CONTACT PROBLEM AND CONTROLLABILITY FOR SINGULAR SYSTEMS IN BIOMEDICAL ROBOTICS

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Abstract. Contact problem for a robotic system in interaction with the environment is challenging especially in the bioengineering applications. In this paper we presented general method for the system modeling. Analyzing system dynamics in the working regime it has been concluded that mathematical model has a singular characteristic in the contact region. Therefore, the system was described as a singular system of differential equations. During the working regime contact surface was treated as a constraint to the system. Geometric conditions for transformation to the state space model have been developed. Controllability analysis for the robotic system was performed using geometric approach. State feedback was introduced to stabilize the system. Both the mathematical conditions for the eigenvalues assignment and the feedback matrix structure were presented.

Key Words. robotic system, controllability, geometric approach, singular system.

1. Introduction

The application of the robotic systems has been widespread over the range of different fields. Biomedical robotics has attracted many researchers who developed and significantly improved different treatments and medical procedures. Varieties of the robotic applications and solutions in medicine have been presented over the past several years. In this paper we have summarized different techniques for adequate modeling and control for the medical robots in the contact with an environment. Contact surface could be either a patient or a device. Here, we have introduced a general theoretic overview and suggested a specific approach to the solution of the insertion problem. For many applications it is enough to consider contact force as a disturbance to the system. Sometimes stochastic character and unexpected range of the contact forces could significantly change or damage contact surface which could be unacceptable for some medical treatments. Furthermore, contact force which has an unknown value and characteristics could produce a compromised medical outcome. Due to the reasons given here, there is a need not only to measure the force, but to control it and to obtain adequate control algorithms which can keep the force within acceptable limits. Contact force, which can occur due to physical constraints to the robotic systems, is able to significantly influence system dynamics. The mathematical model of the system described here is usually a singular system of differential equations. Force control for singular systems is a challenging task. Approach to the force control demands specific mathematical preparations before force feedback is applied. In the robot working
regime there is a problem to obtain qualitative control signal when contact force acts upon the system. The control algorithm is different in the second phase of the working regime, because contact force does not act upon the system. For that phase it is possible to apply the classic control theory. To model the robotic system with the tasks described above, the entire system can be decomposed to the robotic subsystem and the environment subsystem. Modeling of the system by the method mentioned has been proved to be suitable when the force appears as a result of the interaction of the two subsystems. The mathematical model of the system has a singular characteristic. The singular system theory could be applied to the case described. For the second phase in which there is no interaction, the dynamic behavior can be analyzed by the classic theory.

