A NEW APPROACH TO UNIFORM PRACTICAL STABILITY OF DESCRIPTOR SYSTEMS WITH INFINITE TIME DELAYS IN TERMS OF TWO MEASUREMENTS

ZHAN SU\textsuperscript{1}, QINGLING ZHANG\textsuperscript{1}, JUN AI\textsuperscript{2}, AND CHUNYU YANG\textsuperscript{3}

(Communicated by Qingling Zhang)

Abstract. This paper introduces a new approach to the uniform practical stability theory of descriptor systems with infinite time delays in terms of two measurements. An easy-to-test criterion of uniform practical stability is given via Lyapunov functionals and Razumikhin technique. Finally, the application of the results obtained is demonstrated through an analysis of a class of descriptor systems with infinite time delays to illustrate the advantage of the proposed results.

Key Words. Descriptor systems, infinite time delay, uniform practical stability, Razumikhin technique, Lyapunov functions

1. Introduction

In the past two decades, as descriptor system models have come to play an important role in many branches of science and engineering, and due to their comprehensive applications in Leontief dynamic model, electrical model and mechanical models, etc, significant advances have been made in fields of control theory, see [1, 2, 3, 4]. In the meantime, under closer scrutiny, it becomes that a more realistic model would include some of the past states of the system in many applications. And considerable attention has been paid to the study of descriptor systems with time delays [5, 6].

Being much different from stability in the sense of Lyapunov, practical stability is a significant performance specification from engineering point of view, which can be acceptable in many applications for quality analysis. Since it was first introduced by Lasalle in [7], practical stability attracts much attention of many authors. Much works on practical stability have been studied extensively and found many application in different areas [8, 9, 10, 11, 12, 13]. In practice, a system is actually unstable, which might be stable or asymptotically stable in theory, because the stable domain or the domain of the desired attractor is not large enough; or sometimes, the desired state of a system may be mathematically unstable, yet the system may oscillate sufficiently near the state, of which the performance is acceptable, i.e., it is stable in practice.

More recently, the practical stability results of standard state-space systems have been extended to linear descriptor systems. Some sufficient conditions were derived via so-called Lyapunov-like approach and Bellman-Gronwall approach. However, it
is unfortunately that these results cannot be applied to nonlinear descriptor systems with infinite time-delay. Moreover, it is sometime difficult to construct Lyapunov functions of all components of the state variables for such systems. On the other hand, the Razumikhin techniques have been employed to make the improvements. Yet, most of the known examples to demonstrate the method of Lyapunov-Razumikhin functionals are of scalar equations even in finite delay case. It reveals the disadvantage and restricts the applications of Razumikhin techniques to systems of infinite delay differential equations by employing one function containing all components of state variables.

Motivated by the idea of [14], we improve the results of [10] to uniform practical stability of descriptor systems with infinite delays in terms of two measurements. By using a new technique in investigating the uniform practical stability of descriptor systems with infinite delays in terms of two measurements, the Lyapunov functions rather than Lyapunov functionals are adopted via dividing the components of variables into several groups. In this way, construction of the suitable Lyapunov functions is much easier and the imposed conditions guarantee the required practical stability are less restrictive.

This paper is organized as follows. In Section 2, some definitions and preliminaries are introduced. In Section 3, a criteria for uniform practical stability of descriptor systems with infinite delay in two measurements are derived via Lyapunov functions and Razumikhin technique. In Section 4, a class of descriptor system with infinite delay is presented to illustrate the effectiveness of the new approach. The last section summarizes the results.

2. Preliminaries

Consider the following descriptor system with infinite time delay

\[ E \dot{x}(t) = f(t, x(t), x(s)), \quad Ex(t_0) = E\phi, \quad Ex(s) = \phi_s(t), \quad \alpha \leq s \leq t, \quad t \geq t_*, \]

where \( E(t) \in \mathbb{R}^{n \times n} \) with \( \text{rank}(E) = r \leq n \), \(-\infty \leq \alpha \leq t_*\). \( \alpha \) could be \(-\infty\). Denoted by \( x(t; t_0, \phi) \), is a continuous function satisfying (1) through \((t_0, \phi)\).

Firstly, in this paper, we adopt the following notations similar to that in references [14] and [10]:

\[
\begin{align*}
\mathbb{R}^+_t &= \{ t \in \mathbb{R} | t \geq 0 \}; \\
\mathbb{R}^+_t^n &= \{ t \in \mathbb{R} | t \geq \alpha \}; \\
\Gamma^n_\alpha &= \{ h \in C(\mathbb{R}^+_{t_0} \times \mathbb{R}^n, \mathbb{R}^+) | \forall t \geq t_*, \forall x \in \mathbb{R}^n, \inf_{x \in \mathbb{R}^n} h(t, x) = 0 \}; \\
\Gamma^n_\alpha &\subset C(\mathbb{R}^+_{t_0} \times \mathbb{R}^n, \mathbb{R}^+) | \forall t \in \mathbb{R}^+_{t_0}, \inf_{x \in \mathbb{R}^n} h_0(t, x) = 0 \}; \\
C^n_0(t) &= \{ \varphi : [\alpha, t] \to \mathbb{R}^n | \varphi \text{ is continuous and bounded} \}; \\
S^n(h, \rho) &= \{ (t, x) \in \mathbb{R}^+_{t_0} \times \mathbb{R}^n | h(t, x) < \rho, h \in \Gamma^n_\alpha, \rho > 0 \}.
\end{align*}
\]

