CONSENSUS PROBLEM OF SECOND-ORDER MULTI-AGENT SYSTEMS WITH INPUT DELAY

CHENG-LIN LIU AND FEI LIU

Abstract. In this paper, the consensus problem is investigated for the second-order multi-agent systems with input delay. Based on the frequency-domain analysis, the input-delay-dependent consensus conditions are obtained for the continuous-time system and the sampled-data system respectively. Simulation results illustrate the correctness of the theoretical results.

Key Words. Consensus, multi-agent systems, input delay, continuous-time system, sampled-data system

1. Introduction

In recent years, distributed coordination control of multi-agent systems has attracted much attention from various research communities including biology, artificial intelligence, automatic control, etc. Besides, the coordination control of multi-agent systems has broad engineering application, such as automated highway systems, air traffic control, congestion control in communication networks, etc.

Consensus problem, which is one of the most fundamental and important issues in the coordination control of multi-agent systems, requires that the outputs of several spatially distributed agents reach a common value without recourse to a central controller or global communication. Consensus of the first-order multi-agent systems, whose agent’s dynamics is modeled by a single integrator, has been extensively studied [1, 2, 3, 4], and the connectedness conditions have been obtained for the agents asymptotically converging to a consensus with fixed or switching interconnection topology. However, the consensus problem becomes more complicated when the consensus algorithms are extended to the second-order multi-agent systems, whose agent’s dynamics is modeled by a double integrator. Many consensus algorithms have been proposed to solve the consensus problem of the second-order multi-agent systems, and sufficient conditions have been obtained for the system asymptotically converging to the static consensus and the dynamical consensus respectively [5, 6, 7].

In reality, however, time delays cannot be negligible in the coordination control of multi-agent systems. Generally, there exist two kinds of time delays in the multi-agent systems. One is related to the information flow between neighboring agents, which is called communication delay. The other is related to the processing time for the packets arriving at each agent, which is called input delay [8].

The consensus problem under communication delays has been extensively studied for both the first-order and the second-order multi-agent systems in the control...
community based on different methods, such as the Lyapunov functions analysis [10, 11, 13, 12, 14], the frequency-domain analysis [15, 16], the method based on delayed and hierarchical graphs [17, 18], etc. Moreover, communication-delay-independent consensus conditions can be always obtained for the multi-agent systems with the interconnection topology satisfying certain connectedness conditions.

Although input delay problems have been extensively investigated in the classic control theory, the consensus problem under input delays does not attract much attention, to our knowledge. In some reports, the identical communication delay introduced in the multi-agent systems can be treated as the identical input delay [2, 16]. Using frequency-domain analysis, Tian and Liu [8] considered the consensus problem of the first-order multi-agent systems with diverse input delays based on undirected graphs, and obtained the decentralized consensus criterion depending on the input delay. Moreover, the decentralized consensus condition depending only on the input delays is also obtained for the first-order system with both diverse communication delays and input delays in [8]. In [19], Tian and Liu investigated the leader-following consensus of the second-order multi-agent systems with diverse input delays and symmetric coupling weights. Under the double-consensus algorithm, which is composed of the position and the velocity consensus coordination parts, the decentralized consensus conditions with some prerequisites are obtained for the system converging to the leader’s states asymptotically. Furthermore, the robustness of the symmetric system with asymmetric weight perturbation is also studied in [19], and a bound of the largest singular value of the perturbation matrix is obtained as the robust consensus condition.

In the engineering application, each agent uses the sampled information of its neighbor to coordinate with each other, so the consensus analysis of the discrete-time multiple dynamic agents with sampled-data information is much more important [9, 20, 21, 22, 24]. However, the consensus problem of the discrete-time multi-agent systems with input delay has just been studied for the first-order multi-agent systems [8, 20, 21, 22]. To our knowledge, the consensus problem of the second-order multi-agent systems with both input delay and sampled information has not attracted any attention.

In this paper, we consider the consensus problem of the second-order multi-agent systems with input delay and symmetric interconnection topology, and the consensus algorithm consists of the velocity stabilization part, the position consensus coordination part and the velocity consensus coordination part. Firstly, the input-delay-dependent consensus condition is obtained for the continuous-time system converging to a stationary consensus based on the frequency-domain analysis. Then, sufficient and necessary condition, which depends on the input delay and sampling interval, is also obtained for the sampled-data system converging to a stationary consensus. When the sampling interval is larger than the critical value, the sampled-data multi-agent system diverges. Moreover, the results are extended to the system under double-consensus algorithm and another stationary consensus algorithm composed of the velocity stabilization part and the position consensus coordination part, respectively.

2. Preliminaries

2.1. Notions of graph theory. A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order $n$ consists of a set of vertices $\mathcal{V} = \{1, \ldots, n\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ with $a_{ij} \geq 0$. The node indexes belong to a finite index set $\mathcal{I} = \{1, 2, \ldots, n\}$. An directed edge from $i$ to $j$ in $\mathcal{G}$ is denoted by $e_{ij} = (i, j) \in \mathcal{E}$. 

We assume that the adjacency elements associated with the edges of the digraph are positive, i.e., \( a_{ij} > 0 \) \( \Leftrightarrow e_{ij} \in \mathcal{E} \). Moreover, we assume \( a_{ii} = 0 \) for all \( i \in I \). The set of neighbors of node \( i \) is denoted by \( N_i = \{ j \in V : (i, j) \in \mathcal{E} \} \). In \( G \), if \( (i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E} \), we say that \( G \) is undirected graph or bidirectional graph.

In \( G \), the out-degree of node \( i \) is defined as: \( \text{deg}_{\text{out}}(i) = \sum_{j=1}^{n} a_{ij} \). Let \( \mathcal{D} \) be the diagonal matrix with the out-degree of each node along the diagonal and call it the degree matrix of \( G \). The Laplacian matrix of \( G \) is defined as \( L = \mathcal{D} - \mathcal{A} \).

If there is a path in \( G \) from one node \( i \) to another node \( j \), then \( j \) is said to be reachable from \( i \). If not, then \( j \) is said to be not reachable from \( i \). If a node is reachable from every other node in the digraph, then we say it globally reachable.

An undirected graph is connected if it contains a globally reachable node. If there is a path in \( G \) from one node \( i \) to another node \( j \), then \( j \) is said to be reachable from \( i \). If not, then \( j \) is said to be not reachable from \( i \). If a node is reachable from every other node in the digraph, then we say it globally reachable.

