THE CUT SETS, DECOMPOSITION THEOREMS AND REPRESENTATION THEOREMS ON $\bar{R}$-FUZZY SETS

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Abstract. In this paper, some kinds of cut sets for $\bar{R}$-fuzzy sets are introduced and their properties are given. Moreover, the decomposition theorems and representation theorems on $\bar{R}$-fuzzy sets are obtained based on different cut sets. Finally, the axiomatic definition of cut set for $\bar{R}$-fuzzy set is introduced which characterizes the common properties of cut set.

Key Words. $\bar{R}$-fuzzy sets, Cut sets, Decomposition theorem, Representation theorem.

1. Introduction

Since the fuzzy set was introduced by Zadeh [4], many new approaches and theories treating imprecision and uncertainty have been proposed, such as type-II fuzzy sets [6], interval valued fuzzy sets [5] and so on. $L-$fuzzy sets are also expansion of ordinary fuzzy sets which membership function takes value in $L$ [7]. $\bar{R}$-fuzzy set [1], as a special $L$-fuzzy sets, is introduced by Li et al. in the study of fuzzy systems, which plays an important role in constructing fuzzy system.

In theory of fuzzy systems, the cut sets of fuzzy sets play an important role [8–13], which reveals the relationship between fuzzy fuzzy sets and classical sets. Furthermore, decomposition theorems and representation theorems can be obtained based on the cut sets. In [2], the cut set of fuzzy set can also be described by neighborhood relation of fuzzy point and fuzzy set, which has many applications in fuzzy topology and fuzzy algebra.

To best of our knowledge, the decomposition theorem and representation theorem of $\bar{R}$-fuzzy sets are not obtained in the literature. In this paper, we first introduced the cut sets of $\bar{R}$–fuzzy sets based on neighborhood relation of fuzzy point and fuzzy set. Then, we obtain decomposition theorems and representation theorems. Finally, we put forward the axiomatic definition of cut set for $\bar{R}$-fuzzy sets and discuss its properties.

The organization of this paper is as follows. In section 2, we give some notations and definitions. In section 3, the properties of cut sets of $\bar{R}$-fuzzy sets are given. In section 4, the decomposition theorems are obtained. In section 5, the representation theorems are gained. In section 6, the axiomatic definition of cut set for $\bar{R}$-fuzzy sets is put forward and the properties are also given.

2. Preliminaries

Throughout this paper, we use $X$ to denote the universal set, $P(X) = \{A|A \subseteq X\}$, $R$ to stand for real numbers set, and $\bar{R} = R \cup \{-\infty, +\infty\}$. For $A, B \in P(X)$, $A \cup B = \{x \in X|x \in A \text{ or } x \in B\}$, $A \cap B = \{x \in X|x \in A \text{ and } x \in B\}$, and $A^c = \{x \in X|x \notin A\}$. $A \subseteq B \iff \text{for } x \in X, x \in A \implies x \in B$, and

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$A \subset B \iff A \subseteq B$ and $A \neq B$. For $a, b \in \bar{R}$, $a \lor b = \sup \{a, b\}$, $a \land b = \inf \{a, b\}$, and $a^c = -a$. It is easy to verify that $(\bar{R}, \lor, \land, c)$ is fuzzy lattice. We use $T$ to stand for the index set.

The $\bar{R}$-fuzzy set [1] is introduced by Li et al. in the study of generalized fuzzy systems and its definition is as follows:

**Definition 2.1** [1]. The generalized real function $A : X \to \bar{R}$ is called $\bar{R}$-fuzzy set over $X$, and $\bar{R}^X$ stands for the set of all $\bar{R}$-fuzzy sets over $X$.

For $A, B \in \bar{R}$, we define $A \cap B, A \cup B$, and $A^c$ by the membership functions as follows:

$$A \cap B : \quad X \to \bar{R}, \quad A \cup B : \quad X \to \bar{R}, \quad A^c : \quad X \to \bar{R}$$

We also define $A \subseteq B$ by $A(x) \leq B(x)$ for all $x \in X$.

In [2], Yuan has introduced some kinds of cut sets for ordinary fuzzy sets based on the relation of fuzzy point and fuzzy set. Similarly, some different kinds of the cut sets for $\bar{R}$-fuzzy sets can be obtained as follows: For $A \in \bar{R}^X$ and $a \in \bar{R}$,

$$\begin{align*}
(1) & : A_a = \{ x \in X | A(x) \geq a \}, \quad A_a = \{ x \in X | A(x) > a \}, \\
(2) & : A[a] = \{ x \in X | a + A(x) \geq 0 \}, \quad A[a] = \{ x \in X | a + A(x) > 0 \}, \\
(3) & : A^a = \{ x \in X | A(x) \leq a \}, \quad A^a = \{ x \in X | A(x) < a \}, \\
(4) & : A[a] = \{ x \in X | a + A(x) \leq 0 \}, \quad A[a] = \{ x \in X | a + A(x) < 0 \}.
\end{align*}$$

3. The properties of cut sets of $\bar{R}$-fuzzy sets

In this section, we discuss the properties of cut sets for $\bar{R}$-fuzzy sets. We obtain the following properties.