Let \( h_0 \in \Gamma^n_\alpha, \varphi \in C^n_0(t), \forall t_0 \geq t_*, \) for any \( t \in \mathbb{R}^+_{t_0} \), define

\[ h_0(t, \varphi) = \sup_{\alpha \leq s \leq t} h_0(s, \varphi). \]

Then for any \( \rho > 0 \) and \( t \geq t_* \), denote \( C^n_\rho(t) = \{ \varphi \in C^n_0(t) | h_0(t, \varphi) < \rho \} \).

We assume that \( f(t, x(t), \varphi) \) is defined on \([\alpha, \infty) \times \mathbb{R}^n \times C^n_0(t) \) for some \( H > 0 \), \( f(t, 0, 0) \equiv 0, \ t \geq t_0 \geq t_* \). With the following assumption, every neighborhood of the origin has at least one consistent initial condition \( \varphi \) at \( t_0 \), through which there is a solution \( x(t; t_0, \varphi) \).
Theorem 3.1. \( \Gamma = \Gamma \) and define
\[ C(n) = \{ (t, \varphi) \in \mathbb{R}_+^n \times \mathbb{R}_+^m \mid h(t, \varphi) < \rho \}. \]

In what follows, we adopt the same notations and the process as [14]. We will split
\[ \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T = C_n(t) \] into several vectors, that is, \( \tilde{\varphi} = (\varphi_1, \varphi_2)^T, \)
\[ \tilde{\varphi}_2 = (\varphi_{n_1+1}, \ldots, \varphi_{n_1+n_2})^T, \ldots, \tilde{\varphi}_m = (\varphi_{n_1+\ldots+n_{m-1}+1}, \ldots, \varphi_{n_1+\ldots+n_m})^T, \]
\[ \sum_{i=1}^m n_i = n, \] and \( \varphi = (\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_m)^T. \) For \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n, \) we adopt the similar notations as for \( \varphi \in C_n(t). \) For convenience, for the sake of brevity, we denote \( \Gamma = \Gamma^n, \Gamma_j = \Gamma_n^j, \Gamma_y = \Gamma_n^y, \Gamma_{\alpha} = \Gamma_n^\alpha, \Gamma_{\alpha}^j = \Gamma_n^{\alpha_j}, C(t) = C_n(t), \) \( S(h, \rho) = S_n(h, \rho) = S_n^0(t_0), S(h, \rho) = S_n^0(h, \rho). \) Let \( j = (\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_m)^T. \) For any \( t \in \mathbb{R}_+^n, \) define
\[ h_j(t, j) = \sup_{\alpha \leq s \leq t} h_j(s, j), \quad j = 1, \ldots, m. \]

We introduce some definitions as follow.

**Definition 2.1.** [14] A continuous function \( W : \mathbb{R}^+ \to \mathbb{R}^+ \) is called a wedge function if \( W(0) = 0 \) and \( W(s) \) is (strictly) increasing, and denoted by \( K \) the space of all wedge functions.

**Definition 2.2.** [10, 15] Let \( h_0 \in \Gamma_n, h \in \Gamma, \phi \in S_k(t_0) \) and \( h \) is defined as (2). Then system (1) is called to be

- \( PS_1 : (h_0, h) \)-practically stable (P.S.) if for given \( (\lambda, A) \) with \( 0 < \lambda < A \) and some \( t_0 \geq t_\ast, \) we have \( h_0(t_0, E\phi) < \lambda \) implies \( h(t, E\phi) < A, t \geq t_0; \)

- \( PS_2 : (h_0, h) \)-uniformly practically stable (U.P.S.) if \( PS \) holds for all \( t_0 \geq t_\ast; \)

As stated in [10], by appropriate choices of the two measurements, Definition 2.2 reduces to the existing concepts of practical stability in [11], thus, Definition 2.2 is more general. In this paper, we mainly discuss the \( (h_0, h) \)-U.P.S. of the system (1) via Lyapunov functions and Razumikhin-type technique.