In this paper, we just consider the static topology \( G = (V, \mathcal{E}, \mathcal{A}) \), i.e., the connection of the nodes in the digraph \( G \) does not change with time.

### 2.2. Useful lemmas

The following lemmas will be used in the proof of the main results.

**Lemma 1.** Define the function \( f(\omega) = -\omega \tau - \pi + \arctan(\alpha \omega) \), where \( \tau > 0 \) and \( \alpha > 0 \) are two constants.

- When \( \tau < \alpha \), there exists only one constant \( \omega_c > 0 \) that satisfies

\[
\omega_c \tau = \arctan(\alpha \omega_c),
\]

i.e., \( f(\omega_c) = -\pi \), and \( f(\omega) \) is monotonously decreasing for \( \omega \in (\omega_c, \infty) \).

- When \( \tau \geq \alpha \), \( f(\omega) < -\pi \) holds for \( \omega \in (0, \infty) \).

**Proof.** Calculating the derivative of \( f(\omega) \) on \( \omega \) yields

\[
\dot{f}(\omega) = -\tau + \frac{\alpha}{1 + \alpha^2 \omega^2}.
\]

When \( \tau < \alpha \), let \( \omega_0 = \sqrt{\frac{1}{\omega_c^2} - \frac{\omega_c}{\pi}} \), which satisfies \( \dot{f}(\omega_0) = 0 \). Thus, \( f(\omega) \) is monotonously increasing for \( \omega \in (0, \omega_0) \), and \( f(\omega) \) is monotonously decreasing for \( \omega \in (\omega_0, \infty) \). Since \( f(0) = -\pi \), \( f(\omega_0) > -\pi \) and \( \lim_{\omega \to \infty} f(\omega) = -\infty \), there exists only one \( \omega_c \in (\omega_0, +\infty) \) satisfying (1). Hence, \( \omega_c > \omega_0 \) and \( f(\omega) \) is monotonously decreasing for \( \omega \in (\omega_c, \infty) \).

When \( \tau \geq \alpha \), \( \dot{f}(\omega) < 0 \) holds for \( \omega \in (0, \infty) \), i.e., \( f(\omega) \) is monotonously decreasing for \( \omega \in (0, \infty) \). Hence, \( f(\omega) < -\pi \) holds for \( \omega \in (0, \infty) \). Lemma 1 is proved. □

**Lemma 2.** Define the function \( h(\omega) = -\omega D - \frac{\beta}{2} - \pi + \arctan(\beta \tan \frac{\omega_c}{2}) \), \( \omega \in (0, \pi] \), where \( D \) is a positive integer, and \( \beta > 0 \) is a constant.

- When \( 2D + 1 < \beta \) with \( \beta > 1 \), there exists only one constant \( \omega_c \in (0, \pi] \) that satisfies

\[
\omega_c D + \frac{\omega_c}{2} = \arctan(\beta \tan \frac{\omega_c}{2}),
\]

i.e., \( h(\omega_c) = -\pi \), and \( h(\omega) \) is monotonously decreasing for \( \omega \in (\omega_c, \pi] \).

- When \( 2D + 1 \geq \beta \) with \( \beta > 1 \) or \( \beta \leq 1 \), \( h(\omega) < -\pi \) holds for \( \omega \in (0, \pi] \).

**Proof** Calculate the derivative of \( h(\omega) \) on \( \omega \), and we get

\[
\dot{h}(\omega) = -D - \frac{1}{2} + \frac{\beta}{2 + 2(\beta^2 - 1) \sin^2 \frac{\omega_c}{2}}.
\]

To prove the Lemma 2, we divide the proof into the following three cases.

**Case 1.** \( \beta > 1 \),

When \( \beta > 1 \), \( \dot{h}(\omega) \) is monotonously decreasing for \( \omega \in (0, \pi] \).
When $2D+1 < \beta$, we define $\omega_0 = 2 \arcsin \sqrt{\frac{\beta-2D-1}{(2D+1)(\beta-1)}}$, which satisfies $\dot{h}(\omega_0) = 0$. Thus, $h(\omega)$ is monotonously increasing for $\omega \in (0, \omega_0)$, and $h(\omega)$ is monotonously decreasing for $\omega \in (\omega_0, \pi]$. Since $h(0) = -\pi$, $h(\omega_0) > -\pi$ and $h(\pi) = -D\pi - \pi \leq -\pi$, there exists only one $\omega_c \in (\omega_0, \pi]$ satisfying (2). Hence, $\omega_c > \omega_0$ and $h(\omega)$ is monotonously decreasing for $\omega \in (\omega_c, \pi]$.

When $2D + 1 \geq \beta$, $\dot{h}(\omega) < 0$ holds for $\omega \in (0, \pi]$, i.e., $h(\omega)$ is monotonously decreasing for $\omega \in (0, \pi]$, and $h(\omega) < -\pi$ holds for $\omega \in (0, \pi]$.

Case 2. $\beta = 1$.

When $\beta = 1$, $\dot{h}(\omega) = -\omega D - \pi$. $h(\omega)$ is monotonously decreasing for $\omega \in (0, \pi]$, so $h(\omega) < -\pi$ holds for $\omega \in (0, \pi]$.

Case 3. $\beta < 1$.

When $\beta < 1$, $\dot{h}(\omega)$ is monotonously increasing for $\omega \in (0, \pi]$.

When $2D+1 < \frac{1}{\beta}$, we define $\hat{\omega}_0 = 2 \arcsin \sqrt{\frac{\beta-2D-1}{(2D+1)(\beta-1)}}$, which satisfies $\dot{h}(\hat{\omega}_0) = 0$. Thus, $h(\omega)$ is monotonously decreasing for $\omega \in (0, \hat{\omega}_0)$, and $h(\omega)$ is monotonously increasing for $\omega \in (\hat{\omega}_0, \pi]$. Since $h(0) = -\pi$, $h(\hat{\omega}_0) < -\pi$ and $h(\pi) = -D\pi - \pi \leq -\pi$, $h(\omega) < -\pi$ holds for $\omega \in (0, \pi]$.

When $2D + 1 \geq \frac{1}{\beta}$, $\dot{h}(\omega) < 0$ holds for $\omega \in (0, \pi]$, i.e., $h(\omega)$ is monotonously decreasing for $\omega \in (0, \pi]$. Hence, $h(\omega) < -\pi$ holds for $\omega \in (0, \pi]$.

Lemma 2 is proved. □

Lemma 3. $\frac{\omega}{1+\omega^2} < \arctan(y)$ holds for $y > 0$.