**Property 3.1.** For $A \in \bar{R}^X$ and $a \in \bar{R}$, we have

$$\begin{align*}
(1) & : A_a = A_{a^c}, A_{\bar{a}} = A_{a^c} ; \quad (2) & : A[a] = A^c, A[\bar{a}] = A_{a^c} ; \\
(3) & : A^a = A_{\bar{a}}, A^a = A_{a^c} ; \quad (4) & : A[a] = A_{a^c}, A[\bar{a}] = A_{a^c}.
\end{align*}$$

**Proof.** According to (1), the proof is obvious.

**Property 3.2.** Let $A, B \in \bar{R}^X$ and $a \in \bar{R}$, then

$$\begin{align*}
(1) & : A_b \subseteq A_{\bar{b}}, \\
(2) & : a \leq b \implies A_a \supseteq A_b, A_{\bar{a}} \supseteq A_{\bar{b}}, \\
(3) & : A \subseteq B \implies A_a \subseteq A_{\bar{b}}, A_{\bar{a}} \subseteq B_{\bar{b}}, \\
(4) & : (A^c)_a = (A_{\bar{a}})^c, (A_{\bar{a}})^c = (A^c)_a, \\
(5) & : (A \cup B)_a = A_a \cup B_a, (A \cup B)_{\bar{a}} = A_{\bar{a}} \cup B_{\bar{a}}, (A \cap B)_a = A_a \cap B_a, (A \cap B)_{\bar{a}} = A_{\bar{a}} \cap B_{\bar{a}}, \\
(6) & : \bigcup_{t \in T} (A_t)_a \subseteq \bigcup_{t \in T} A_t, \bigcup_{t \in T} A_t \subseteq \bigcup_{t \in T} (A_t)_a, \bigcap_{t \in T} (A_t)_a \subseteq \bigcap_{t \in T} A_t, \bigcap_{t \in T} (A_t)_{\bar{a}} \subseteq \bigcap_{t \in T} A_t \subseteq \bigcup_{t \in T} (A_t)_a, \bigcup_{t \in T} A_t \subseteq \bigcup_{t \in T} (A_t)_{\bar{a}}, \\
(7) & : A_{a \lor b} = A_a \lor A_b, A_{a \land b} = A_a \land A_b, A_{a \land b} = A_a \lor A_b, A_{a \lor b} = A_a \land A_b, \\
(8) & : a_t \in \bar{R}(t \in T), a = \bigcap_{t \in T} a_t \text{ and } b = \bigcup_{t \in T} a_t, \text{ then} \\
A_b = \bigcap_{t \in T} A_{a_t}, A_{\bar{b}} \subseteq \bigcap_{t \in T} A_{\bar{a}_t}, A_a \supseteq \bigcup_{t \in T} A_{a_t}, A_{\bar{a}} = \bigcup_{t \in T} A_{\bar{a}_t}, \\
(9) & : A_{-\infty} = X, A_{\infty} = \emptyset.
\end{align*}$$

**Proof.** (1)-(5) is obvious.
(6) (i) $\forall x \in \bigcup_{t \in T} (A_t)_a \exists t \in T$ such that $x \in (A_t)_a$. It follows that $(\bigcup_{t \in T} A_t)(x) \geq A_t(x) \geq a$, that is $x \in (\bigcup_{t \in T} A_t)_a$. Hence, $\bigcup_{t \in T} (A_t)_a \subseteq (\bigcup_{t \in T} A_t)_a$.

(ii) $\forall x \in \bigcup_{t \in T} (A_t)_b \exists t \in T$ such that $x \in (A_t)_b$. It follows that $(\bigcup_{t \in T} A_t)(x) \geq A_t(x) > a$, that is $x \in (\bigcup_{t \in T} A_t)_b$. Hence, $\bigcup_{t \in T} (A_t)_b \subseteq (\bigcup_{t \in T} A_t)_b$. On the other hand, for $x \in (\bigcup_{t \in T} A_t)_b$, we have $(\bigvee_{t \in T} A_t)(x) > a$. It follows that there exists a $t \in T$ such that $A_t(x) > a$, namely, $x \in (A_t)_b \subseteq (\bigcup_{t \in T} A_t)_b$. So we have $\bigcup_{t \in T} (A_t)_b = (\bigcup_{t \in T} A_t)_b$. Similarly, we can prove $(\bigcap_{t \in T} A_t)_a = \bigcap_{t \in T} (A_t)_a$ and $(\bigcap_{t \in T} A_t)_b \subseteq \bigcap_{t \in T} (A_t)_b$.