The definition of the upper right-hand Dini derivative of an Lyapunov function \( V(t, x) \in C(\mathbb{R}_+^n \times \mathbb{R}^m, \mathbb{R}^+). \) is expressed as
\[ D^+V(t, x) = \lim_{\delta \to 0^+} \sup_{\delta \to 0^+} \{ \delta^{-1}[V(t + \delta, x(t + \delta)) - V(t, x(t))] \}. \]

**Lemma 2.3.** [10] Let \( y = Ex \) and Lyapunov function \( V(t, y) \in C(\mathbb{R}_+^n \times \mathbb{R}^m, \mathbb{R}^+) \) be locally Lipschitzian in \( y, \) then \( D^+V(t, y(t)) \) along the solution of system (1) is given by
\[ D^+V(t, y(t)) = \lim_{\delta \to 0^+} \sup_{\delta \to 0^+} \{ \delta^{-1}[V(t + \delta, y(t) + \delta y) - V(t, y(t))] \}. \]

According to the Lemma 2.3, split \( y = Ex \) into \( m \) part, i.e., \( y = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m)^T, \) we define
\[ D^+V_j(t, \tilde{y}_j) = \lim_{\delta \to 0^+} \sup_{\delta \to 0^+} \{ \delta^{-1}[V_j(t + \delta, \tilde{y}_j + \delta \tilde{y}_j) - V_j(t, \tilde{y}_j)] \}, \quad j = 1, \ldots, m. \]

3. Main results

For simplicity, we start with the case of \( m = 2, \) and establish the following criterion on the U.P.S.. In this section, let \( y = Ex \) for the following theorems.

**Theorem 3.1.** Given \( (\lambda, A) \) with \( 0 < \lambda < A \) and \( h \in \Gamma, h_0 \) defined by Eq.(3) are given, \( x(t) = (\tilde{x}_1(t), \tilde{x}_2(t)) \) is a solution of (1). Let \( W_{ij}(i = 1, 2, 3, 4; j = 1, 2) \in K, \Phi_j(j = 1, 2) \in C(\mathbb{R}^+, \mathbb{R}^+), \Phi_j \in L^1[0, +\infty), \Phi_j(t) \leq K_j \) for \( t \geq 0 \) with some
constant $K_j(j = 1, 2) > 0$, $J_j = \int_0^\infty \Phi_j(s)ds(j = 1, 2)$. If there exist continuous functions $V_j(j = 1, 2) : [\alpha, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ and $\psi_j(j = 1, 2) \in \mathcal{K}$ such that

(i) $h_j \leq \psi_j(h_{j0})$, $\psi_j(\lambda) < A$, for all $(t, x) \in \mathbb{R}^+_n \times S(h_{j0}, \lambda),$

(ii) $W_{2j}(\lambda) + W_{3j}(J_jW_{4j}(\lambda)) < a^*$, where $a^* = \min_{j=1,2} \{W_{1j}(A)\},$

(iii) for all $(t, x) \in \mathbb{R}^+_n \times S^j(h_j, A)$, $j = 1, 2$, 

\[ W_{1j}(h_j(t, \tilde{y}_j(t))) \leq V_j(t, \tilde{y}_j(\cdot)) \leq W_{2j}(h_{j0}(t, \tilde{y}_j(\cdot))) \]

\[ + W_{3j} \left[ \int_0^t \Phi_j(t - s)W_{4j}(h_{j0}(s, \tilde{y}_j(\cdot))) ds \right], \]

(iv) when $V_1(t, \tilde{y}_1(\cdot)) \geq V_2(t, \tilde{y}_2(\cdot))$, for all $(t, x) \in \mathbb{R}^+_n \times S^{(1)}(h_1, A)$, $D^+V_1(t, \tilde{y}_1(\cdot)) \leq 0$, provided $V_1(s, \tilde{y}_1(\cdot)) \leq V_1(t, \tilde{y}_1(\cdot))$, $\forall s \in [\alpha, t]$; when $V_1(t, \tilde{y}_1(\cdot)) \leq V_2(t, \tilde{y}_2(\cdot))$, for all $(t, x) \in \mathbb{R}^+_n \times S^{(2)}(h_2, A)$, $D^+V_2(t, \tilde{y}_2(\cdot)) \leq 0$, provided $V_2(s, \tilde{y}_2(\cdot)) \leq V_2(t, \tilde{y}_2(\cdot))$, $\forall s \in [\alpha, t]$.

Then the zero solution of (1) is $(h_0, h)$-U.P.S.

**Proof.** For all $t \in \mathbb{R}^+_n$, for $x(t) = (\tilde{y}_1(t), \tilde{y}_2(t))$, we define

\[ h(t, x) = \max_{j=1,2} \{h_{j0}(t, \tilde{y}_j(\cdot))\}, \quad h_0(t, x) = \max_{j=1,2} \{h_{j0}(t, \tilde{y}_j(\cdot))\}, \]

and define a function

\[ V(t) = \max_{j=1,2} \{V_j(t, \tilde{y}_j(\cdot))\}. \]

Hence, following the definitions of $h$, $h_0$ in (7) and $V(t)$ in (8), one has

\[ h(t, y) < A \Leftrightarrow h_j(t, \tilde{y}_j) (j = 1, 2) < A, \]

\[ h_0(t, y) < \lambda \Leftrightarrow h_{j0}(t, \tilde{y}_j) (j = 1, 2) < \lambda, \]

and $V(t)$ is continuous for all $(t, x) \in \mathbb{R}^+_n \times S(h, A)$.