Proof. Define $q(y) = \frac{\omega}{1+\omega^2} - \arctan(y)$, and we get the derivative of $q(y)$ on $y$ as $\dot{q}(y) = -\frac{2\omega^2 y}{(1+y^2)^2}$. Obviously, $q(y)$ is monotonously decreasing for $y > 0$. Since $q(0) = 0$, $q(y) < 0$ holds with $y > 0$. Lemma 3 is proved. □

Lemma 4. Let $\phi(\lambda) = \arctan(\frac{K_1+\gamma}{\psi(\lambda)})$, where $\kappa > 0, \gamma > 0$, and $\psi(\lambda) > 0$ is monotonously increasing for $\lambda > 0$. Then, $\phi(\lambda)$ is monotonously decreasing for $\lambda > 0$.

Proof. Calculating the derivative of $\phi(\lambda)$ on $\lambda$ yields

$$\dot{\phi}(\lambda) = \frac{\dot{\psi}(\lambda)}{\psi(\lambda)} \left( \frac{K_1+\gamma}{\psi(\lambda)} \right)^2 - \frac{\arctan(\frac{K_1+\gamma}{\psi(\lambda)})}{1 + \left( \frac{K_1+\gamma}{\psi(\lambda)} \right)^2}.$$  

Since $(\frac{K_1+\gamma}{\psi(\lambda)}) > 0$ and $\dot{\psi}(\lambda) > 0$, we obtain that $\dot{\phi}(\lambda) < 0$ holds with $\lambda > 0$ from Lemma 3. Lemma 4 is proved. □

3. Consensus of continuous-time system

In a multi-agent system, each agent can be considered as a node of a digraph, and the information flow between neighboring agents can be regarded as a directed edge between the neighboring nodes in the digraph. Thus, the interconnection topology of the multi-agent systems is usually described as a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$.

First of all, consider the continuous-time system with each agent’s dynamics described as

$$\dot{x}_i(t) = v_i(t),$$
$$\dot{v}_i(t) = u_i(t),$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position, velocity and acceleration, respectively, of the agent $i$. 

With non-negligible input delay, the system (3) becomes
\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= u_i(t - \tau),
\end{align*}
\]
where \(\tau > 0\) is the input delay of each agent.

We say that the agents in the system (4) converge to a stationary consensus asymptotically, if the agents’ states satisfy
\[
\lim_{t \to \infty} x_i(t) = c, \quad \lim_{t \to \infty} v_i(t) = 0, \forall i \in I,
\]
where \(c\) is a constant. Furthermore, if
\[
\lim_{t \to \infty} x_i(t) = \frac{1}{n} \left( \sum_{i=1}^{n} x_i(0) + t \sum_{i=1}^{n} v_i(0) \right), \quad \lim_{t \to \infty} v_i(t) = \frac{1}{n} \sum_{i=1}^{n} v_i(0), \forall i \in I,
\]
we say that the agents converge to a dynamical average consensus asymptotically.

For the system (4), we adopt a consensus algorithm analogous to that in [5]
\[
u_i = -\kappa v_i(t) - \sum_{j \in N_i} a_{ij}(x_i(t) - x_j(t)) - \gamma \sum_{j \in N_i} a_{ij}(v_i(t) - v_j(t)),
\]
where \(\kappa > 0, \gamma > 0, N_i\) denotes the neighbors of the agent \(i\), and \(a_{ij} > 0\) is the adjacency element of \(A\) in the digraph \(\mathcal{G} = (V, E, A)\).

**Remark 1.** To our knowledge, consensus problem has been studied extensively for the second-order continuous-time multi-agent systems under fixed and switching topology without input delays [5, 6, 7], and the consensus analysis under input delay has been analyzed only for the system with the consensus algorithm: \(u_i = -\sum_{j \in N_i} a_{ij}(x_i(t) - x_j(t)) - \sum_{j \in N_i} a_{ij}(v_i(t) - v_j(t))\) [14, 16, 19]. However, the algorithm (5) is another important consensus protocol for the second-order multi-agent systems, so we will analyze the consensus of the system (4) with the algorithm (5) in the following.

With (5), the closed-loop form of the system (4) is given by
\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= -\kappa v_i(t - \tau) - \sum_{j \in N_i} a_{ij}(x_i(t - \tau) - x_j(t - \tau)) \\
&\quad - \gamma \sum_{j \in N_i} a_{ij}(v_i(t - \tau) - v_j(t - \tau)).
\end{align*}
\]

**Proposition 1.** Consider the network of \(n\) agents (6) with a static interconnection topology \(\mathcal{G} = (V, E, A)\) that is undirected and connected, and the topology graph has symmetric weights, i.e., \(a_{ij} = a_{ji}\). When \(\tau < \frac{k + \gamma \lambda_{\text{max}}}{\lambda_{\text{max}}}\), where \(\lambda_{\text{max}}\) is the largest eigenvalue of the Laplacian matrix \(L\), the system (6) asymptotically converges to a stationary consensus, if and only if
\[
\frac{\sqrt{\lambda_i^2 + (k + \gamma \lambda_i)^2 \omega_{ci}^2}}{\mu_{ci}} < 1, \quad \forall i \in I,
\]
where \(\lambda_i\) is the eigenvalue of \(L\), and \(\omega_{ci}\) satisfies
\[
\begin{align*}
\omega_{ci} &\in \left( \frac{\pi}{2\tau}, \lambda_i \right) \quad \text{if } \lambda_i = 0; \\
\omega_{ci} &\in \left( \arctan \frac{k + \gamma \lambda_i}{\lambda_i}, \omega_{ci} \right), \quad \lambda_i > 0, \lambda_i \neq 0.
\end{align*}
\]

The system (6) diverges, if \(\tau \geq \frac{k + \gamma \lambda_{\text{max}}}{\lambda_{\text{max}}}\).
Then, we investigate the roots of the following equation for $\lambda$

$$(11) \quad \text{det}(s^2 + (\kappa + \gamma) se^{-st} + \lambda e^{-st}) = 0.$$  

Because the interconnection topology $G$ is undirected and connected, 0 is a simple eigenvalue of $L$, i.e., $\text{rank}(L) = n - 1$, and the other eigenvalues are all positive numbers for the symmetry of the coupling weights $[23]$. Denote the eigenvalues of $L$ as $\lambda_i, i = 1, \cdots, n$, and we assume $\lambda_1 = 0$ and $\lambda_i > 0, i \geq 2$. Thus, (10) equals

$$(12) \quad s^2 + (\kappa + \gamma \lambda_i) se^{-st} + \lambda_i e^{-st} = 0, i \in I.$$

To prove Proposition 1, we divide the proof into the following two cases.