(7) It is easy to verify that $A_a \lor b = \{ x | A(x) \geq (a \lor b) \} = \{ x | A(x) \geq a \text{ and } A(x) \geq b \} = A_a \cap A_b$ and $A_a \land b = \{ x | A(x) \geq (a \land b) \} = \{ x | A(x) \geq a \text{ or } A(x) \geq b \} A_a \cup A_b$. Similarly, we can prove $A_a \lor b = A_a \cap A_b$ and $A_a \land b = A_a \cup A_b$.

(8) It is easy to verify that

$$A_b = \{ x | A(x) \geq \bigvee_{t \in T} a_t \} = \bigcap_{t \in T} A_{a_t}, \quad A_b = \bigcup_{t \in T} (A_t)_a.$$  

Since $A(x) > a_t$ (for $t \in T$) $\Rightarrow$ $A(x) > \bigvee_{t \in T} a_t$ and $A(x) \geq \bigwedge_{t \in T} a_t \Rightarrow$ there exists $t \in T$ such that $A(x) \geq a_t$, we have

$$A_b \subseteq \bigcap_{t \in T} A_{a_t}, A_a \supseteq \bigcup_{t \in T} A_{a_t}.$$  

(9) is obvious.

It is easy to get the following properties by Property 3.1 and Property 3.2.

**Property 3.3.** Let $A, B \in \bar{R}^X$ and $a \in \bar{R}$, then

(1) $A^a \subseteq A^a$.

(2) $a \leq b \Rightarrow A^a \subseteq A^b$, $A^b \subseteq A^a$.

(3) $A \subseteq B \Rightarrow A^a \subseteq B^a$, $A^b \subseteq B^b$.

(4) $(A^c)^a = (A^a)^c$, $(A^c)^b = (A^b)^c$.

(5) $(A \cup B)^a = A^a \cap B^a$, $(A \cup B)^b = A^b \cap B^b$, $(A \cap B)^a = A^a \cup B^a$, $(A \cap B)^b = A^b \cup B^b$.

(6) $(\bigcup_{t \in T} A_t)^a = \bigcap_{t \in T} A_t^a$, $(\bigcup_{t \in T} A_t)^b = \bigcap_{t \in T} A_t^b$, $(\bigcap_{t \in T} A_t)^a = \bigcup_{t \in T} A_t^a$, $(\bigcap_{t \in T} A_t)^b = \bigcup_{t \in T} A_t^b$.

(7) $A_a \lor b = A^a \cup A^b$, $A_a \lor b = A^a \cup A^b$, $A_a \land b = A^a \cap A^b$, $A_a \land b = A^a \cap A^b$.

(8) Let $a_t \in \bar{R}(t \in T)$, $a = \bigwedge_{t \in T} a_t$ and $b = \bigvee_{t \in T} a_t$, then

$$A^a = \bigcap_{t \in T} A^a, \quad A^b \subseteq \bigcap_{t \in T} A^a, \quad A^b \supseteq \bigcup_{t \in T} A^a, \quad A^b = \bigcup_{t \in T} A^a.$$  

(9) $A_{-\infty} = \emptyset$, $A_{+\infty} = X$.

**Property 3.4.** Let $A, B \in \bar{R}^X$ and $a \in \bar{R}$, then

(1) $A_{[a]} \subseteq A_{[a]}$.

(2) $a \leq b \Rightarrow A_{[a]} \subseteq A_{[b]}$, $A_{[a]} \subseteq A_{[b]}$.

(3) $A \subseteq B \Rightarrow A_{[a]} \subseteq B_{[a]}$, $A_{[a]} \subseteq B_{[a]}$.

(4) $(A^c_{[a]})^c = (A^c_{[a]})^c$, $(A^c_{[a]})^c = (A^c_{[a]})^c$. 

(5) \((A \cup B)_a = A_a \cup B_a, (A \cup B)_b = A_b \cup B_b, (A \cap B)_a = A_a \cap B_a, (A \cap B)_b = A_b \cap B_b\).

(6) \(\bigcup_{t \in T} (A_t)_a \subseteq (\bigcup_{t \in T} A_t)_a, \bigcup_{t \in T} (A_t)_b = \bigcup_{t \in T} A_t = \bigcup_{t \in T} (A_t)_a, \bigcap_{t \in T} A_t \subseteq \bigcap_{t \in T} (A_t)_a\),

\((\bigcap_{t \in T} A_t)_b \subseteq \bigcap_{t \in T} (A_t)_b\).

(7) \(A_{a \lor b} = A_a \cup A_b, A_{a \land b} = A_a \cap A_b, A_{a \lor b} = A_a \cup A_b, A_{a \land b} = A_a \cap A_b\).

(8) Let \(a_t \in R(t \in T), a = \bigwedge_{t \in T} a_t\) and \(b = \bigvee_{t \in T} a_t\), then

\[A_a = \bigcap_{t \in T} A_{a_t}, A_b = \bigcap_{t \in T} A_{b_t}, A_{a \lor b} = \bigcup_{t \in T} A_{a_t}, A_{a \land b} = \bigcap_{t \in T} A_{a_t}, A_{a \lor b} = \bigcup_{t \in T} A_{a_t}, A_{a \land b} = \bigcap_{t \in T} A_{a_t}\].