2. Chronological literature review

McClamroch [16] was one of the first authors who used the singular system of differential equations for mathematical modeling of the robotic system with the contact problems. The author suggested several classes of the robotic systems in which the singular system approach was suitable to be applied. Hogan [6] used the previous technique to investigate the stability of the robots with contact tasks. That article has brought unique mathematical formulations of the external and internal boundedness that led us to solve singular differential equations. Huang and McClamroch [8] defined optimal control for robots tracking predefined contours. This model included algebraic equations and together with differential equations formed the singular system of differential equations. Contact force was regulated by the optimal control algorithm. Hui and Goldenberg [9] presented the singular mathematical model of the manipulator in contact with rigid environments. They defined conditions for the feedback design. McClamroch and Wang, [18] solved stabilization and tracking problems for constrained robotic systems. They developed feedback by introducing a generalized error and generalized velocity. Furthermore, they investigated stability conditions for such a system. By using the described method it was possible to determine the effects of the contact force to the system dynamics. After publishing an article [19] on the similar topic, Mills joined the group of the authors who had done respectable work in the field of constrained robotics. After that he published several related articles with significant results. In the article [19] he described the dynamics of constrained robots as well as the hybrid control. Carelli and Kelly [2] introduced the adaptive controller for constrained robots described by the singular system of differential equations. The adaptive control law was based on actuator torque calculations. McClamroch [15] applied the well-known singular perturbation technique to the constrained mechanical system. Using appropriate assumptions, he concluded that dynamical behavior of the system depends on the small system parameters, such as system damping coefficient or system stiffness. Mills and Goldenberg [27] continued the work on force and position control for manipulators with contact tasks. In that article the authors described a unique method for feedback parameter calculations. The solution was applied to the tracking problem. Their system was decomposed to the slow and fast subsystems. Furthermore, influence on the singularities to the new perturbed system was investigated. One year later, Krishnan and McClamroch [12] presented a new approach to the contact force regulation. The method was applied to the mathematical model which was linearized in the surroundings of the contact point. Mills [21] investigated the robotic system which transitied from the nominal regime to the contact regime. For that specific working cycle the system stability was
analyzed with special emphasis on the transition moments. In the article [26] Mills introduced the investigation on the singular perturbed system which was extended to the force feedback development. A force sensor was installed to detect the torque at the joint on which contact force reacts. The dynamical behavior was investigated and it was analytically shown that high frequency dynamic could be neglected during the system control. In the article [24] by Mills and Lokhorst the series of experimental results with different control laws was presented. Their experiments were undertaken using two-degree-of-freedom robot. Mills and Lui [26] continued the investigation of robotic systems being in contact with environments. They applied dumping control for generalized contact forces and positions. The influence of the friction was neglected. Singular perturbation method was applied during the mathematical modeling. Goddard et.al [4] solved sliding and rolling problems of the end-effector. The rigid contact surface was fixed. This method was applied to industrial robots for insertion and parts assembly. Mills [23] investigated the stability for robotic systems with elastic joint in the constrained environment. He introduced conditions for the asymptotic stability. Krishnan and McClamroch in the articles [11] and [10] presented a nonlinear mathematical model of the constrained robots. The mathematical model was singularly perturbed. Consequently, the system was transformed to slow and fast subsystems. Control design was performed using a composite approach. Mills and Lokhorst [25] proposed a control methodology that addresses the problem of the control of robotic manipulators during a general class of tasks that requires the manipulator to make a transition from noncontact motion to contact motion and contact motion to noncontact motion. During noncontact motion, a control suitable for the noncontact phase of motion is applied; during contact, another control, suitable for contact motion, is applied. These different control schemes are applied to the manipulator in such a way that the overall control is discontinuous in nature, [25]. In the article [34] Wapenhas et.al presented a complete design procedure for determining an optimal force control for assembly tasks while assuring stability. Based on a constrained motion model of the robot, including elasticity of joints and force sensors, a custom design scheme was applied for individual types of mating tasks. Hu and Davison in the article [7] studied the problem of position tracking and contact force control for a constrained manipulator, the end-effector of which is required to move along a preassigned trajectory with a specified contact force. The basic procedure for controller design was divided into two steps: first, the nonlinear descriptor system is linearized into a linear system; then various control schemes for different cases are proposed based on the resultant linear model. Mills [22] analyzed the problem of control of generalized contact forces with a manipulator controller that has traditionally been regarded as a noncontact task trajectory controller. Experimental results of a two-degree-of-freedom direct drive manipulator during contact with a one-degree-of-freedom linear mechanical impedance illustrated the usefulness of the proposed method. Chan et.al [3] developed the impedance control scheme for robot manipulators performing assembly tasks which required interactions with the environment. Lui and Goldenberg in the article [14] presented a new robust control approach for robot manipulators based on a decomposition of model uncertainty. Parameterized uncertainty is distinguished from unparameterized uncertainty. Prokop and Pfeiffer [29] suggested a method for planning of robotic assembly by numerical optimization of position and joint controller coefficients. Together with constraints ensuring practical applicability a nonlinear vector optimization problem is stated for a peg-in-hole insertion task and that problem was solved. Stokic and Vukobratovic [30] solved the problem of the
practical stabilization of robots being in contact with a dynamic environment. The goal of another article [31] of Vukobratovic et al. was to shed light on the control problem of constrained robot motion from the aspect of the dynamical nature of the environment with which the robot was in contact. In the [32] Vukobratovic et al. presented the current state of the art in the adaptive control of single rigid robotic manipulators in the constrained motion tasks. A complete mathematical model of a single rigid robotic manipulator in contact with dynamic environment was presented. The same author in the [33] investigated the problem of impendence control using a unified approach to contact tasks control in robotics, where the interpretation of the contact between the robot and its known environment were based on a sufficiently correct treating of interactions between the environment and robot dynamics. McClamroch et al. [17] analyzed control problems for a specific mechanical system consisting of a rigid base body with an unactuated internal degree of freedom. The key assumptions were that the translational and rotational motions of the base body can be completely controlled by external forces and moments, while the internal degree of freedom is unactuated. The presented idea can apply to the system in the contact task. Park and Kim [28] presented a differential geometric analysis of manipulability for holonomic multiple robot systems containing active and passive joints. The system was described using singular differential equations. The authors used a geometric approach. Kuzmina in the [13] worked on the analysis methods in complex system dynamics. The generalized approach, based on Lyapunov's methods, is considered unified for mechanical systems. The previous two references presented general approaches to elaborate the technology of modeling in mechanics using singular systems theory. Ho et al. [5] presented the model for the constrained robot dynamics, incorporating constraint uncertainties. The rigid constraining surfaces were represented by a set of the algebraic functions.