Case 1: $\tau < \frac{k+\gamma\lambda_{\max}}{\sqrt{\omega^2}}$.

When $\lambda_1 = 0$, (12) becomes

$$(13) \quad s(s + \kappa e^{-st}) = 0.$$

It is obvious that (13) has a simple root at $s = 0$. When $s \neq 0$, we consider the roots of following equation

$$(14) \quad 1 + \frac{\kappa e^{-st}}{s} = 0.$$

Based on the Nyquist stability criterion, the roots of (14) lie on the open left half complex plane, if and only if if the curve $\frac{\kappa e^{-jt\omega}}{j\omega}$ does not enclose the point $(-1, j0)$ for $\omega \in R$. When $\omega \in (0, +\infty)$, $|\frac{\kappa e^{-jt\omega}}{j\omega}| = \frac{\kappa}{\omega}$ and $\text{arg}(\frac{\kappa e^{-jt\omega}}{j\omega}) = -\omega \tau - \frac{\pi}{2}$ are both monotonously decreasing, where $\text{arg}(\cdot)$ denotes the phase, and $\frac{\kappa e^{-jt\omega}}{j\omega}$ crosses the negative real axis for the first time at $\omega_{c1} = \frac{\pi}{2\omega}$. Thus, the roots of (14) all lie on the open left half complex plane, if and only if \[ \frac{\kappa}{\omega_{c1}} < 1, \] which holds for (7).

When $\lambda_i > 0$, (12) can be written as

$$(15) \quad 1 + \frac{(\kappa + \gamma \lambda_i)s + \lambda_i}{s^2}e^{-st} = 0.$$

The roots of (15) lie on the open left half complex plane, if and only if \[ \frac{((\kappa + \gamma \lambda_i)s + \lambda_i)e^{-j\omega \tau}}{-\omega^2} \] does not enclose the point $(-1, j0)$ for $\omega \in R$. Defining

$$g_i(\omega) = \frac{j((\kappa + \gamma \lambda_i)\omega + \lambda_i)}{-\omega^2}e^{-j\omega \tau},$$

we obtain $|g_i(\omega)| = \frac{\sqrt{\omega^2((\kappa + \gamma \lambda_i)\omega + \lambda_i)^2}}{\omega^2}$ and $\text{arg}(g_i(\omega)) = -\omega \tau - \pi + \arctan(\frac{((\kappa + \gamma \lambda_i)\omega + \lambda_i)}{\omega})$. Obviously, $|g_i(\omega)|$ is monotonously decreasing for $\omega \in (0, \infty)$.

When $\tau < \frac{k+\gamma\lambda_{\max}}{\sqrt{\omega_{c2}^2}}$, $g_i(\omega)$ crosses the real axis for the first time at $\omega_{ci}$, defined as (8b), and $\text{arg}(g_i(\omega)) < -\pi$ holds for $\omega \in (\omega_{ci}, +\infty)$ from Lemma 1. Hence, the roots of (15) all lie on the open left half complex plane, if and only if \[ \frac{\sqrt{\omega^2((\kappa + \gamma \lambda_i)\omega + \lambda_i)^2}}{\omega_{ci}} < 1, \] i.e., (7) holds.
As proved above, the roots of (10) all lie on the open left half complex plane except for a root at \( s = 0 \). Therefore, the state \( x_i(t) \) of the system (6) converges to a steady state, i.e., \( \lim_{t \to \infty} x_i = x_i^* \), \( i \in \mathcal{I} \), and \( \lim_{t \to \infty} v_i(t) = 0 \), \( \forall i \in \mathcal{I} \) holds for (6). Thus, it is obtained from (6) that \( L[x_1^*, \ldots, x_n^*]^T = 0 \). Since \( \text{rank}(L) = n - 1 \) and \( L[1, \ldots, 1]^T = 0 \) based on the definition of \( L \), the roots of \( Lx^* = 0 \) can be expressed as \( x^* = c[1, \ldots, 1]^T \), where \( c \) is a constant. Therefore, the system (6) converges to a stationary consensus.

**Case 2:** \( \tau \geq \frac{k + \gamma \lambda_{\max}}{\lambda_{\max}} \).

In this case, \( \arg(g_i(\omega)) \) with \( \lambda_i = \lambda_{\max} \) is monotonously decreasing for \( \omega \in (0, \infty) \), and \( \arg(g_i(\omega)) < -\pi \) with \( \lambda_i = \lambda_{\max} \) holds for \( \omega \in (0, \infty) \) from Lemma 1. Based on the Nyquist stability criterion, the equation (15) with \( \lambda_i = \lambda_{\max} \) has at least one root on the open right half complex plane, so the system (6) diverges.

**Proposition 1** is proved. □

**Remark 2.** From (8a) and (8b), we obtain

\[
\tau = \frac{\arctan(\kappa + \gamma \lambda_i \omega_{ci})}{\omega_{ci}}.
\]

Calculating the derivative of \( \tau \) on \( \omega_{ci} \) yields

\[
\frac{d\tau}{d\omega_{ci}} = \frac{1}{\omega_{ci}} \left( \frac{(\kappa + \gamma \lambda_i)\omega_{ci}}{1 + (\kappa + \gamma \lambda_i)^2} - \arctan(\kappa + \gamma \lambda_i \omega_{ci}) \right).
\]

Based on Lemma 3, \( \frac{d\tau}{d\omega_{ci}} < 0 \) holds with \( \omega_{ci} > 0 \) for \( \frac{(\kappa + \gamma \lambda_i)\omega_{ci}}{1 + (\kappa + \gamma \lambda_i)^2} \). Hence, based on (8a) and (8b), \( \tau \) decreases when \( \omega_{ci} \) increases. In addition, the condition (7) equals

\[
\omega_{ci} = \sqrt{\frac{(k + \gamma \lambda_i)^2 + \sqrt{(k + \gamma \lambda_i)^4 + 4\lambda_i^2}}{2}},
\]

and we obtain

\[
\tau < \frac{\arctan(\kappa + \gamma \lambda_i \sqrt{\frac{(k + \gamma \lambda_i)^2 + \sqrt{(k + \gamma \lambda_i)^4 + 4\lambda_i^2}}{2}})}{\sqrt{\frac{(k + \gamma \lambda_i)^2 + \sqrt{(k + \gamma \lambda_i)^4 + 4\lambda_i^2}}{2}}},
\]

Based on Remark 2 and Lemma 4, the sufficient and necessary conditions in Proposition 1 can be replaced by a simple algebraic condition on the input delay subjected to the control parameters and the largest eigenvalue of the Laplacian matrix in the following theorem.