In the following, for the sake of brevity, for $j = 1, 2$, we denote,

\[ h_j(t) = h_j(t, \tilde{y}_j(t)), \quad V_j(t) = V_j(t, \tilde{y}_j(\cdot)), \]

\[ h_{j0}(t) = h_{j0}(t, \tilde{y}_j(\cdot)), \quad D^+V_j(t) = D^+V_j(t, \tilde{y}_j(\cdot)), \]

\[ h_{j0}(t) = h_{j0}(t, \tilde{y}_j(\cdot)). \]

Firstly, we claim that for any $(t, x) \in \mathbb{R}^+_n \times S(h, A)$,

\[ \max_{j=1,2} \{W_{1j}(h_j(t))\} \leq V(t) \leq \max_{j=1,2} \left\{ W_{2j}(h_{j0}(t)) \right\} \]

\[ + W_{3j} \left[ \int_0^t \Phi_j(t - s)W_{4j}(h_{j0}(s)) ds \right]. \quad (10) \]

In fact, when $V_1(t) \geq V_2(t)$, by (8), $V(t) = V_1(t)$. If $W_{11}(h_1(t)) \geq W_{12}(h_2(t))$, then by condition (iii), we have $V(t) = V_1(t) \geq W_{11}(h_1(t))$; while if $W_{11}(h_1(t)) \leq W_{12}(h_2(t))$, then $V(t) = V_1(t) \geq V_2(t) \geq W_{12}(h_2(t))$. Therefore,

\[ V(t) = V_1(t) \geq \max_{j=1,2} \{W_{1j}(h_j(t))\}; \]

whereas, when $V_1(t) \leq V_2(t)$, we also have the left-hand inequality in (10). On the other hand, the right-hand inequality in (10) obviously holds.
A NEW APPROACH TO UPS OF DS WITH ITD IN TERMS OF TWO MEASUREMENTS

Secondly, we show that \( \forall t_0 \geq t_* \), for each \( t \geq t_0 \), \( x \in S(h, A) \),
\[
D^+ V(t) \leq 0, \quad \text{provided } V(s) \leq V(t), \quad \alpha \leq s \leq t.
\]

In fact, suppose \( V_1(t_0) \geq V_2(t_0) \) and define \( t_1 = \inf \{ t > t_0 : V_1(t) < V_2(t) \} \), then by (8), for \( t \in [t_0, t_1) \), \( V(t) = V_1(t) \). Consider each \( s \in [\alpha, t] \). If \( V_1(s) \geq V_2(s) \), then \( V(s) = V_1(s) \), and hence, \( V(s) \leq V(t) \) implies \( V_1(s) \leq V_1(t) \). While if \( V_1(s) < V_2(s) \), then \( V(s) = V_2(s) \), and hence, \( V(s) \leq V(t) \) implies \( V_1(s) \leq V_2(s) = V(s) \leq V(t) = V_1(t) \). Therefore, by condition (iv) and (9), we have for all \( t \in [t_0, t_1) \), \( x \in S(h, A) \),
\[
D^+ V(t) = D^+ V_1(t) \leq 0, \quad \text{provided } V(s) \leq V(t), \quad \forall s \in [\alpha, t].
\]

If \( t_1 = \infty \), then (11) is proved. Otherwise, define \( t_2 = \inf \{ t > t_1 : V_1(t) > V_2(t) \} \), then for \( t \in [t_1, t_2) \), \( V(t) = V_2(t) \), and by (8), we have \( V(t) = V_2(t) \). By a similar analysis to the above, we have for all \( t \in [t_1, t_2) \), \( x \in S(h, A) \),
\[
D^+ V(t) = D^+ V_2(t) \leq 0, \quad \text{provided } V(s) \leq V(t), \quad \forall s \in [\alpha, t].
\]

If \( t_2 = \infty \), then (11) holds. Otherwise, by repeating the above arguments, we may define \( \{ t_k \} (k = 0, 1, 2, 3, \ldots) \) and there exists \( K \in \mathbb{Z}^+ \) such that \( t_K = \infty \), then for all \( t \in [t_0, \infty) = \bigcup_k [t_k, t_{k+1}) \), \( x \in S(h, A) \), (11) holds.

As for the case of \( V_1(t) \leq V_2(t) \) for \( t \in [t_0, t_1) \), it is the similar to derive (11) and thus omitted.

Next, we are in a position to show that U.P.S. of the zero solution of (1).
\( \forall t_0 \geq t_* \), by definition of \( h_0 \) in (2), for \( t \in [\alpha, t_0] \), we have \( h_0(t) = \sup_{\alpha \leq s \leq t} \{ h_0(s) \} \leq \sup_{\alpha \leq s \leq t_0} \{ h_0(s) \} = h_0(t_0) \). By the definition of wedge function and condition (i) and (ii), there exists a scalar \( \epsilon_1(\lambda, A) > 0 \) such that for all \( \epsilon_2 \in (0, \epsilon_1) \) and \( j = 1, 2, \), \( \psi_j(\lambda) < A - \epsilon_2 \); there also exist scalars \( \epsilon_3(\lambda, A) > 0 \), \( \psi_3(\lambda) < A - \epsilon_3 \), and \( \psi_4(\lambda) < A - \epsilon_4 \). Then, \( \psi_j(\lambda) < A - \epsilon_2 \) for all \( \epsilon_2 \in (0, \epsilon_1) \) and \( j = 1, 2, \), \( \psi_3(\lambda) < A - \epsilon_3 \), and \( \psi_4(\lambda) < A - \epsilon_4 \). Denote \( x(t) = x(t; t_0, \phi) \). For \( \forall t_0 \geq t_* \), \( \forall \phi \in S_k(t_0) \cap S(h_0, \lambda) \), it follows condition (i) and (9) that
\[
h_j(t, \psi_j) \leq \psi_j(h_{j0}(t_0, \psi_j)) \leq \psi_j(\lambda) < A - \delta, \quad \text{for } t \in [\alpha, t_0], \quad \forall t_0 \geq t_*.
\]