3. Singular systems and mathematical modeling

As suggested above, the most accurate mathematical model for constrained robots should include dynamics of the system due to interaction between the robot and the surface. General guidelines for mathematical modeling together with the basic equations are presented. The model of the manipulator with its constraints is shown in Figure 1. Generally speaking, open kinematic chain with \( n \) joints is analyzed.
The generalized coordinates vector is denoted by $q$, $q \in \mathbb{R}^n$, the contact force vector is denoted by $f$. Force $f \in \mathbb{R}^n$ appears when end-effector touches constrain surface $c$. The differential equation which describes the influence on the contact force to the system is

$$
M(q)\ddot{q} + G(q, \dot{q}) = \tau + J^T(q)f.
$$

$M(q) \in \mathbb{R}^{n \times n}$ denotes inertia matrix function and $G(q) \in \mathbb{R}^n$ is vector function which describes Coriolis, centrifugal and gravitational effects. $\tau$ is torque vector of the joints, $\tau \in \mathbb{R}^n$. $J(q) \in \mathbb{R}^{n \times n}$ is defined as Jacobian matrix function. If the position vector of the contact point in fixed coordinate system is denoted by $p \in \mathbb{R}^m$, than the algebraic equation for contact surface is written as

$$
\phi(p) = 0,
$$
in which $\phi: \mathbb{R}^m \rightarrow \mathbb{R}^1$ is scalar function. For further mathematical modeling it is necessary to introduce three assumptions.

**Assumption 1**: Friction between end-effector and surface at point $T$ could be neglected. 

**Assumption 2**: First derivative of the function $\phi(p)$ is a smooth function. 

**Assumption 3**: Matrix $M$ is symmetric and positive defined. 

Having in mind Assumptions 1-3, contact force vector can be presented as [16]

$$
f = D^T(p)\lambda,
$$
where $\lambda$ is scalar multiplier for constrained function, and $D(p)$ is constrained function gradient. Expression for $D(p)$ is

$$
D(p) = \frac{\partial \phi(p)}{\partial p}.
$$

Establishing direct kinematic relationship between end-effector and point $T$ on the contact surface, using generalized coordinates, one can write $p = H(q)$, where $H(q)$ is vector function. In that case, Jacobian is of the following form

$$
J(q) = \frac{\partial H(p)}{\partial q}.
$$

As a result of combination of the equations (3)-(5) with (1) general dynamic equations for the robotic system in contact with environment is obtained [16],

$$
\begin{bmatrix}
M(q) & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q} \\
\dot{\lambda}
\end{bmatrix}
= 
\begin{bmatrix}
-G(q, \dot{q}) + \tau + J^T(q)D^T(H(q))\lambda \\
\phi(H(q))
\end{bmatrix}.
$$

Equation (6) consisted of the $n$ differential equations and one algebraic equation with $n+1$ unknown values, $n$ generalized coordinates and scalar multiplier $\lambda$. Based on the structure of equation (6) the system is represented as a singular system of differential equations. The physical explanation is the following: the mathematical model includes contact surface due to its influence on the system dynamics.
4. Linearization procedure and matrix representation

