURE 13

In the Figure 14 and 15 are shown the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change (from 110C to 127 C) in the fluid inlet temperature.

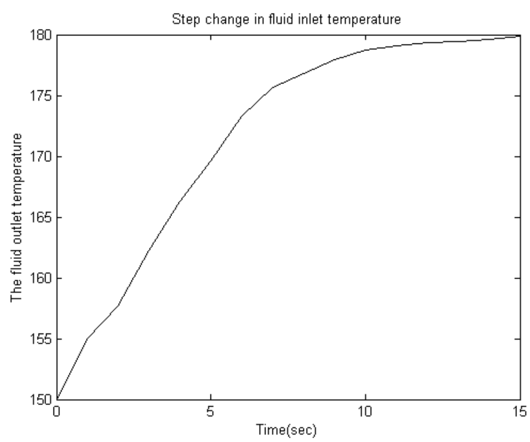

FIGURE 14

Finally, it should be pointed out that temperatures changes obtained by application of the transport approach are in total coordinates while step responses obtained by application of previous methods are in relative obviations.

7. Geometric optimization

The configuration of thermal systems is usually designed based on its steady-state performance subject to global and local constraints, where its configuration is optimized for maximization of global performance or for minimization of total cost [8]. This is usually done by assuming the system configuration and then simulating its operation under various conditions to determine its best operating conditions.

Recently, another approach has been used which consists in searching for the best design parameter to determine the optimal geometry that would produce the

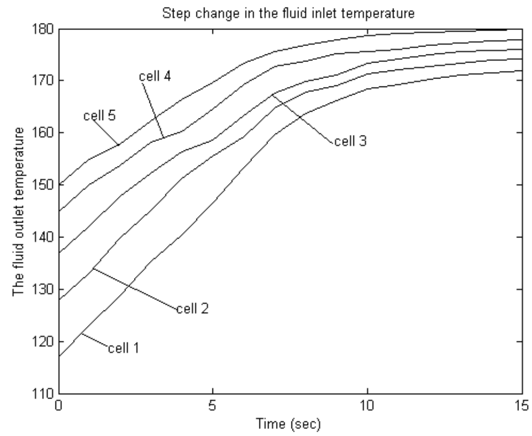

FIGURE 15

maximum performance [4]. In this procedure the configuration of the system is simulated and the effect of the system geometric parameters on its performance is analyzing to determine the optimal geometry for this system. The maximum performance is usually determined by the maximum output of the system.

The optimal geometry of the heat exchanger is searching in order to achieve the best controllability of the observed system. Therefore, the system controllability, the ability of a system to be control by the available control input in the form of a inlet temperatures, is the criterion used for optimization. This can be done by investigating its state controllability and output controllability with the system geometry for a better system design. The configuration of cross-flow heat exchanger is optimized in this work to produce the best geometry in terms of the system controllability. The air inlet temperature and the fluid inlet temperature are used as control inputs.

The problem of geometric optimization in this paper is analyzed for the mathematical model of the cross-flow heat exchanger derived using the method of physical dicretization, Eq. (45).

It is known [9] that output controllability conditions for the system, Eq.(45), can be expressed in the following form:

$$U = \left[\begin{array}{ccccc} \mathbf{c}B & \mathbf{c}AB & \mathbf{c}A^2B & \dots & \mathbf{c}A^{n-1}B \end{array} \right], n = 1, 2, ..., 10 \tag{57}$$

In the present case it can be shown that

$$rankU = rank \left[\begin{array}{ccccc} \mathbf{c}B & \mathbf{c}AB & \mathbf{c}A^2B & \dots & \mathbf{c}A^{n-1}B \end{array} \right] = 1 \tag{58}$$

indicating that the output of the system is controllable.

The condition number of U, $\wp_U(A, B, C)$, is used as an indicator of the degree to which the system may be controlled with a bounded input. The condition number of U can be expressed as ratio of the largest to smaller singular values of U:

$$\wp_U(A, B, C) = \frac{\sigma_{\max}(U)}{\sigma_{\min}(U)},\ U = U(A, B, C) \tag{59}$$

where $\sigma_{\max}$, $\sigma_{\min}$ are maximum and minimum singular values of U.