It is follows (9) that \( h(t, x(t)) < A - \delta \) for \( t \in [\alpha, t_0] \), \( \forall t_0 \geq t_* \). Then, by condition (ii), (iii) and (10), \( \forall t_0 \geq t_* \), for all \( t \in [\alpha, t_0] \),
\[
\max_{j=1,2} \{ W_j(h_j(t)) \} \leq V(t) \leq \max_{j=1,2} \left\{ W_{j2}(h_{j0}(t_0)) + W_{j3} \left( \int_{t_0}^{\infty} \Phi_j(t_0 - s) W_{j4}(h_{j0}(s)) \, ds \right) \right\} \leq \max_{j=1,2} \left\{ W_{j2}(\lambda) + W_{j3} \left( W_{j4}(\lambda) \int_{t_0}^{t_0} \Phi_j(t_0 - s) \, ds \right) \right\} \leq \mu a^*_\lambda.
\]

We claim that for all \( \forall t_0 \geq t_* \), \( t \geq t_0 \),
\[
\max_{j=1,2} \{ W_j(h_j(t)) \} \leq V(t) \leq \mu a^*_\lambda, \quad \text{and } h(t) < A - \delta.
\]

Suppose (12) is not true.

Case 1, there exist some \( t_0 \geq t_* \) and \( t_1 > t_0 \) such that \( h(t_1) \leq A - \delta \) and
\[
V(t) \leq V(t_1), \quad \text{for } t \in [\alpha, t_1], \quad V(t_1) > \mu a^*_\lambda, \quad \text{and } D^+ V(t_1) > 0.
\]
But by (11) we have $D^+V(t_1) \leq 0$. This is a contradiction.

Case 2, there exist some $t_0 \geq t_*$ and $t_1 > t_0$ such that $h(t_1) = A - \delta$ and for all $t \geq t_0$,

$$\max_{j=1,2} \{W_{1j}(h_j(t))\} \leq V(t) \leq \mu a^*_\delta.$$ 

Then we have

$$a^*_\delta \leq \max_{j=1,2} \{W_{1j}(h_j(t_1))\} \leq V(t_1) \leq \mu a^*_\delta < a^*_\delta.$$ 

This is a contradiction.

Hence, (12) is true and it follows that

$$h_0(t_0, \phi) < \lambda \quad \text{implies} \quad h(t, Ex) < A, \quad t \geq t_0, \forall t \geq t_*.$$ 

i.e., the zero solution of (1) is $(h_0, h)$-U.P.S.. $\square$

Furthermore, we may develop the ideas behind Theorems 3.1 to obtain the following more general results.

**Theorem 3.2.** Given $(\lambda, A)$ with $0 < \lambda < \Lambda$ and $h_j \in \Gamma^j$, $h_{j0}$ defined by (3) are given, $x(t) = (\bar{x}_1(t), \ldots, \bar{x}_m(t))$ is a solution of (1). Let $W_{ij}(i = 1, 2, 3, 4; j = 1, \ldots, m) \in K$, $\Phi_j(j = 1, \ldots, m) \in C(R^+, R^+)$, $\Phi_j \in L^1[0, +\infty)$, $\Phi_j(t) \leq K_j$ for $t \geq 0$ with some constant $K_j(j = 1, \ldots, m) > 0$, $J_j = \int_0^\infty \Phi_j(s)ds(j = 1, \ldots, m)$. If there exist continuous functions $V_j(j = 1, \ldots, m) : \alpha, \infty \times R^{+1} \rightarrow R^+$ and $\psi_j(j = 1, \ldots, m) \in K$ such that

1. $h_j \leq \psi_j(h_{j0})$, $\psi_j(\lambda) < A$ for all $(t, x) \in R^+ \times S(h_{j0}, \lambda)$,
2. $W_{2j}(\lambda) + W_{3j}(J_jW_{4j}(\lambda)) < a^*$, where $a^* = \min_{j=1,\ldots,m} \{W_{1j}(A)\}$,
3. for all $(t, x) \in R^+ \times S^j(h_j, A)$, $j = 1, \ldots, m$,
   $$W_{1j}(h_j(t, \bar{y}_j(t))) \leq V_j(t, \bar{y}_j(\cdot)) \leq W_{2j}(h_{j0}(t, \bar{y}_j(\cdot)))$$
   $$+W_{3j} \left[ \int_0^t \Phi_j(t-s)W_{4j}(h_{j0}(s, \bar{y}_j(\cdot))) ds \right],$$
4. when $V_j(t, \bar{y}_j(\cdot)) \geq V_k(t, \bar{y}_k(\cdot))$, for all $j \neq k, j, k \in \{1, \ldots, m\}$,
   $$D^+V_j(t, \bar{y}_j(\cdot)) \leq 0, \quad (t, x) \in R^+ \times S^j(h_j, A),$$
   provided $V_j(s, \bar{y}_j(\cdot)) \leq V_j(t, \bar{y}_j(\cdot))$, $\forall s \in [\alpha, t]$.