Using transformation given at [16] it is possible to transform equation (6) into the form shown below:

\[
\begin{bmatrix}
  I & 0 & 0 \\
  0 & M(q) & 0 \\
  0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  \dot{q} \\
  \ddot{q} \\
  \lambda \\
\end{bmatrix} = \begin{bmatrix}
  0 & I & 0 \\
  0 & 0 & J^T D^T \\
  0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  q \\
  \dot{q} \\
  \lambda \\
\end{bmatrix} + \begin{bmatrix}
  -H(q, \dot{q}) - G(q) \\
  \phi(p) \\
  0 \\
\end{bmatrix} + \begin{bmatrix}
  0 \\
  I \\
  0 \\
\end{bmatrix} \tau
\]

In the following discussion we assume that force acts upon the constrained surface at a certain point where the position, velocity and acceleration amplitudes do not change significantly in the surroundings of that point. The point is called nominal and it is denoted by

\[(8) \ (q_0, \dot{q}_0, \lambda_0)^T,\]

where \(q_0\) is a generalized coordinate in nominal system configuration. First derivation of the \(q_0\) is nominal joint velocity, and \(\lambda_0\) is a nominal value of the multiplier. For this case, linearization problem is to represent equation (7) around nominal state (8). At the nominal point (8) the robotic system is at rest, so both nominal velocity and nominal acceleration are equal to zero. Expanding equation (7) in the surroundings of the nominal point, using Taylor series it can be described with the [27]

\[(9) \ \delta \tau = M(q_0) \delta \ddot{q} + \partial/\partial q(G - J^T D^T \lambda)|_0 \delta q - J^T D^T|_0 \delta \lambda,\]

In which variances are defined as

\[(10) \ \delta q = q - q_0, \ \delta \dot{q} = \dot{q} - \dot{q}_0, \ \delta \ddot{q} = \ddot{q} - \ddot{q}_0, \ \delta \tau = \tau - \tau_0, \ \delta \lambda = \lambda - \lambda_0.\]

Combining equations (7)-(10) the linearized equation for the system, described by (1), can be written as

\[(11) \ \begin{bmatrix}
  I & 0 & 0 \\
  0 & M(q_0) & 0 \\
  0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  \delta \dot{q} \\
  \delta \ddot{q} \\
  \delta \lambda \\
\end{bmatrix} = \begin{bmatrix}
  0 & I & 0 \\
  \partial/\partial q(G - J^T D^T \lambda)|_0 & 0 & J^T D^T|_0 \\
  DJ|_0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  \delta q \\
  \delta \dot{q} \\
  \delta \lambda \\
\end{bmatrix} + \begin{bmatrix}
  0 \\
  -H(q, \dot{q}) - G(q) \\
  \phi(p) \\
\end{bmatrix} + \begin{bmatrix}
  0 \\
  I \\
  0 \\
\end{bmatrix} \delta \tau \]

All coefficients of the system (11) have constant values and they are not time dependent. To calculate nominal values of the generalized torques \(\tau_0\), it is necessary to calculate \(\tau\) from equation (7) for nominal point (8). Result is as follows in equation (12).

\[(12) \ \tau_0 = G(q_0) - J^T(q_0) D^T(L(q_0)) \lambda_0\]

Analyzing characteristics of the contact force, it can be concluded that end-effector acts upon the surface and surface reacts at the point T. Former force is called active force and latter is denoted as the passive force. During working cycle it is important to control passive force because that can damage the surface. For surgery robots,
reactive force could be responsible for tissue deformation as well as unnecessary damage of the healthy organs. Passive force could be expressed as

\[ D^T (L(q_0)) \lambda_0 = f. \]

Combining equations (12) and (13) nominal value for generalized torque is

\[ \tau_0 = G(q_0) - J^T(q_0) f. \]

Multiplier \( \lambda_0 \) could be calculated from equation (12), using (14)

\[ \lambda_0 = (DD^T)^{-1} Df, \]

Linearized equation (11) together with (12)-(15) represents equation of motion for the robotic system in contact with working environment. Described equations are complements of the differential and algebraic equations and consequently, they represent a singular mathematical model of the system described. The obtained results could be summarized with the following theorem.

**Theorem 1**: The special class of the nonlinear singular system described by differential equation (7) is equivalent to its own transformed system (11), linearized in the surroundings of the nominal point (8), if and only if there are equations (13) and (14) and multiplier \( \lambda_0 \) defined by equation (15), for which (11) is fulfilled on constrained domain \( \Delta \).