(9) \(A_{-\infty} = \emptyset, A_{+\infty} = X\).

**Property 3.5.** Let \(A, B \in R^X\) and \(a \in R\), then

(1) \(A_b \subseteq A_a\).

(2) \(a \leq b \implies A_a \supseteq A_b, A_{a \lor b} \supseteq A_b\).

(3) \(A \cap B \subseteq B_a \subseteq A_a, B_b \subseteq A_b\).

(4) \((A^{\lor})_a = (A^{\lor})_b, (A^{\land})_a = (A^{\land})_b\).

(5) \((A \cup B)_a = A_a \cup B_a, (A \cup B)_b = A_b \cup B_b, (A \cap B)_a = A_a \cap B_a, (A \cap B)_b = A_b \cap B_b\).

(6) \(\bigcup_{t \in T} (A_t)_a = \bigcup_{t \in T} A_t = \bigcup_{t \in T} (A_t)_a\),

\(\bigcap_{t \in T} (A_t)_a = \bigcap_{t \in T} A_t \subseteq \bigcap_{t \in T} (A_t)_a\),

\((\bigcap_{t \in T} A_t)^a = \bigcap_{t \in T} A_t = \bigcap_{t \in T} (A_t)_a\).

(7) \(A_{a \lor b} = A_a \lor A_b, A_{a \land b} = A_a \land A_b, A_{a \lor b} = A_a \lor A_b, A_{a \land b} = A_a \land A_b\).

(8) Let \(a_t \in R(t \in T), a = \bigwedge_{t \in T} a_t\) and \(b = \bigvee_{t \in T} a_t\), then

\[A_a = \bigcap_{t \in T} A_{a_t}, A_b = \bigcap_{t \in T} A_{b_t}, A_{a \lor b} = \bigcup_{t \in T} A_{a_t}, A_{a \land b} = \bigcap_{t \in T} A_{a_t}, A_{a \lor b} = \bigcup_{t \in T} A_{a_t}, A_{a \land b} = \bigcap_{t \in T} A_{a_t}\].

(9) \(A_{-\infty} = X, A_{+\infty} = \emptyset\).

4. **The decomposition theorems of \(R\)-fuzzy sets**

In this section, the decomposition theorems of \(R\)-fuzzy sets are obtained.

For \(A \in P(X)\) and \(a \in R\), we define \(aA, a \in R^X\) as follow:

\[(aA)(x) = \begin{cases} a, x \in A; \\ -\infty, x \notin A; \\ +\infty, x \notin A. \end{cases}\]

\[(a \bullet A)(x) = \begin{cases} a, x \in A; \\ -\infty, x \notin A; \\ +\infty, x \notin A. \end{cases}\]

**Theorem 4.1**

(1) \(\bigcup_{a \in R} aA_a = A^c = \bigcap_{a \in R} a^c \bullet A_a\);

(2) \(\bigcup_{a \in R} aA_b = A^e = \bigcap_{a \in R} a^e \bullet A_b\);

(3) Let \(H : R \to P(X)\) such that \(A_b \subseteq H(a) \subseteq A_a\), then

(i) \(\bigcup_{a \in R} aH(a) = A^c = \bigcap_{a \in R} a^c \bullet H(a)\);

(ii) \(a < b(a, b \in R), \implies H(a) \supseteq H(b)\).
(iii) \( A_a = \bigcap_{\alpha < a} H(\alpha), A_b = \bigcup_{\alpha > a} H(\alpha) \)

**Proof.** (1) We have \( \bigcup_{a \in R} aA_a(x) = \bigvee_{a \in R} [aA_a(x)] = (\bigvee_{a \in R} [aA_a(x)] = A(x) \text{ and} \)
\[ (\bigcap_{a \in R} a^c \cdot A_a(x)) = \bigwedge_{a \in R} (a^c \cdot A_a(x)) = \bigwedge_{A(x) \geq a} a^c = A(x). \]

With the same reason, we can prove (2).

(3)(i) From (1), (2) and \( A_b \subseteq H(a) \subseteq A_a \), (i) of (3) is obvious. (ii)Since \( a < b \) and \( A_b \subseteq H(a) \subseteq A_a \), we have \( H(a) \supseteq A_b \subseteq A_a \supseteq H(b) \). (iii) From (8) of Property 3.2 and \( A_b \subseteq H(a) \subseteq A_a \), we have \( A_a = \bigcap_{\alpha < a} \bigcup_{\alpha < a} H(\alpha) \supseteq \bigcap_{\alpha < a} A_a = A_a \). With same reason, we can prove \( A_b = \bigcup_{\alpha > a} H(\alpha) \).