**Theorem 1.** Consider the network of \( n \) agents (6) with a static interconnection topology \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) that is undirected and connected, and the topology graph has symmetric weights. The system (6) asymptotically converges to a stationary consensus, if and only if

\[
\tau < \frac{\arctan(\kappa + \gamma \lambda_{\max} \sqrt{\frac{(k + \gamma \lambda_{\max})^2 + \sqrt{(k + \gamma \lambda_{\max})^4 + 4\lambda_{\max}^2}}{2}})}{\sqrt{\frac{(k + \gamma \lambda_{\max})^2 + \sqrt{(k + \gamma \lambda_{\max})^4 + 4\lambda_{\max}^2}}{2}}},
\]

where \( \lambda_{\max} \) is the largest eigenvalue of the Laplacian matrix \( L \).

When consensus algorithm (5) without the velocity stabilization part, (5) becomes a dynamical consensus algorithm as follows

\[
u_i(t) = - \sum_{j \in N_i} a_{ij}(x_i(t) - x_j(t)) - \gamma \sum_{j \in N_i} a_{ij}(v_i(t) - v_j(t)), \quad (17)
\]
where $\gamma > 0$, $a_{ij} > 0, j \in N_i$. Obviously, the results in Theorem 1 can be easily extended to the consensus of the system (4) with (17). With (17), the closed-loop form of the system (4) is given by

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = -\sum_{j \in N_i} a_{ij}(x_i(t) - x_j(t)) - \gamma \sum_{j \in N_i} a_{ij}(v_i(t) - v_j(t)).$$

**Corollary 1.** Consider the network of $n$ agents (18) with a static interconnection topology $G = (V, E, A)$ that is undirected and connected, and the topology graph has symmetric weights. The system (18) asymptotically converges to a dynamical average consensus, if and only if

$$\tau < \text{arctan}\left(\sqrt{\frac{\gamma^2 \lambda_{\text{max}}^2 + \lambda_{\text{max}} \sqrt{\gamma^4 \lambda_{\text{max}}^4 + 4}}{\gamma^2 \lambda_{\text{max}}^2 + \sqrt{\gamma^4 \lambda_{\text{max}}^4 + 4}}}, \right)$$

where $\lambda_{\text{max}}$ is the largest eigenvalue of the Laplacian matrix $L$.

**Remark 3.** The consensus analysis of the system (18) has also been presented in the reference [16]. Different from [16], Corollary 1 gives another expression of the sufficient and necessary consensus condition, which is equal to that in [16].

Besides the algorithms (5) and (17), another stationary consensus algorithm as follows can be applied when the neighbors' velocities can not be obtained for each agent

$$u_i(t) = \kappa v_i(t) - \sum_{j \in N_i} a_{ij}(x_i(t) - x_j(t)),$$

where $\kappa > 0$, and $a_{ij} > 0, j \in N_i$. Similarly, the results in Theorem 1 can be easily extended to the consensus of the system (4) with (20). The closed-loop form of the system (4) with the algorithm (20) is given by

$$\dot{x}_i(t) = v_i(t),$$
$$\dot{v}_i(t) = -\kappa v_i(t - \tau) - \sum_{j \in N_i} a_{ij}(x_i(t - \tau) - x_j(t - \tau)).$$

**Corollary 2.** Consider the network of $n$ agents (21) with a static interconnection topology $G = (V, E, A)$ that is undirected and connected, and the topology graph has symmetric weights. The system (21) asymptotically converges to a stationary consensus, if and only if

$$\tau < \text{arctan}\left(\frac{\kappa}{\lambda_{\text{max}} \sqrt{\lambda_{\text{max}}^2 + \sqrt{\lambda_{\text{max}}^4 + 4}}}, \right)$$

where $\lambda_{\text{max}}$ is the largest eigenvalue of the Laplacian matrix $L$.

4. Consensus of sampled-data system

In this section, the consensus problem is considered for the second-order multi-agent systems based on sampled-data control. Similar to [9, 24], we assume that each agent in the system receives the information of its neighboring agents synchronously with the same time interval.
Consider the sampled-data model with zero-order hold of the system (4)

\[
\begin{align*}
x_i((k+1)T) &= x_i(kT) + Tv_i(kT) + \frac{T^2}{2}u_i(kT - DT), \\
v_i((k+1)T) &= v_i(kT) + Tu_i(kT - DT),
\end{align*}
\]

where \( T > 0 \) is the sampling interval, \( D > 0 \) is a positive integer and \( DT = \tau > 0 \) is the input delay.

For simplicity, (22) is rewritten as a normal discrete-time model

\[
\begin{align*}
x_i(k+1) &= x_i(k) + Tv_i(k) + \frac{T^2}{2}u_i(k - D), \\
v_i(k+1) &= v_i(k) + Tu_i(k - D),
\end{align*}
\]

where the integer \( D > 0 \) denotes the input delay of the system (23).

The system (23) is said to be achieving a dynamical average consensus asymptotically, if the agents’ states satisfy

\[
\lim_{k \to \infty} x_i(k) = c, \quad \lim_{k \to \infty} v_i(k) = 0, \forall i \in \mathcal{I},
\]

where \( c \) is a constant. Moreover, if

\[
\lim_{k \to \infty} x_i(k) = \frac{1}{n} \left( \sum_{i \in \mathcal{I}} x_i(0) - T \sum_{i \in \mathcal{I}} v_i(0) \right), \quad \lim_{k \to \infty} v_i(k) = \frac{1}{n} \sum_{i \in \mathcal{I}} v_i(0), \forall i \in \mathcal{I},
\]

the system (23) achieves a dynamical average consensus asymptotically.

For the system (23), we adopt the same consensus algorithm analogous to (5)

\[
\tag{24} u_i(k) = -\kappa v_i(k) - \sum_{j \in \mathcal{N}_i} a_{ij}(x_i(k) - x_j(k)) - \gamma \sum_{j \in \mathcal{N}_i} a_{ij}(v_i(k) - v_j(k)),
\]

where \( \kappa > 0 \) and \( \gamma > 0 \), \( a_{ij} > 0 \), \( j \in \mathcal{N}_i \).

**Remark 4.** For the importance of sampled-data control and input delays, much attention has been paid on the first-order sampled-data multi-agent systems under input delays \([8, 20, 21, 22]\). The consensus problem of the second-order sampled-data multi-agent systems (23) without input delay has been analyzed in \([9, 24]\). To our knowledge, however, the consensus of the system (23) with input delay has not attracted any attention, so we will analyze the consensus of the system (23) with (24) in the following.