**Proof**: Having equation (9) and the condition (16) in mind

\[ (\partial/\partial q) H(q, \dot{q})|_0 = 0, \]

together with equations for contact surface (2) and geometric characteristics of the surface

\[ \frac{\partial \phi(p)}{\partial p} \frac{\partial p}{\partial q}|_0 \delta q = 0, \]

expressions (13) and (14) can be calculated from (12). Calculating the value for multiplier \( \lambda_0 \) at a nominal point (8), it can be concluded that (11) represents the linearized equation of the system (7). From that fact it can be concluded that both equations (7) and (11) are equivalent, q.e.d.

To determine the appropriate control law for the special class of the system described, it is of the primary interest to represent the system (11) in the state space form.

Analyzing equation (14) it can be assumed that nominal torques could change their values in the surroundings of the nominal point (8)

\[ \hat{\tau}_0 = \tau_0 + \Delta \tau. \]

Dynamic torque \( \Delta \tau \) represents gravitational influence on the system during the working regime. The value \( \Delta \tau \) shows changes of the other dynamic parameters and consequently, in the system dynamics. State space vector can be adopted as

\[ x = \begin{pmatrix} \delta q \\ \dot{\delta q} \\ \delta \lambda \end{pmatrix}. \]
Using (18) and (19), equation (11) can be transformed as follows

\[
\begin{bmatrix}
    I & 0 & 0 \\
    0 & M(q_0) & 0 \\
    0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    \delta q \\
    \delta \dot{q} \\
    \delta \lambda
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    I \\
    0
\end{bmatrix}
\begin{bmatrix}
    \delta q \\
    \delta \dot{q} \\
    \delta \lambda
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    I \\
    0
\end{bmatrix}
\delta \tau +
\begin{bmatrix}
    0 \\
    \Delta \tau
\end{bmatrix}.
\]

Now it is possible to represent the robotic system (7) which is in contact with working environment in its state space form (21) with vector \( \delta \tau \) as a disturbance

\[
E \dot{x}(t) = Ax(t) + Bu(t) + d,
\]
where corresponding matrices are defined as

\[
E = \begin{bmatrix}
    I & 0 & 0 \\
    0 & M(q_0) & 0 \\
    0 & 0 & 0
\end{bmatrix},
A = \begin{bmatrix}
    0 & I \\
    \frac{\partial}{\partial q}(G - J^T D^T \lambda)|_0 & 0 \\
    DJ|_0 & 0
\end{bmatrix},
B = \begin{bmatrix}
    0 \\
    I \\
    0
\end{bmatrix},
d = \begin{bmatrix}
    0 \\
    0 \\
    \Delta \tau
\end{bmatrix}.
\]

### 4.1. Geometric conditions for system transformation.

Equation (22) can be transformed to the nonsingular and singular parts. The following equation is a result of the described transformation

\[
\begin{align*}
\dot{x}(t) &= A'x(t) + B'u(t) = \\
C'x(t) &= \begin{bmatrix}
    0 \\
    -M^{-1}K \\
    -M^{-1}L
\end{bmatrix} x + \begin{bmatrix}
    0 \\
    I \\
    M^{-1}L
\end{bmatrix} u,
\end{align*}
\]

where \( Py \geq d' \) is the surface equation in the state space, \( K, L \) and \( P \) are corresponding matrices. The compound equivalent equation is of the following form

\[
\dot{x} = A'x + B'u
\]

\[
0 \leq C'x + d
\]

To analyze geometric conditions for system transformation it is necessary to connect equations (24) and (21). Result is

\[
E = \begin{bmatrix}
    I & 0 \\
    0 & 0
\end{bmatrix},
A = \begin{bmatrix}
    A' & 0 \\
    C' & 0
\end{bmatrix},
B = \begin{bmatrix}
    B' & 0 \\
    0 & 0
\end{bmatrix},
D = \begin{bmatrix}
    0 \\
    d
\end{bmatrix}.
\]

For system subspace analysis we use a geometric approach to show whether the described transformation exists.