If the controllability matrix U is singular and not of full rank the condition number $\wp_U(A,B,C)$ is infinite. Also at the minimum $\wp_U(A,B,C)$ the system is most controllable. Thus the condition number will provide the information on the best situation for control of the cross-flow heat exchanger.

The cross-flow heat exchanger has a three dimensional parameters that can be varied: inner diameter d_i, outer diameter $d_o$and the length L which result in a three geometric degree of freedom. In order to reduce this degree of freedom, two geometric constraints will be taken into account, the total volume and the mass of the heat exchanger. The total volume can be expressed in the following form:

$$V=\frac{\pi d_o}{4}L. \tag{60}$$

The volume of the heat exchanger constraint can be represented by the volume occupied by the solid part of the tube:

$$V_t=\frac{\left(d_o^2-d_i^2\right)\pi}{4}L. \tag{61}$$

A non-dimensional parameter can be used to represent the ratio of the tube volume to the total volume:

$$\varphi=\frac{V_t}{V}=1-\frac{d_i^2}{d_o^2}. \tag{62}$$

These two constraints will decrease the number of geometric degree of freedom to one, that will be presented by the dimensionless aspect ratio d_o/L.

The optimization process starts by numerically simulate the cross-flow heat exchanger and minimize its controllability matrix condition number $\wp_U(A,B,C)$ by varying its aspect ratio d_o/L while keeping all other parameters constant as shown in Fig.16. In this figure it can be seen that there is a minimum $\wp_U$ with respect to the d_o/L which means that the system is best controllable at this aspect ratio. There are two extreme cases that is hard to control, when the aspect ratio d_o/L is large and when it is small. In both cases the condition number values are increasing.

This procedure illustrated in Fig.16 was repeated for different values of the fluid flow rate G_f. The minimum condition number in each case labeled $\wp_{U_{\min}}$is drawn against G_f as shown in Fig. 17. It is clear that the already minimized $\wp_{U_{\min}}$has a minimum value with respect to G_f which is mathematically expressed as:

$$\frac{\partial^2}{\partial\left(d_s/L\right)}\left(\wp_U\right)=0. \tag{63}$$

Analyzing results,Fig. 17 it can be concluded that there are two extreme cases: a case of high fluid mass flow rate and a case of low fluid mass flow rate. Both of this cases are hard to control which is well illustrated in increasing of the minimized condition number.

8. Conclusion

In this paper some characteristic methods in mathematical modeling of the heat exchangers are considered.

Analyzing the analytical approach using Laplace transform it can be seen that system of PDE is very complicated to be solved analytically, and it seems that this model has only theoretical signification while its practical meaning is insignificant.