Similarly, we can obtain the following theorems:

**Theorem 4.2**
1. \( A = \bigcap_{a \in R} a \cdot A^a, A^c = \bigcup_{a \in R} a^c A^a \)
2. \( A = \bigcap_{a \in R} a \cdot A^b, A^c = \bigcup_{a \in R} a^c A^b \)
3. Let \( H : R \rightarrow P(X) \) such that \( A^b \subseteq H(a) \subseteq A^a \) then
   (i) \( A = \bigcap_{a \in R} a \cdot H(\alpha), A^c = \bigcup_{a \in R} a^c H(\alpha) \)
   (ii) \( a < b, a, b \in R \), \( a \cdot H(\alpha) \supseteq H(b) \)
   (iii) \( A^a = \bigcap_{a \in R} H(\alpha), A^a = \bigcup_{a \in R} H(\alpha) \)

**Theorem 4.3**
1. \( A = \bigcup_{a \in R} a^c A_{[a]} A^c = \bigcap_{a \in R} a \cdot A_{[a]} \)
2. \( A = \bigcup_{a \in R} a^c A_{[a]} A^c = \bigcap_{a \in R} a \cdot A_{[a]} \)
3. Let \( H : R \rightarrow P(X) \) such that \( A_{[a]} \subseteq H(a) \subseteq A_{[a]} \), then
   (i) \( A = \bigcup_{a \in R} a^c H(\alpha), A^c = \bigcap_{a \in R} a \cdot H(\alpha) \)
   (ii) \( a < b, a, b \in R \), \( a \cdot H(\alpha) \supseteq H(b) \)
   (iii) \( A_{[a]} = \bigcap_{a \in R} H(\alpha), A_{[a]} = \bigcup_{a \in R} H(\alpha) \)

**Theorem 4.4**
1. \( A = \bigcap_{a \in R} a^c \cdot A[a], A^c = \bigcup_{a \in R} aA[a] \)
2. \( A = \bigcap_{a \in R} a^c \cdot A[a], A^c = \bigcup_{a \in R} aA[a] \)
3. Let \( H : R \rightarrow P(X) \) such that \( A[a] \subseteq H(a) \subseteq A[a] \) then
   (i) \( A = \bigcap_{a \in R} a^c \cdot H(\alpha), A^c = \bigcup_{a \in R} aH(\alpha) \)
   (ii) \( a < b, a, b \in R \), \( a \cdot H(\alpha) \supseteq H(b) \)
   (iii) \( A[a] = \bigcap_{a \in R} H(\alpha), A[a] = \bigcup_{a \in R} H(\alpha) \)
5. The representation theorems of $\bar{R}$-fuzzy sets

Let $\mathcal{W}(X) = \{H|H$ is inverse order nested set\}. The operations $\cap, \cup,$ and $^c$ on $\mathcal{W}(X)$ are defined as follows:

$$\bigcap_{t \in T} H_t : (\bigcap_{t \in T} H_t)(\lambda) = \bigcap_{t \in T} H_t(\lambda)$$

(2) $$\bigcup_{t \in T} H_t : \bigcup_{t \in T} H_t(\lambda) = \bigcup_{t \in T} H_t(\lambda)$$

$$H^c : H^c(\lambda) = (H(\lambda))^c$$

Then $(\mathcal{W}(X), \cup, \cap, ^c)$ is fuzzy lattice.

We can get the representation theorems as follows:

**Theorem 5.1**

Let $T : \mathcal{W}(X) \rightarrow \bar{R}^X$, $H \mapsto T(H) = \bigcup_{\lambda \in R} \lambda H(\lambda)$, then,

1. $T(H)_{\Delta} \subseteq H(\lambda) \subseteq T(H)_{\lambda}$.
2. $T(H)_{\lambda} = \bigcap_{a < \lambda} H(a)$, $T(H)_{\Delta} = \bigcup_{a < \lambda} H(a)$

3. $T$ is a surjective homomorphism from $\mathcal{W}(X)$ to $\bar{R}^X$, that is $T$ is surjection and $T\big(\bigcup_{t \in T} H_t\big) = \bigcup_{\lambda \in R} \lambda T(H_t)$, $T\big(\bigcap_{t \in T} H_t\big) = \bigcap_{\lambda \in R} \lambda T(H_t)$, $T(H^c) = (T(H))^c$.

**Proof.** (1) For $\forall x \in T(H)_{\Delta}$, we have $T(H)(x) = \bigvee_{\lambda \in R} \lambda H(\lambda)(x) > \lambda$. It follows that $\exists \lambda_0 > \lambda$ such that $x \in H(\lambda_0) \subseteq H(\lambda)$. On the other hand, $H(\lambda) \subseteq T(H)_{\lambda}$ is obvious. So we have $T(H)_{\lambda} \subseteq H(\lambda) \subseteq T(H)_{\Delta}$.