**Theorem 2:** System (1), with equation of the constrained surface (2) is transformable to system (21) if and only if conditions (26) are fulfilled

\[
C'B' = 0, \quad \mathcal{N}(B') \subseteq \mathcal{R}(C'),
\]

where \( \mathcal{N}(\cdot) \) and \( \mathcal{R}(\cdot) \) denote null space and range of the corresponding matrices, respectively.
Proof: To prove conditions (26) it is sufficient to prove equivalency of each condition itself. From the structure of the matrices $C'$ and $B'$ it can be concluded that their product is equal to zero. From the geometric characteristics of the orthogonal invariant subspaces it can be stated that condition $\mathcal{N}(B') \subseteq \mathcal{R}(C')$ is fulfilled. To confirm the statement, null space for the matrix $B'$ is calculated and the result is $\mathcal{N}(B') = \{0\}$. The range of the matrix $C'$ contains zero-vector, $\mathcal{R}(C') > \{0\}$. Consequently, one can conclude that geometric condition (26) for transformation of the robotic system (1), with equation of the contained surface (2) is fulfilled if and only if transformation is possible, $q.e.d.$

5. Geometric structure and controllability

Here we introduce state space feedback described by

\begin{align}
    u &= Kx(t),
\end{align}

where $K$ is the matrix. Applying feedback (27) to the system characterized by matrices (25), controlled system is obtained as follows

\begin{align}
    E\dot{x} &= (A + BK)x \\
    y &= (C + DK)x.
\end{align}

Definition 1: System (28) is controllable if the matrix pencil

\begin{align}
    C(s) &= \begin{bmatrix}
    sE - A & B
    \end{bmatrix}
\end{align}

does not have definite or indefinite zeros.

Theorem 3: Systems characterized by matrices (25) are controllable if and only if none of the matrix eigenvalues $C(s)$ is equal to zero. That condition is represented as

\begin{align}
    C(s) &= \begin{bmatrix}
    \frac{\partial}{\partial q}(G - J^T D^T \lambda)\mid_0 & -I & 0 & 0 \\
    -DJ\mid_0 & sM(q_0) & -J^T D^T \mid_0 & I
    \end{bmatrix}.
\end{align}

Proof: Equation (22) represents structure of the robotic system. Combining equations (22), (25) and (29) it can be concluded that condition (30) is fulfilled, $q.e.d.$

Definition 2: Systems characterized by matrices (25) are reachable if (29) is fulfilled and if

\begin{align}
    \text{rang} \begin{bmatrix}
    E & B
    \end{bmatrix} = n.
\end{align}

Theorem 3: Systems characterized by matrices (25) are reachable if and only if condition (32) is fulfilled:

\begin{align}
    \text{rang} \begin{bmatrix}
    I & 0 & 0 & 0 \\
    0 & M(q_0) & 0 & I \\
    0 & 0 & 0 & 0
    \end{bmatrix} = n
\end{align}

Proof: The procedure is similar to the proof of the Theorem 3, using the condition of Definition 2.

Corollary 1: Analyzing condition (32), it can be concluded that controllability of the linearized robotic system (21) in contact with environments depends on inertia matrix in the surroundings of the contact point.
This conclusion is significant because the design of the system with contact tasks can influence the controllability and consequently the system stability. Condition (32) could be used for potential stability checking during the system design.

**Definition 3:** Systems characterized by matrices (25) are infinite controllable if (29) does not have indefinite zeros.

In the following part, eigenstructure and eigenvalues assignments for the robotic system are analyzed. State space feedback was used and geometrical approach was applied to solve controllability problem.

**Theorem 4:** Assume that system (21) is controllable by eigenvalues assignment procedure. Let the \( \{ \sigma_i \} \) \( i \in 1, 2 \ldots h \), \( h = \text{rang } E \) be a symmetric set of the definite complex numbers. Assume that subspace \( \mathcal{Z} = \text{span } \{ v_i \} \), \( i \in 1, 2 \ldots h \) exists, so the following conditions are true

1. \( v_i \in \mathcal{Z} \) if \( \sigma_i \in \mathbb{R}^n \) and \( v_i = v_i^* \) is complex conjugate if and only if \( \sigma_i = \sigma_i^* \).
2. Vectors \( \{ v_i \} \) are linearly independent and \( v_i \in \mathcal{Z} = (\sigma_i E - A)B = [0 0 - (1/F D^T)I_n 0]^T \).
3. \( \mathcal{Z} \cap \mathcal{I}_3 = 0 \).