Then the zero solution of (1) is $(h_0, h)$-U.P.S..

We only need to mention two points in the proof of Theorem 3.2, the rest is the same as in the proof of Theorem 3.1, and thus, we omitted.

First, for $x(t) = (\bar{x}_1(t), \ldots, \bar{x}_m)$, we define

$$V(t) = \max_{j=1,\ldots,m} \{V_j(t, \bar{y}_j)\}.$$ 

Second, instead of (10), we can claim that for all $t \geq \alpha$,

$$\max_{j=1,\ldots,m} \{W_{1j}(h_j(t))\} \leq V(t) \leq \max_{j=1,\ldots,m} \{W_{2j}(h_{j0}(t)) + W_{3j} \left[ \int_0^t \Phi_j(t-s)W_{4j}(h_{j0}(s)) ds \right] \}.$$
4. Application

To illustrate the applications of the obtained results, we consider a class of descriptor systems with infinite time delays as follows:

\[
\begin{aligned}
\dot{x}_1(t) &= -a_1(t)x_1^3(t) + a_2(t)x_1(t)x_2^2(t) + a_3(t)x_1(t)x_3(t) \\
&\quad + d_1(t)x_1(t)x_2^2(t - \tau_1(t)) + \int_{-\infty}^{0} g_1(t, \mu, x_1(t + \mu))d\mu, \\
\dot{x}_2(t) &= b_1(t)x_2^3(t)x_2(t) - b_2(t)x_2^3(t) + b_3(t)x_2(t)x_3(t) \\
&\quad + d_2(t)x_2^3(t)x_2(t - \tau_2(t)) + \int_{-\infty}^{0} g_2(t, \mu, x_2(t + \mu))d\mu, \\
0 &= c_1(t)x_1^2(t) + c_2(t)x_2^2(t) + c_3(t)x_3(t).
\end{aligned}
\]

where \(a_i(t), b_i(t), c_i(t), d_j(t), \tau_j(t), \) and \(g_j(t, \mu, x_j), \) \((i = 1, 2, 3; j = 1, 2)\) are all continuous functions, \(c_3(t) \neq 0.\) Assume that \(g_j(t, \mu, x_j)\) satisfies certain conditions to guarantee the global existence and uniqueness of solutions. Let \(x(t) = (x_1(t), x_2(t), x_3(t))^T,\) where \(x_1(t), x_2(t)\) are state variables and \(x_3(t)\) is algebraic variable. Then, above systems can be written in the form of (1) as follows:

\[
E\dot{x}(t) = F_1(t, x) + F_2(t, x_r) + F_3 \left( \int_{-\infty}^{0} g(t, \mu, x(t + \mu))d\mu \right),
\]

where

\[
E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2(t, x_r) = \begin{pmatrix} d_1(t)x_1(t)x_2^2(t - \tau_1(t)) \\ d_2(t)x_2^3(t)x_2(t - \tau_2(t)) \\ 0 \end{pmatrix},
\]

\[
F_1(t, x) = \begin{pmatrix} -a_1(t)x_1^3(t) + a_2(t)x_1(t)x_2^2(t) + a_3(t)x_1(t)x_3(t) \\ b_1(t)x_2^3(t)x_2(t) - b_2(t)x_2^3(t) + b_3(t)x_2(t)x_3(t) \\ c_1(t)x_1^2(t) + c_2(t)x_2^2(t) + c_3(t)g_2(x_3(t)) \end{pmatrix},
\]

\[
F_3 \left( \int_{-\infty}^{0} g(t, \mu, x(t + \mu))d\mu \right) = \begin{pmatrix} \int_{-\infty}^{0} g_1(t, u, x_1(t + \mu))d\mu \\ \int_{-\infty}^{0} g_2(t, u, x_2(t + \mu))d\mu \\ 0 \end{pmatrix},
\]

\(n = 3,\) and \(\text{rank}(E) = 2 < 3.\) Set \(\alpha = -\infty\) and \(t_\ast = 0.\) Denote \(y = (y_1, y_2, y_3)^T = Ex = (x_1, x_2, x_3)^T, \) \(i = 1, \ldots, m.\) Slip \(x\) and \(y\) into 2 parts, respectively, i.e., \(x = (\tilde{x}_1, \tilde{x}_2)\) where \(\tilde{x}_1 = x_1, \tilde{x}_2 = (x_2, x_3)^T,\) \(y = (\tilde{y}_1, \tilde{y}_2)\) where \(\tilde{y}_1 = x_1, \tilde{y}_2 = (x_2, 0)^T.\)