(2) From (8) of Property 3.2 and (1), we have $T(H)_{\lambda} = \bigcap_{a < \lambda} (T(H)_a) \supseteq \bigcap_{a < \lambda} H(a) \supseteq \bigcap_{a < \lambda} H(\lambda)_{\Delta} = T(H)_{\lambda}$. With the same reason, we have $T(H)_{\Delta} = \bigcup_{\lambda \in R} H(a)$.

(3) For $\forall A \in \bar{R}^X$, let $H_A : \bar{R} \rightarrow P(X)$ be defined by $H_A(a) = A_a$, known by (1) of Property 3.2 and (1) of Theorem 3.1, $H_A$ is inverse order nested set and $T(H_A) = \bigcup_{a \in R} a A_a = \bigcup_{a \in R} a H_A(a)$. Hence $T$ is surjection. $T\big(\bigcup_{t \in T} H_t\big) = \bigcup_{\lambda \in R} \lambda \big(\bigcup_{t \in T} H_t(\lambda)\big) = \bigcup_{\lambda \in R} \lambda T(H_t)$, $T\big(\bigcap_{t \in T} H_t\big) = \bigcap_{\lambda \in R} \lambda T(H_t)$. With the same reason, we have $T\big(\bigcap_{t \in T} H_t\big) = \bigcap_{\lambda \in R} T(H_t)_{\lambda}$ and $T(H^c) = (T(H))^c$.

Similarly, we can get the following theorems.

Let $\mathcal{W}(X) = \{H|H$ is order nested set\}. The operations $\cap, \cup,$ and $^c$ are same as (2), then $(\mathcal{W}(X), \cap, \cup, ^c)$ is also fuzzy lattice. We also can get the following theorem.

**Theorem 5.2** Let $T : \mathcal{W}(X) \rightarrow \bar{R}^X$, $H \mapsto T(H) = \bigcup_{\lambda \in R} \lambda^c H(\lambda)$. Then,

1. $T(H)_{\Delta} \subseteq H(\lambda) \subseteq T(H)_{\lambda}$.
2. $T(H)_{\lambda} = \bigcap_{a \in R} H(a)$, $T(H)_{\Delta} = \bigcup_{a \in R} H(a)$

3. $T$ is a surjective homomorphism from $\mathcal{W}(X)$ to $\bar{R}^X$, that is $T$ is surjection and $T\big(\bigcup_{t \in T} H_t\big) = \bigcup_{\lambda \in R} T(H_t)$, $T\big(\bigcap_{t \in T} H_t\big) = \bigcap_{\lambda \in R} T(H_t)$, $T(H^c) = (T(H))^c$. 


However, if the operations $\bigcap, \bigcup$, and $^c$ on $\mathcal{F}(x)$ is defined as

\[
\bigcap_{t \in T} H_t : \bigcap_{t \in T} H_t(\lambda) = \bigcup_{t \in T} H_t(\lambda)
\]

(3)

\[
\bigcup_{t \in T} H_t : \bigcup_{t \in T} H_t(\lambda) = \bigcap_{t \in T} H_t(\lambda)
\]

\[
H^c : \quad H_t(\lambda) = (H(\lambda))^c,
\]

then, $(\mathcal{F}(X), \bigcap, \bigcup, ^c)$ is also a fuzzy lattice, and we can get the following theorem:

**Theorem 5.3** Let $T : \mathcal{F}(X) \rightarrow \bar{R}^X$, $H \mapsto T(H) = \bigcap_{\lambda \in R} H(\lambda)$. Then,

1. $T(H) \subseteq H(\lambda) \subseteq T(H)^\lambda$.
2. $T(H)^\lambda = \bigcap_{\lambda \leq a} H(a)$.
3. $T$ is a surjective homomorphism from $\mathcal{F}(X)$ to $\bar{R}^X$, that is $T$ is surjection and $T(\bigcup_{t \in T} H_t) = \bigcup_{t \in T} T(H_t)$, $T(\bigcap_{t \in T} H_t) = \bigcap_{t \in T} T(H_t)$, $T(H^c) = (T(H))^c$.

If the operations $\bigcap, \bigcup$, and $^c$ on $\mathcal{F}(X)$ are same as (3), then $(\mathcal{F}(X), \bigcap, \bigcup, ^c)$ is also a fuzzy lattice and we have the following theorem.

**Theorem 5.4** Let $T : \mathcal{F}(X) \rightarrow \bar{R}^X$, $H \mapsto T(H) = \bigcap_{\lambda \in R} H(\lambda)$. Then,

1. $T(H)^\lambda \subseteq H(\lambda) \subseteq T(H)^\lambda$.
2. $T(H)^\lambda = \bigcap_{\lambda \leq a} H(a)$.
3. $T$ is a surjective homomorphism from $\mathcal{F}(X)$ to $\bar{R}^X$, that is $T$ is surjection and $T(\bigcup_{t \in T} H_t) = \bigcup_{t \in T} T(H_t)$, $T(\bigcap_{t \in T} H_t) = \bigcap_{t \in T} T(H_t)$, $T(H^c) = (T(H))^c$.