With (24), the closed-loop form of the system (23) is given by

\[
\begin{align*}
x_i(k+1) &= x_i(k) + Tv_i(k) + \frac{T^2}{2}(-\kappa v_i(k - D) - \sum_{j \in \mathcal{N}_i} a_{ij}(x_i(k - D) - x_j(k - D))) \\
&\quad - \gamma \sum_{j \in \mathcal{N}_i} a_{ij}(v_i(k - D) - v_j(k - D)), \\
v_i(k+1) &= v_i(k) + T(-\kappa v_i(k - D) - \sum_{j \in \mathcal{N}_i} a_{ij}(x_i(k - D) - x_j(k - D)) \\
&\quad - \gamma \sum_{j \in \mathcal{N}_i} a_{ij}(v_i(k - D) - v_j(k - D)).
\end{align*}
\]

**Proposition 2.** Consider the network of \( n \) agents (25) with a static interconnection topology \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) that is undirected and connected, and the topology graph has symmetric weights. Assume \( \frac{2(\kappa + \gamma \lambda_{\text{max}})}{\lambda_{\text{max}} T} > 1 \), where \( \lambda_{\text{max}} \) is the largest
eigenvalue of $L$. When $2D + 1 < \frac{2(k+\lambda_{\max})}{\lambda_{\max} T}$, the system (25) asymptotically converges to a stationary consensus, if and only if

$$\sqrt{(\frac{\lambda_i T^2}{4 \sin \frac{\omega}{2} \tan \frac{\omega}{2}})^2 + (\frac{\kappa + T \gamma \lambda_i}{2 \sin \frac{\omega}{2}})^2} < 1, \; i \in I,$$

where $\lambda_i$ is the eigenvalue of $L$, and $\omega_{ci}$ satisfies

$$\omega_{ci} = \frac{\pi}{2D + 1}, \; \lambda_i = 0;$$

$$\omega_{ci} D + \omega_{ci} = \arctan(\frac{2(k + \gamma \lambda_i)}{\lambda_i T} \tan \frac{\omega_{ci}}{2}), 0 < \omega_{ci} < \pi, \; \lambda_i \neq 0.$$

When $\frac{2(k+\gamma \lambda_{\max})}{\lambda_{\max} T} \leq 1$ or $2D + 1 \geq \frac{2(k+\gamma \lambda_{\max})}{\lambda_{\max} T}$ with $\frac{2(k+\gamma \lambda_{\max})}{\lambda_{\max} T} > 1$, the system (25) diverges.

**Proof.** Rewrite the system (25) as

$$x(k + 1) = x(k) + T v(k) + \frac{T^2}{2} (-\kappa v(k - D) - Lx(k - D) - \gamma L v(k - D)),$$

$$v(k + 1) = v(k) + T (-\kappa v(k - D) - Lx(k - D) - \gamma L v(k - D)),$$

where $x = [x_1, \ldots, x_n]^T$ and $v = [v_1, \ldots, v_n]^T$. Taking the $z$ transform of the system (28), we get that the characteristic equation about $x(k)$ is given by:

$$\det((z - 1)^2 I + (T \kappa I + T \gamma L)(z - 1)^{-D} + \frac{T^2}{2} (z + 1)^{-D} L) = 0.$$

The following proof is similar to that of Proposition 1.

Since the interconnection topology $G$ is undirected and connected, 0 is a simple eigenvalue of $L$ [23], and the other eigenvalues are all positive numbers for the symmetric coupling weights. The eigenvalues of $L$ are $\lambda_i, \; i = 1, \ldots, n$, and we assume $\lambda_1 = 0$ and $\lambda_i > 0, \; i \geq 2$. Thus, (29) becomes

$$\prod_{i=1}^{n} ((z - 1)^2 + (T \kappa + T \gamma \lambda_i)z^{-D}(z - 1) + \frac{T^2}{2} (z + 1)z^{-D} \lambda_i) = 0.$$

Then, we investigate the roots of the following equation for $\lambda_i, \; i \in I.$

$$\prod_{i=1}^{n} ((z - 1)^2 + (T \kappa + T \gamma \lambda_i)z^{-D}(z - 1) + \frac{T^2}{2} (z + 1)z^{-D} \lambda_i) = 0.$$

Then, we investigate the roots of the following equation for $\lambda_i, \; i \in I.$

$$\prod_{i=1}^{n} ((z - 1)^2 + (T \kappa + T \gamma \lambda_i)z^{-D}(z - 1) + \frac{T^2}{2} (z + 1)z^{-D} \lambda_i) = 0.$$

**Case 1:** $2D + 1 < \frac{2(k+\gamma \lambda_{\max})}{\lambda_{\max} T}$ with $\frac{2(k+\gamma \lambda_{\max})}{\lambda_{\max} T} > 1.$

When $\lambda_1 = 0$, (30) becomes

$$(z - 1)((z - 1) + T \kappa z^{-D}) = 0.$$

Obviously, (31) has a simple root at $z = 1$. When $z \neq 1$, we consider the roots of following equation

$$1 + T \kappa \frac{z^{-D}}{z - 1} = 0.$$

When $\omega \in (0, \pi], \; |T \kappa e^{\frac{-j \omega}{\omega D}}| = \frac{T \kappa}{2 \sin \frac{\omega}{2} D}$ and $\arg(T \kappa e^{\frac{-j \omega}{\omega D}}) = -\omega D - \frac{\omega}{2} - \frac{\pi}{2}$ are all monotonously decreasing, and $T \kappa e^{\frac{-j \omega}{\omega D}}$ crosses the negative real axis for the first time at $\omega_{c1} = \frac{\pi}{2D}$. Thus, based on the Nyquist stability criterion, the roots of (32) have modulus less than unity, if and only if $\frac{T \kappa}{2 \sin \frac{\omega}{2} D} < 1$, which holds for (26).
When $\lambda_i > 0$, (30) can be written as
\begin{equation}
1 + \frac{(T\kappa + T\gamma \lambda_i)(z - 1) + \frac{T^2}{2} \lambda_i(z + 1)}{(z - 1)^2} z^{-D} = 0.
\end{equation}

Defining

$$
\tilde{g}_i(\omega) = \frac{(T\kappa + T\gamma \lambda_i)(e^{j\omega} - 1) + \frac{T^2}{2} \lambda_i(e^{j\omega} + 1)}{(e^{j\omega} - 1)^2} e^{-j\omega D},
$$
we obtain $|\tilde{g}_i(\omega)| = \sqrt{\left(\frac{\lambda_i T^2}{2 \sin \frac{T}{2}}\right)^2 + \left(\frac{T\kappa + T\gamma \lambda_i}{2 \sin \frac{T}{2}}\right)^2}$ and $\arg(\tilde{g}_i(\omega)) = -\omega D - \frac{\omega}{2} - \pi + \arctan\left(\frac{2(\kappa + \gamma \lambda_i)}{\lambda_i T}\right) \tan \frac{T}{2}$. Obviously, $|\tilde{g}_i(\omega)|$ is monotonously decreasing for $\omega \in (0, \pi]$.