Define \(h_{00}(t, \cdot) = \| \cdot \|_{\infty}\) and by (3), \(h_{00}(t, \cdot) = \sup_{-\infty \leq s \leq t} \{ h_{00}(s, \cdot) \}, \)

\(i = 1, 2.\) \(h_i(t, x) = \max_{j=1,2} \{ h_{j0}(t, \tilde{y}_j(t)) \}, h_{00}(t, x) = \max_{j=1,2} \{ h_{j0}(t, \tilde{y}_j(t)) \}.\) Obviously,

\[h_i(t, \cdot) \leq h_{00}(t, \cdot) \leq h_{i0}(t, \cdot).\]

Suppose that the initial condition of systems (13) is

\[x_0 = (x_{10}, x_{20}, x_{30})^T = \phi, \quad \phi \in C((-\infty, 0], \mathbb{R}^3),\]

then, the consistent initial condition of system (13) is

\[S_k(0) = \{ \phi \mid c_1(t)\phi_1(0) + c_2(t)\phi_2(0) + c_3(t)\phi_3(0) = 0 \}\]

For \(\forall \phi \in S_k(0)\) system (13) has a unique continuous solution in \((-\infty, \infty)\) through \((0, \phi).\) And the solution is denoted by \(x(t).\)

For system (13), we have the following results by Theorem 3.1.

**Corollary 4.1.** Given \((\lambda, A)\) with \(0 < \lambda < A, \) \(i = 1, 2.\) Assume that \(m_i(\cdot) \in C(\mathbb{R}^-, \mathbb{R}^+), \) \(m_i \in L^1((-\infty, 0], m_i(t) \leq K_i \) for \(t \leq 0\) with some constant \(K_i > 0,\)
\[ J = \max_{i=1,2} \{ \int_{-\infty}^{0} m_i(s)ds \} \] is bounded. If there exists a scalar \( \kappa \in (\lambda^2/A^2, A^2/\lambda^2) \), such that

\[ \begin{align*}
(\text{i}) & \quad \tau_1(t) \geq 0 \text{ and } h_{10}(t + \mu, g_i(t, \mu, x_i)) \leq m_i(\mu)h_{10}^3(t + \mu, x_i), \quad t > 0. \\
(\text{ii}) & \quad \text{for } (t, x) \in \mathbb{R}_+^* \times S'(h_j, A), \\
& \quad z_1(t) = a_1(t) + a_3(t)c_3^{-1}(t)c_1(t) - |d_1(t)| - \int_{-\infty}^{0} m_1(\mu)d\mu \\
& \quad -\kappa |(a_2(t) - a_3(t)c_3^{-1}(t)c_2(t))| \geq 0; \\
& \quad z_2(t) = b_2(t) + b_3(t)c_3^{-1}(t)c_2(t) - |d_2(t)| - \int_{-\infty}^{0} m_2(\mu)d\mu \\
& \quad -\kappa |b_1(t) - b_3(t)c_3^{-1}(t)c_1(t)| \geq 0.
\end{align*} \]

Then, the zero solution of system (13) is \((h_0, h)\)-U.P.S.

**Proof.** There exist positive scalars \( p_1, p_2 \) that \( p_1^2 = p_2^2 \kappa \). Choose \( V_1(t, \tilde{y}_1) = p_1^2 \tilde{y}_1^T \tilde{y}_1, V_2(t, \tilde{y}_2) = \tilde{y}_2^T \left( \begin{array}{cc} p_2^2 & * \\ * & * \end{array} \right) \tilde{y}_2 = p_2^2 \tilde{x}_2^T \tilde{x}_2 \), where \( \begin{pmatrix} p_2^2 & * \\ * & * \end{pmatrix} > 0, * \) stands for a matrix irrelevant to the following analysis. When \( \kappa \in (\lambda^2/A^2, 1] \), \( p_1 \leq p_2 \); whereas \( \kappa \in (1, A^2/\lambda^2), p_1 > p_2 \). Thus, there exists a scalar \( \epsilon > 0 \) such that \( \max_{i=1,2} \{ \lambda^2 + \epsilon J^2 \lambda^2 \} < \min_{i=1,2} \{ p_i^2 \} A^2 \). Let \( W_{1t}(\xi) = \min_{i=1,2} \{ p_i^2 \} \xi^2, W_{2t}(\xi) = \max_{i=1,2} \{ p_i^2 \} \xi^2, W_{3t}(\epsilon) = \xi^2, W_{3t}(\epsilon) = \xi^2 \in C(R^+, R^+), \Phi_i(t) = m_i(-t) \). Then, \( W_{1t}, W_{2t}, W_{3t}, W_{4t} \) are wedge functions, \( \Phi_i(-t) \) satisfy the assumption of Theorem 3.1. Thus, conditions (i), (ii), (iii) of Theorem 3.1 are all satisfied for given \( 0 < \lambda < A \). Moreover, when \( V_1(t) \geq V_2(t) \), we have \( p_1h_{10}(t, \tilde{y}_1) \geq p_2h_{20}(t, \tilde{y}_2) \), if \( V_3(s) < V_1(t) \) for \( s \in (-\infty, t) \), then we have \( h_{10}(t - \tau_1(t, \tilde{y}_1)) \leq h_{10}(t, \tilde{y}_1) \) and \( h_{10}(t + \mu, \tilde{y}_1) \leq h_{10}(t, \tilde{y}_1) \). Then by condition (i) and (ii), we can derive that