### 6. The Axiomatic Definition of the Cut Sets

The axiomatic definition of cut sets for ordinary fuzzy sets is introduced by Yuan and Wu [3] and the common properties of cut sets are discussed. Similarly, we give axiomatic definition of $\bar{R}$-fuzzy sets and discuss its properties.

For $x \in X$, $A \in \bar{R}^X$ and $\lambda \in \bar{R}$, we define $\lambda A, \lambda \circ A, \lambda \bullet A, \lambda \Delta A \in \bar{R}^X$ as follows:

For $x \in X$,

\[
(\lambda A)(x) = \lambda \wedge A(x), \quad \lambda \circ A(x) = \lambda \vee A(x);
\]

\[
(\lambda \bullet A)(x) = \lambda \vee (A(x))^c, \quad (\lambda \Delta A)(x) = \lambda \wedge (A(x))^c.
\]

**Definition 6.1.** The function $f : \bar{R} \times \bar{R}^X \rightarrow P(X)$, $(a, A) \mapsto f(a, A)$ is called $f$-cut sets of $A$.

**Theorem 6.1.** If the $f$-cut set $f$ satisfies the following conditions

1. $f(\lambda, \bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(\lambda, A_t)$;
2. $f(\lambda, A) = A$, for $A \in P(X)$ and $\lambda < +\infty$;
3. $f(\lambda, A)(a) = \begin{cases} \emptyset, & a \leq \lambda; \\ f(\lambda, A), & a > \lambda \end{cases}$ for $A \in \bar{R}^X$ and $a, \lambda \in \bar{R}$.

then, $f(\lambda, A) = A_\lambda$ for $\forall \lambda \in \bar{R}$ and $A \in \bar{R}^X$. 
Proof. Case 1: $\lambda < +\infty$. We have
\[ f(\lambda, A) = f(\lambda, \bigcup_{a \in R} aA_\emptyset) = \bigcup_{a > \lambda} f(\lambda, aA_\emptyset) = \bigcup_{a > \lambda} f(\lambda, A_{\emptyset}) = A_{\emptyset}. \]

Case 2: $\lambda = +\infty$. Let $a = +\infty$, then it is easy to verify that $f(\lambda, A) = \emptyset = A_{\emptyset}$.

From case 1 and case 2, we have $f(\lambda, A) = A_{\emptyset}$ for $\forall \lambda \in \hat{R}$, $A \in \hat{R}^X$.

Theorem 6.2. If the $f$-cut set $f$ satisfies the following conditions
(1) $f(\lambda, \bigcap_{t \in T} A_t) = \bigcap_{t \in T} f(\lambda, A_t)$;
(2) $f(\lambda, A) = A$, for $A \in P(X)$ and $\lambda > -\infty$;
(3) $f(\lambda, a \circ A) = \left\{ X, \begin{cases} a > \lambda; & f(\lambda, A), \quad a < \lambda. \end{cases} \right.$

then, $f(\lambda, A) = A_\lambda$ for $\forall \lambda \in \hat{R}$ and $A \in \hat{R}^X$.

Proof. Case 1: $\lambda > -\infty$. We have
\[ f(\lambda, A) = f(\lambda, \bigcap_{a \in R} a \circ A_\emptyset) = f(\lambda, a \circ A_\emptyset) = \bigcup_{a > \lambda} f(\lambda, A_{\emptyset}) = A_{\emptyset}. \]

Case 2: $\lambda = -\infty$. Let $a = -\infty$, then it is easy to verify that $f(\lambda, A) = f(\lambda, a \circ A) = X = A_\lambda.$

Theorem 6.3. If the $f$-cut set $f$ satisfies the following conditions
(1) $f(\lambda, \bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(\lambda, A_t)$;
(2) $f(\lambda, A) = A$, for $A \in P(X)$ and $\lambda > -\infty$;
(3) $f(\lambda, a \circ A) = \left\{ X, \begin{cases} a > -\lambda; & f(\lambda, A), \quad a < -\lambda. \end{cases} \right.$

then, $f(\lambda, A) = A_\lambda$ for $\forall \lambda \in \hat{R}$ and $A \in \hat{R}^X$.

Proof. Case 1: $\lambda > -\infty$. We have
\[ f(\lambda, A) = f(\lambda, \bigcup_{a \in R} a^{\circ} A_{[\emptyset]}) = f(\lambda, a^{\circ} A_{[\emptyset]}) = \bigcup_{a > \lambda} f(\lambda, A_{[\emptyset]}) = A_{[\emptyset]}. \]

Case 2: $\lambda = -\infty$. Let $a = +\infty$, then it is easy to verify that $f(\lambda, A) = \emptyset = A_{[\emptyset]}.$

From case 1 and case 2, we have $f(\lambda, A) = A_{[\emptyset]}$ for $\forall \lambda \in \hat{R}$, $A \in \hat{R}^X$.