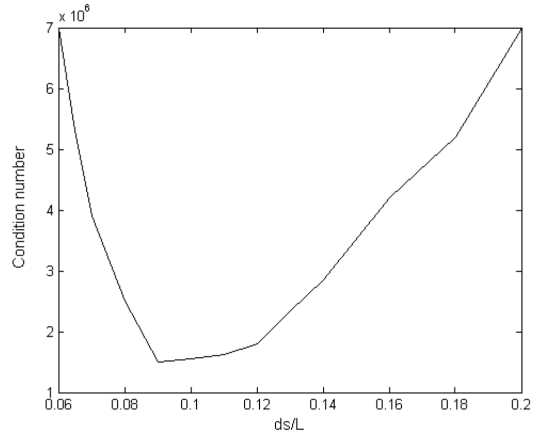

FIGURE 16

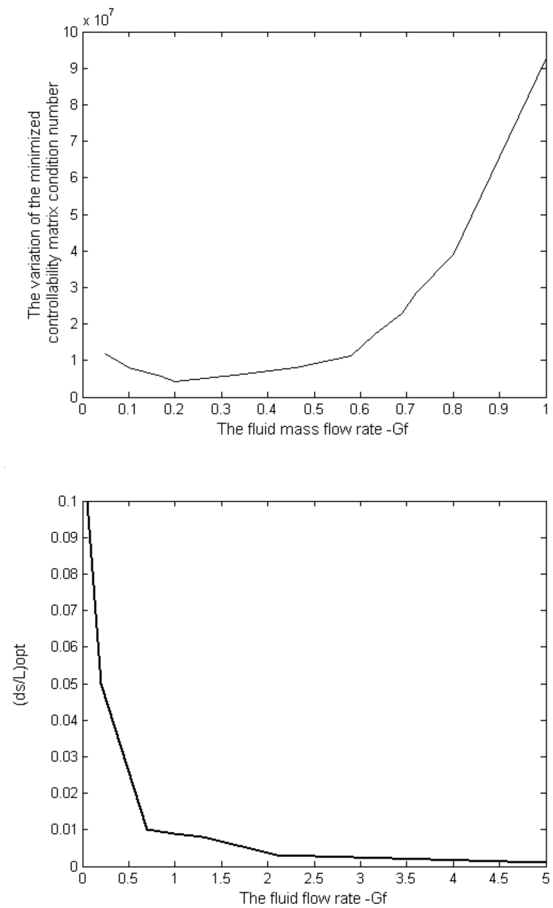

FIGURE 17

In order to overcome the solution difficulties, the procedure of differential discrete modeling is applied, leading to the set of ordinary differential equations of rather high order. This procedure is based on discretization of spatial coordinates using finite differences method.

One of the possible methods to avoid partial differential equations is the method of physical discretization which implies dividing observed heat exchanger into appropriate number of cells. If the number of cells is big enough it can be assumed that process within one cell is process with certain parameters. As a result of this procedure the ordinary differential equations of high order are obtained which are still simple enough to be solved.

The transport approach is based both on the spatial and time discretization which transforms the system of partial differential equations to the system of the algebraic equations which can be solved using numerical, iterative procedures. This procedure should be carried out taking into account the conditions for convergence of numerical procedures.

The graphical results of simulations of the different mathematical models of cross-flow heat exchangers are presented in this section.

Comparing the step responses obtained by application of different methods for mathematical modeling of the heat exchanger it can be concluded that it fits very well to each other.

The aim of this work is also to study the ability of optimizing the geometry of heat exchangers with respect to its dynamic controllability which is quantified in the controllability matrix condition number. The controllability matrix condition number was minimized with respect to the tube aspect ratio and the fluid mass flow rate. This procedure showed that optimizing the geometry of heat exchangers with respect to its controllability is important and should be taken into account in the design of the heat exchangers.

References

[1] Debeljkovi,Lj.D, G.V. Simeunovi, V.S. Muli, Mathematical Model of Objects and Plants in Automatic Control Systems,Planeta print, Belgrade, 2006.
[2] Stevi,Z.D, A Contribution to Comparative Dynamic Analysis and Geometric Optimization of Modern Recuperative Heat Exchangers, Dissertation, Belgrade, 2010.
[3] Gruji, Lj. T.,D.Lj. Debeljkovi, Differential Discrete Modelling and Controllability and Observability Analysis of Heat Exchangers,Technika, No 7-8, 1978.
[4] Alotaibi, S., M. Sen, B. Goodwine, K.T. Yang, Controllability of Cross-Flow Heat Exchangers, International Journal of Heat and Mass Transfer, Vol. 47, 913-924, 2004.
[5] Spiga, G., M. Spiga, Two- Dimensional Transient Solution for Cross-Flow Heat Exchangers with Neither Gas Mixe,Journal of Heat Transfer, Vol. 109, May,281-286, 1987.
[6] Romie, F. E., Transient Response of the Counter-Flow Heat Exchanger, J. Heat Transfer 106 (3), 620-626, 1984.
[7] Roetzel, W., Y. Xuan, Transient/Response of Parallel and Counter-Flow Heat Exchangers, J. Heat Transfer, 114 (2), 510-512, 1992.
[8] A. Bejan, Entropy Generation Minimization, CRC Press, 1996.
[9] Klamka J., Controllability of Dynamical Systems, Kluwer Academic Publishers, Netherlands, 1991.
[10] Abate, J.,P.P. Valko, Multi-precision Laplace Inversion, Int. J. Numer. Meth.,Engng. 60,979-993, 2004.
[11] Ili, K., Dynamical behaviour of the heat ehchangers, M.Sc. Theasys, Faculty of Mechanical Engineering, Sarajevo,1986.

University of Belgrade, Faculty of Mechanical Engineering, Department of Control Engineering, 11000, Belgrade, Serbia
E-mail: dalibor1976@ptt.rs