If the conditions (i)-(iii) are fulfilled, real matrix \( F \) exists, so the matrix pair \( (sE - A - BF) \) is regular, \( (A + BF) v_i = \sigma_i E v_i \).

**Proof:** Matrix \( P_\sigma = [\sigma E - A]B \) can be associated with complex number \( \sigma \). \( Q_\sigma \) is a compatible matrix which columns span null space of the \( P_\sigma \), \( \mathcal{N}(P_\sigma) \). \( Q_\sigma \) is defined as

\[
Q_\sigma = \begin{bmatrix} N_\sigma \\ R_\sigma \end{bmatrix}.
\]

From the structure of the matrices (22) it can be concluded that \( \text{rang } B = m \), which implies that columns of the \( N_\sigma \) are linearly independent.

From the statement \( v_i \in \mathcal{Z} = \mathcal{N}(N_\sigma) \), it can be stated that \( v_i = N_\sigma K_i \), for some \( K_i \), defined as

\[
(\sigma_i E - A)N_\sigma K_i + BR_\lambda K_i = 0.
\]

\( F_0 \) is defined as the transformation of subspaces, \( F_0 : \mathcal{Z} \to \Psi \) as it is in (35)

\[
(35) \quad F_0 v_i = -R_\sigma K_i, \quad i \in 1, 2 \ldots n.
\]

In order to achieve regularity of the matrix pencil \( (sE - A - BF) \) it is necessary to define the extension of \( F \) to \( F_0 \). Dimension of the subspace \( \mathcal{Z} \) is \( \text{dim } \mathcal{Z} = h \), therefore \( \text{dim } \mathcal{N}(E) = n - h \) and \( \mathcal{Z} \cap \mathcal{N}(E) = 0 \). Consequently,

\[
(36) \quad \mathcal{Z} \oplus \mathcal{N}(E) = \mathcal{Z}.
\]

It can also be concluded that equation (37) is fulfilled,

\[
(37) \quad E \mathcal{Z} = E(\mathcal{Z} \oplus \mathcal{N}(E)) = E \mathcal{Z} = \mathcal{R}(E).
\]

Assume that \( x \in \mathcal{Z} \) and condition (36) is fulfilled. Subspace transformation \( F_{\mathcal{Z}} : \mathcal{Z} \to \Psi \) is defined as the extension on the \( F_0 \). \( E \mathcal{X} \) and \( (A + BF) \) is represented by decomposition (38)

\[
(38) \quad E \mathcal{Z} \oplus \mathcal{Z}^* = \mathcal{Z}.
\]
where \( \mathcal{V}^* \) is any subspace of the dimension \( n - h \) which supplements \( \Re(E) \) to the \( \mathcal{V} \). In the decompositions (36) and (38) matrices \( E \) and \( \mathcal{A} + BF' \) can be represented as

\[
E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A + BF' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.
\]

Statement (39) can be confirmed with equation (22), because \( \dim I = h \) and eigenvalues of the \( A_{11} \) are \( \{\sigma_i\}, i \in 1, 2 \ldots h \). Matrix pencil \( (sE - \mathcal{A} - BF') \) is regular if and only if \( \mathcal{A}_{22} \) is a nonsingular matrix.

Assume that \( \mathcal{Q}_i \) is projection on the \( \mathcal{V}^* \) along \( \Re(E) \). Nonsingular character of the matrix \( \mathcal{A}_{22} \) is equivalent to the nonsingular value of the expression (40)

\[
\mathcal{Q}_i (A + BF')|\Re(E).
\]

Furthermore, it can be written

\[
\mathcal{Q}_i (A + BF')|\Re(E) = \mathcal{Q}_i A_i |\Re(E) + \mathcal{Q}_i BF_i,
\]

where \( F_i = F|\Re(E) \). Matrix \( F_i : \Re(E) \rightarrow \Psi \) exists, therefore expression (40) is nonsingular if and only if all eigenvalues of the \( \mathcal{Q}_i A_i |\Re(E) \) are controllable with \( \mathcal{Q}_i B \).

\[
\mathcal{Q}_i (A|\Re(E) + B) = \mathcal{V}^*.
\]

The system (21) is controllable; therefore, it is possible to define \( F_1 \). By doing so, expression (40) becomes nonsingular. Now, \( F : \mathcal{V} \rightarrow \Psi \) is defined as

\[
F|\mathcal{S} = F_0, \quad F|\Re(E) = F_1.
\]

Defining real transformation represented by matrix \( F \), it is possible to adjust eigenvalues of the system (21). In that case, system (21) became controllable, q.e.d.\( \Box \).