When $2D + 1 < \frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T}$, $\tilde{g}_i(\omega)$ crosses the real axis for the first time at $\omega_{ci}$ defined as (27b), and $\arg(\tilde{g}_i(\omega)) < -\pi$ is monotonously decreasing for $\omega \in (\omega_{ci}, \pi]$ from Lemma 2. Thus, the roots of (33) all have modulus less than unity, if and only if $|\tilde{g}_i(\omega_{ci})| < 1$, i.e., (26) holds.

As proved above, the roots of (29) have modulus less than unity except for a root at $z = 1$. Therefore, the state $x_i(k)$ of the system (25) converges to a steady state, i.e., $\lim_{k \to \infty} x_i(k) = x^*_i, \forall i \in I$, and $\lim_{k \to \infty} v_i(k) = 0, \forall i \in I$ holds for (25). Therefore, similar to the proof of Proposition 1, the system (25) converges to a stationary consensus for the connected interconnection topology.

Case 2: $\frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T} \leq 1$ or $2D + 1 \geq \frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T}$ with $\frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T} > 1$.

In this case, $\arg(\tilde{g}_i(\omega)) < -\pi$ with $\lambda_i = \lambda_{\max}$ holds for $\omega \in (0, \pi]$ from Lemma 2. Thus, based on the Nyquist stability criterion, the equation (30) with $\lambda_i = \lambda_{\max}$ has at least one root whose modulus is larger than unity, i.e., the system (25) diverges.

Proposition 2 is proved. $\square$

**Remark 5.** When $2D + 1 < \frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T}$ with $\frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T} > 1$, where $\lambda_{\max}$ is the eigenvalue of $L$, similar to the comments in Remark 2, $D$ decreases when $\omega_{c1}$ increases based on the conditions (27a) and (27b), and the condition (26) determines the least value of $\omega_{c1}$. Thus, the conditions (26), (27a) and (27b) equal
\begin{equation}
2D + 1 < \frac{\arctan\left(\frac{2(\kappa + \gamma \lambda_i)}{\lambda_i T} \tan(\xi(\lambda_i))\right)}{\xi(\lambda_i)},
\end{equation}
where
\begin{equation}
\xi(\lambda_i) = \arcsin\left(\frac{T}{2} \sqrt{\frac{(\kappa + \gamma \lambda_i)^2}{8} - \frac{\lambda_i^2 T^2}{8}} + \sqrt{\left(\frac{(\kappa + \gamma \lambda_i)^2}{2} - \frac{\lambda_i^2 T^2}{8} + \lambda_i^2\right)^2 - \lambda_i^2}\right).
\end{equation}

When $\frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T} > 1$, $\xi(\lambda_i)$ is monotonously increasing for $\lambda_i > 0$.

According to Remark 5 and Lemma 4, we get the main result for the system (25) as follows.

**Theorem 2.** Consider the network of $n$ agents (25) with a static interconnection topology $G = (V, E, A)$ that is undirected and connected, and the topology graph has symmetric weights. The system (25) asymptotically converges to a stationary consensus, if and only if
\begin{equation}
2D + 1 < \frac{\arctan\left(\frac{2(\kappa + \gamma \lambda_{\max})}{\lambda_{\max} T} \tan(\zeta)\right)}{\zeta}
\end{equation}
holds with $T < \frac{2(\kappa + \gamma \lambda_{\text{max}})}{\lambda_{\text{max}}}$, where $\lambda_{\text{max}}$ is the largest eigenvalue of the Laplacian matrix $L$ and

$$
\zeta = \arcsin \left( \frac{T}{2} \sqrt{\frac{(\kappa + \gamma \lambda_{\text{max}})^2}{2} - \frac{\lambda_{\text{max}}^2 T^2}{8}} + \sqrt{\left( \frac{(\kappa + \gamma \lambda_{\text{max}})^2}{2} - \frac{\lambda_{\text{max}}^2 T^2}{8} \right)^2 + \lambda_{\text{max}}^2} \right).
$$

**Remark 6.** Different from the continuous-time system, the consensus convergence of the sampled-data system with input delay depends on not only the control parameters, the eigenvalues of the Laplacian matrix and the input delay, but also the sampling interval. When the sampling interval crosses the critical value, the agents’ states diverge without any relationship to the input delay.

Then, we consider the dynamical consensus for the second-order multi-agent systems (23) under the consensus algorithm similar to (17) as follows

$$
u_i(k) = -\sum_{j \in N_i} a_{ij} (x_i(k) - x_j(k)) - \gamma \sum_{j \in N_i} a_{ij} (v_i(k) - v_j(k)),
$$

where $\gamma > 0, a_{ij} > 0, j \in N_i$. With (35), the closed-loop form of the system (23) is given by

$$
x_i(k+1) = x_i(k) + T v_i(k) + \frac{T^2}{2} \left( -\sum_{j \in N_i} a_{ij} (x_i(k-D) - x_j(k-D)) \right) - \gamma \sum_{j \in N_i} a_{ij} (v_i(k-D) - v_j(k-D)),
$$

$$
v_i(k+1) = v_i(k) + T \left( -\sum_{j \in N_i} a_{ij} (x_i(k-D) - x_j(k-D)) \right) - \gamma \sum_{j \in N_i} a_{ij} (v_i(k-D) - v_j(k-D)).
$$

**Corollary 3.** Consider the network of $n$ agents (36) with a static interconnection topology $G = (V, E, A)$ that is undirected and connected, and the topology graph has symmetric weights. The system (36) asymptotically converges to a dynamical average consensus, if and only if

$$
2D + 1 < \frac{\arctan \left( \frac{2\gamma}{T} \tan(\zeta) \right)}{\zeta}
$$

holds with $T < 2\gamma$, where

$$
\zeta = \arcsin \left( \frac{T}{2} \sqrt{\frac{(\gamma \lambda_{\text{max}})^2}{2} - \frac{\lambda_{\text{max}}^2 T^2}{8}} + \sqrt{\left( \frac{(\gamma \lambda_{\text{max}})^2}{2} - \frac{\lambda_{\text{max}}^2 T^2}{8} \right)^2 + \lambda_{\text{max}}^2} \right),
$$

and $\lambda_{\text{max}}$ is the maximum eigenvalue of the Laplacian matrix $L$.