\[ D^+ V_1(t) = 2p_1^2 \tilde{x}_1(t)\tilde{x}_1(t) = -2p_1^2(a_1(t) + a_3(t)c_3^{-1}(t)c_1(t))x_1^2(t) + 2p_1^2(a_2(t) - a_3(t)c_3^{-1}(t)c_2(t))x_1^2(t)x_2^2(t) + 2p_1^2d_1(t)x_2^2(t)(-\tau_1(t)) + 2p_1^2x_1(t)\int_{-\infty}^{0} g_1(t, \mu, x_1(t + \mu))d\mu \leq -2p_1^2\{ (a_1(t) + a_3(t)c_3^{-1}(t)c_1(t)) - \kappa |a_2(t) - a_3(t)c_3^{-1}(t)c_2(t)| - |d_1(t)| - \int_{-\infty}^{0} m_1(\mu)d\mu \}x_1^2(t) \leq 0. \]

Similarly, when \( V_1(t) \leq V_2(t) \), if \( V_2(s) \leq V_2(t) \) for \( s \in (-\infty, t) \), we have

\[ D^+ V_2(t) = 2p_2^2 \tilde{x}_2(t)\tilde{x}_2(t) \leq -2p_2^2z_2(t)x_2^2(t) \leq 0. \]

Thus, condition (iv) in Theorem 3.1 is also satisfied. Therefore, the zero solution of system (13) is \((h_0, h)\)-U.P.S. \( \square \)
Example. Given $\lambda = 0.05$, $A = 0.0501$. Consider the following systems:

$$\begin{align*}
\dot{x}_1(t) &= -3(5 + \cos(t))x_1^3(t) + a(8 + \sin(t))x_1(t)x_2^2(t) + 4e^{\sin(t)}x_1(t)x_3(t) \\
&\quad + 3\cos(t)x_1(t)x_1^2(t - e^{-\cos(t)}) + \int_{-\infty}^{0} e^{-1.5\mu}x_1^3(t + \mu)d\mu, \\
\dot{x}_2(t) &= 2.5\sin(t)x_1^2(t)x_2(t) - 4(2 + \cos(t))x_2^3(t) + 2e^{\sin(t)}x_2(t)x_3(t) \\
&\quad + 3\sin(t)x_2^2(t)x_2(t - e^{-\sin(t)}) + \int_{-\infty}^{0} e^{-3.5\mu}x_2^3(t + \mu)d\mu, \\
0 &= 2(1 + \cos(t))x_1^2(t) + \sin(t)x_2^2(t) + 4e^{\sin(t)}x_3(t).
\end{align*}$$

Let $m_1(\mu) = e^{-1.5\mu}$, $m_2(\mu) = e^{-3.5\mu}$, $\mu \in (-\infty, 0]$. Then, $J = \max\{1.5^{-1}, 3.5^{-1}\}$ is bounded. For $a = 2.5$, one can calculate that the norm of solution is greater than 0.0501 on time interval of $[0.45, 7.4]$, and hence is not $(h_0, h)$-U.P.S., see Figure 1a. For $a = 1$, consider conditions in Corollary 4.1. Figure 2 shows the profiles of $z_1(t)$, $z_2(t)$ with $\kappa = 1$ on $[-15, 15]$. We can also calculate that the values of $z_1(t)$ and $z_2(t)$ are both non-negative. By Corollary 4.1, system in Example is $(h_0, h)$-U.P.S.. The state response curve is shown in Figure 1b.

**Figure 1.** State response curves of Example system

**Figure 2.** Profiles of $z_1(t)$, $z_2(t)$ with $\kappa = 1$. 

*Example.*
5. Conclusion

In this paper, inspired by the idea of [14] we have developed a new technique in investigating the uniform practical stability of descriptor systems with infinite delays in terms of two measurements via dividing the components of variables into several groups. It is easy to obtain results in a quite effective way, especially for systems of infinite delay equations. Through an analysis of a class of descriptor systems with infinite time delays, the application of the results obtained is discussed to illustrate the advantage of the proposed results. It is easy to see that, if we apply the usual Razumikhin technique, by which we usually put state variables in one Lyapunov function, then, the discussion to obtain the desired practical stability results would be much more complicated and the conditions would be more restrictive.

References

1 Institute of Systems Science, Northeastern University, Shenyang, Liaoning province, 110004, PR China
E-mail: su.zhan@hotmail.com and qlzhang@mail.neu.edu.cn

2 Lab of embedded technology, Northeastern University, Shenyang, Liaoning province, 110004, PR China
E-mail: aijun@live.com

3 College of Information Science and Engineering, Northeastern University, Shenyang, Liaoning province, 110004, PR China
E-mail: y-chunyu@sohu.com