Comment: Theorem 4 gives possibility for derivation of the opposite statement. If the \( \{\sigma_i\}, i \in 1, 2 \ldots h \) is the set of the finite eigenvalues, it is not necessary for \( (sE - \mathcal{A} - BF') \) to be regular. In that case

\[
(A + BF)v_i = \sigma_i Ev_i,
\]

for some vectors \( v_i, i \in 1, 2 \ldots l \). It can be proved that vectors \( v_i, i \in 1, 2 \ldots l \) fulfill conditions (i)-(iii) for robotic system described by equation (21). Also, it can be proved that subspace \( \mathcal{S} \) from the statement in Theorem 4 exists for any symmetric set of the complex numbers \( \{\sigma_i\}, i \in 1, 2 \ldots h \). Due to that fact, system (21) is controllable in both finite and infinite eigenvalues. In that case, matrix \( F \) exists, so the \( (sE - \mathcal{A} - BF') \) is a regular expression. Furthermore, infinite eigenvalues of the \( (sE - \mathcal{A} - BF') \) are \( \{\sigma_i\}, i \in 1, 2 \ldots h \). That implies existence of the \( v_i \in \mathcal{V}, i \in 1, 2 \ldots h \), so it can be concluded that

\[
\text{span}\{v_i\} \cap \Re(E) = 0.
\]

The possible method to define subspace \( \mathcal{S} \) is to define linearly independent set of the vectors \( v_i \in \mathcal{S}, \) but \( v_i \notin \Re(E) \). In that case \( \mathcal{S} = \text{span} \{v_i\} \) and \( \mathcal{S} \cap \Re(E) = 0. \) From
the practical point of view, this means that it is possible to adjust eigenvalues and eigenvectors to the system (21) using feedback defined by matrix $F$.

6. Conclusion

In this article general method for mathematical modeling of constrained robots has been presented. The rigid constrained surface is represented by a set of the algebraic functions. Therefore, the mathematical model of the robotic system is the singular system of differential equations. Nonlinear system is transformed to the state space system. Linearization was performed in the surroundings of the contact point. A geometric approach was used for both the controllability analysis and the eigenstructure assignment. System transformation conditions are presented. It can be concluded that system transformation is possible only if the range of the transformed matrix $C$ contains the null space of the transformed matrix $B$. Robotic system was controllable only if the matrix $C$ eigenvalues were not zeros. Controllability of the linearized robotic system depends on inertia matrix in the surroundings of the contact point. Structure of the space feedback matrix was analyzed. Conditions for system stabilization using space feedback were derived.

7. Appendix

In Appendix, algorithm for the null space and range calculation related to the corresponding matrices are presented. Algorithm is used in the proof of Theorem 4. More detailed calculations of the different matrix subspaces and algorithms applied in the geometric theory can be found in [1].

- Direct sum
  
  function A = dirsum(A,varargin)
  for k=1:length(varargin)
    (n,m) = size(A);
    (o,p) = size(varargin(k));
    A = [A zeros(n,p); zeros(o,m) varargin(k)];
  end

- Null subspace
  
  function Q = ker(A)
  comment: Q=ker(A) is an orthonormal basis for kerA.
  Q = ortco(A');

  function Q = ortco(A)
  comment: ORTCO Complementary orthogonalization.
  comment: Q=ortco(A) is an orthonormal basis for the orthogonal complement of imA.
  (ma,na) = size(A);
  if norm(A,'fro')<0.0001, Q = eye(ma); return, end
  (ma,na) = size(imA); RR = imA([A/norm(A,'fro'),eye(ma)],0);
  Q = RR(:,na+1:ma);
  if isempty(Q)
    Q = zeros(ma,1);
  end

  Q = zeros(ma,1);
function Q = sums(A,B)
comment: Q = sums(A,B) is an orthonormal basis for subspace im[A \ B] = imA + imB.
Q = ima([A B],0);

References

   Available: http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm


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