Theorem 6.4. If the $f$-cut set $f$ satisfies the following conditions
(1) $f(\lambda, \bigcap_{t \in T} A_t) = \bigcap_{t \in T} f(\lambda, A_t)$;
(2) $f(\lambda, A) = A$, for $A \in P(X)$ and $\lambda < +\infty$;
(3) $f(\lambda, a \circ A) = \left\{ X, \begin{cases} a \geq -\lambda; & f(\lambda, A), \quad a < -\lambda. \end{cases} \right.$

then, $f(\lambda, A) = A_\lambda$ for $\forall \lambda \in \hat{R}$ and $A \in \hat{R}^X$. 
for

then,\( f(\lambda, A) = \bigcup_{\alpha \in R} f(\lambda, A_{\alpha}) \) for \( \lambda < +\infty \).

**Theorem 6.6.** If the \( f \)-cut set \( f \) satisfies the following conditions

1. \( f(\lambda, \bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(\lambda, A_t) \);
2. \( f(\lambda, A) = A^c \), for \( A \in P(X) \) and \( \lambda < +\infty \);
3. \( f(\lambda, a \bullet A) = \left\{ \begin{array}{ll}
X, & a \leq \lambda; \\
\bigcap_{t \in T} f(\lambda, A_t), & a > \lambda;
\end{array} \right. \) for \( \forall A \in \bar{R}^X \) and \( a, \lambda \in \bar{R} \).

then, \( f(\lambda, A) = A^\lambda \) for \( \forall A \in \bar{R} \) and \( A \in \bar{R}^X \).

**Proof.** Case 1: \( \lambda < +\infty \). We have

\[
f(\lambda, A) = f(\lambda, \bigcup_{a \in R} a \bullet A_{\alpha}) = \bigcup_{a \in R} f(\lambda, a \bullet A_{\alpha}) = \bigcup_{a < \lambda} f(\lambda, A_{\alpha})
\]

\[
= \bigcap_{a < \lambda} A_{\alpha} = A_{\bigcap a = A_{|\lambda|}}.
\]

Case 2: \( \lambda = +\infty \). Let \( a = -\infty \), then it is easy to verify that \( f(\lambda, A) = f(\lambda, A_{\alpha}) = X = A_{|\lambda|} \).

From case 1 and case 2, we have \( f(\lambda, A) = A_{|\lambda|} \) for \( \forall \lambda \in \bar{R}, A \in \bar{R}^X \).

**Theorem 6.5.** If the \( f \)-cut set \( f \) satisfies the following conditions

1. \( f(\lambda, \bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(\lambda, A_t) \);
2. \( f(\lambda, A) = A^c \), for \( A \in P(X) \) and \( \lambda > -\infty \);
3. \( f(\lambda, a \bullet A) = \left\{ \begin{array}{ll}
\emptyset, & a \geq \lambda; \\
f(\lambda, A^c), & a < \lambda.
\end{array} \right. \) for \( \forall A \in \bar{R}^X \) and \( a, \lambda \in \bar{R} \).

then, \( f(\lambda, A) = A^\lambda \) for \( \forall A \in \bar{R} \) and \( A \in \bar{R}^X \).

**Proof.** Case 1: \( \lambda > -\infty \). We have

\[
f(\lambda, A) = f(\lambda, \bigcup_{a \in R} a \bullet A_{\alpha}) = \bigcup_{a \in R} f(\lambda, a \bullet A_{\alpha}) = \bigcup_{a < \lambda} f(\lambda, (A_{\alpha})^c)
\]

\[
= \bigcup_{a < \lambda} (A_{\alpha})^c = A_{\bigcup a = A_{|\lambda|}}.
\]

Case 2: \( \lambda = -\infty \). Let \( a = -\infty \), then it is easy to verify that \( f(\lambda, A) = \emptyset = A_{|\lambda|} \).

From case 1 and case 2, we have \( f(\lambda, A) = A_{|\lambda|} \) for \( \forall \lambda \in \bar{R}, A \in \bar{R}^X \).

**Theorem 6.7.** If the \( f \)-cut set \( f \) satisfies the following conditions

1. \( f(\lambda, \bigcup_{t \in T} A_t) = \bigcup_{t \in T} f(\lambda, A_t) \);
2. \( f(\lambda, A) = A^c \), for \( A \in P(X) \) and \( \lambda < +\infty \);
3. \( f(\lambda, a \bullet A) = \left\{ \begin{array}{ll}
\emptyset, & a \geq -\lambda; \\
f(\lambda, A^c), & a < -\lambda.
\end{array} \right. \) for \( \forall A \in \bar{R}^X \) and \( a, \lambda \in \bar{R} \).

then, \( f(\lambda, A) = A^\lambda \) for \( \forall A \in \bar{R} \) and \( A \in \bar{R}^X \).
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