For the system (23), we adopt the same consensus algorithm as (20)

$$
u_i(k) = -\kappa v_i(k) - \sum_{j \in N_i} a_{ij} (x_i(k) - x_j(k)),
$$
where $\kappa > 0$, $a_{ij} > 0$, $j \in N_i$. With (37), the closed-loop form of the system (23) is given by

$$x_i(k + 1) = x_i(k) + Tu_i(k) + \frac{T^2}{2}(-\kappa v_i(k - D)) - \sum_{j \in N_i} a_{ij}(x_i(k - D) - x_j(k - D)))$$

$$v_i(k + 1) = v_i(k) + T(-\kappa v_i(k - D)) - \sum_{j \in N_i} a_{ij}(x_i(k - D) - x_j(k - D))).$$

**Corollary 4.** Consider the network of $n$ agents (38) with a static interconnection topology $G = (V, E, A)$ that is undirected and connected, and the topology graph has symmetric weights. The system (38) asymptotically converges to a stationary consensus, if and only if

$$2D + 1 < \frac{\arctan(\frac{2\kappa}{\lambda_{\max}T} \tan(\zeta))}{\zeta}$$

holds with $T < \frac{2\kappa}{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of the Laplacian matrix $L$ and

$$\zeta = \arcsin\left(\frac{T}{2} \sqrt{-\frac{\kappa^2}{2} + \frac{\lambda_{\max}^2T^2}{8}} + \sqrt{\left(\frac{\kappa^2}{2} - \frac{\lambda_{\max}^2T^2}{8}\right)^2 + \lambda_{\max}^2}\right).$$

**5. Simulation**

To illustrate the correctness of the our comments, we consider the consensus problem of the continuous-time and the sampled-data multi-agent systems respectively in the following simulations.

**Example 1.** Continuous-time system. Consider a network of six agents described by (6), and the interconnection topology is described in Figure 1. The graph is undirected and connected. The symmetric weights of the edges are: $a_{12} = a_{21} = 0.1$, $a_{23} = a_{32} = 0.4$, $a_{26} = a_{62} = 0.3$, $a_{34} = a_{43} = 0.9$, $a_{36} = a_{63} = 0.7$, $a_{45} = a_{54} = 0.5$. The eigenvalues of $L$ are: $\lambda_1 = 0$, $\lambda_2 = 0.104$, $\lambda_3 = 0.3737$, $\lambda_4 = 1.0948$, $\lambda_5 = 1.3408$, $\lambda_6 = 2.8868$. Choose the control parameters $\kappa = 2.0$ and $\gamma = 0.5$, and we get $\tau < 0.3782(s)$ from (15) in Theorem 1. When $\tau < 0.3782(s)$, the system (6) converges to a stationary consensus (see, Figure 2). When $\tau \geq 0.3782(s)$, the agents’ states oscillate with $\tau = 0.3782(s)$ (see, Figure 3) or diverge with $\tau > 0.3782(s)$ (see, Figure 4), and no consensus can be achieved.
Figure 2. Continuous-time system: stationary consensus.

Figure 3. Continuous-time system: oscillations of the agents’ states.

Figure 4. Continuous-time system: divergence of the agents’ states.
Then, we investigate the dynamical consensus for the continuous-time system (4). Consider a network of six agents described by (18). For simplicity, we choose the same interconnection topology and linked weights as above described in Figure 1. Choosing the control parameter $\gamma = 0.8$, we get $\tau < 0.435(s)$ from (19) in Corollary 1. When $\tau < 0.435(s)$, the system converges to a dynamical average consensus (see Figure 5), and the final consensus states of the agents are consistent with Corollary 1. When $\tau \geq 0.435(s)$, the agents’ states oscillate or diverge.

In the same way, Corollary 2 for the continuous-time second-order multi-agent systems with another stationary consensus algorithm (21) can also be proved by the simulations, but the simulations are omitted here for the length of the paper.

**Example 2.** Sampled-data system. Consider a network of six agents described by (25). The interconnection topology is described in Figure 6. The graph is undirected and connected. The symmetric weights of the edges are: $a_{12} = a_{21} = 0.1$, $a_{16} = a_{61} = 0.3$, $a_{23} = a_{32} = 0.4$, $a_{34} = a_{43} = 0.9$, $a_{36} = a_{63} = 0.7$, $a_{45} = a_{54} = 0.5$. The eigenvalues of $L$ are: $\lambda_1 = 0$, $\lambda_2 = 0.261$, $\lambda_3 = 0.4889$, $\lambda_4 = 0.7673$, $\lambda_5 = 1.3757$, $\lambda_6 = 2.9072$. Choose the control parameters $\kappa = 0.3$ and $\gamma = 0.5$, and the system (25) diverges when $T \geq \frac{2(\kappa + \lambda_{\text{max}})}{\lambda_{\text{max}}} \simeq 1.2064(s)$ from Theorem 2 (See Figure 7). Thus, we choose the sampling interval $T = 0.05(s)$. With $T = 0.05(s)$, we obtain $D \leq 7$ from (34), and the system (25) asymptotically converges to a stationary consensus (See Figure 8). When $D \geq 8$, the system diverges.

Moreover, simulations can also demonstrate the correctness of Corollary 3 and Corollary 4, but we don’t illustrate the simulations here for the length of the paper.
6. Conclusion

In this paper, the consensus problem of second-order multi-agent systems with input delay is investigated based on the symmetric interconnection topology. For the continuous-time system, sufficient and necessary consensus conditions, which depend on the input delay, are obtained for the system asymptotically converging to the stationary consensus and the dynamical average consensus respectively. Then, sufficient and necessary consensus conditions, which depend on the input delay and sampling interval, are also obtained for the sampled-data system converging to the stationary consensus and the dynamical average consensus respectively. When the sampling interval is larger than the critical value, the sampled-data system diverges without any relationship to the input delay.

References


Institute of Automation, Jiangnan University, Wuxi 214122, China
E-mail: liucl@jiangnan.edu.cn and fliu@jiangnan.edu.cn