ON THE SUPERCONVERGENCE OF BILINEAR FINITE ELEMENT FOR SECOND ORDER ELLIPTIC PROBLEMS WITH GENERAL BOUNDARY CONDITIONS

MINGXIA LI AND SHIPENG MAO

Abstract. In this paper, the superconvergence of bilinear finite element for general second order elliptic problems with general boundary conditions is studied to avoid a sub-optimal superconvergence, we construct an auxiliary equation to eliminate the effect of boundary terms. As a result, the $O(h^2)$ superconvergence rate is obtained under the regularity assumption that $u \in H^3(\Omega)$ when almost uniform rectangular meshes is employed, which improves the previous results.

Key Words. finite element methods; general boundary conditions; superconvergence

1. Introduction

Since Oganesyan and Rukhovetz [15] first proved the superconvergence for linear element approximations on a uniform triangle mesh in 1969, superconvergence for the finite element solutions has been an active research area in numerical analysis for more than thirty years. The main target in the superconvergence study is to improve the existing approximation accuracy by applying certain postprocessing techniques which are easy to implement. Roughly speaking, the superconvergence analysis has developed into several different methods in recent years, for example: superconvergence on locally point-symmetric meshes [1, 17, 18]; element analysis method [7]; error expansions with special integral identities [11, 12]; the Zienkiewicz-Zhu gradient patch recovery method (SPR) [10, 21, 22, 23, 24]; $L^2$ projection method [19, 20]; computer based methods and etc., interested readers are referred to [2, 8, 9] for the surveys on the development and various results of superconvergence. Furthermore, superconvergence is also recognized as a useful tool in a posterior error estimations, mesh refinement and adaptivity.

In this paper, we aim to give a minor complementarity of the superconvergence of the lowest order rectangular element, i.e, the conforming bilinear element. It is known that bilinear finite element approximation is often used for it’s simplicity and nice structure accompanied with high accuracy. Its superconvergence for the second order elliptic problems have been obtained by many authors (for example, in [7, 11, 12, 18] and etc.). When the exact solution $u \in H^3(\Omega)$, $O(h^2)$ superconvergence rate on uniform rectangular meshes can be proved for pure Dirichlet conditions; however, as to other boundary conditions, for example, the Neumann or the Robin boundary conditions, only $O(h^{3/2})$ superconvergence rate has been proved if under the same assumptions. The object of this paper is to improve the latter result when general boundary conditions are considered. Generally, after the process of
asymptotic expansions, there will appear some residual boundary terms which can not be eliminated. The classical technique is to transform these boundary terms to the whole domain by the trace like theorems, which results in a non-optimal $O(h^3)$ convergence rate under $H^3$ regularity (quasi-optimal convergence rate $O(h^2|\ln h|^{\frac{1}{2}})$ can be proved if the solution is more smoother, i.e., $u \in H^4(\Omega)$, cf. [11, 12]). We overcome this drawback by introducing an auxiliary equation to convert the residual boundary terms into the integral on whole domain, and optimal convergence rate of $O(h^2)$ can be obtained only with $H^3$ regularity on almost uniform rectangular meshes.

The rest of this paper is organized as follows. In Section 2, after introducing the general second order elliptic problems and some notations, we are dedicated the superconvergence of bilinear finite element approximation based on the work in [11, 12]. In Section 3, we carry out a numerical test to verify our result.

2. Superconvergence for the general boundary conditions

Let $\Omega$ be a convex polygonal domain in the $xy$-plane. Throughout this paper, we will use the standard Sobolev space $W^{m,p}(\Omega)$ ($m \in \mathbb{N}$) with the norm and the seminorm (see, for example, [3]) given by

$$\|\psi\|_{m,p,\Omega} = \left( \sum_{0 \leq |\alpha| \leq m} \int_{\Omega} |D^\alpha \psi|^p \, dx \, dy \right)^{\frac{1}{p}},$$

and

$$|\psi|_{m,p,\Omega} = \left( \sum_{|\alpha| = m} \int_{\Omega} |D^\alpha \psi|^p \, dx \, dy \right)^{\frac{1}{p}},$$

respectively. As usual, we will drop the index $p$ when it is 2, and write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The two-dimensional general second order elliptic problem with general boundary conditions can be written as: find $u$, such that

$$\begin{cases} -\text{div}(A \nabla u) + \beta \cdot \nabla u + \gamma u = f, & \text{in } \Omega, \\ u = g_1, & \text{on } \Gamma_1, \\ A \nabla u \cdot n = g_2, & \text{on } \Gamma_2, \\ A \nabla u \cdot n + \sigma u = g_3, & \text{on } \Gamma_3, \end{cases} \quad (2.1)$$

where

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

$$\beta = (\beta_1, \beta_2),$$

$\alpha_{ij}$ ($i,j = 1, 2$), $\beta_i$ ($i = 1, 2$), $\gamma, \sigma$ are given functions, $\alpha_{12} = \alpha_{21}$, and $A$ is uniformly elliptic, i.e.,

$$\sum_{i,j=1}^{2} \alpha_{i,j} \xi_i \xi_j \geq \theta |\xi|^2, \quad \forall \xi \in \mathbb{R}^2 \text{ a.e. in } \Omega, \quad (2.2)$$

where $\theta$ is a positive constant. In this paper, we assume $\alpha_{i,j}$ ($i,j = 1, 2$), $\beta_i$ ($i = 1, 2$) $\in W^{1,\infty}(\Omega)$ and $\gamma, \sigma \in L^{\infty}(\Omega)$.

For the purpose of this paper, we assume $\Omega$ to be a rectangular domain. The right, top, left and bottom boundaries are denoted by $\partial \Omega_i$ ($i = 1, \cdots, 4$), respectively (as shown in Figure 2.1).
Let $T_h$ be a rectangular partition of $\Omega$ and $e \in T_h$ an arbitrary element with the edges

\begin{align*}
  l_{e,1} &= \{ (x, y) : x = x_e + h_e, y_e - k_e \leq y \leq y_e + k_e \}; \\
  l_{e,2} &= \{ (x, y) : x_e - h_e \leq x \leq x_e + h_e, y = y_e + k_e \}; \\
  l_{e,3} &= \{ (x, y) : x = x_e - h_e, y_e - k_e \leq y \leq y_e + k_e \}; \\
  l_{e,4} &= \{ (x, y) : x_e - h_e \leq x \leq x_e + h_e, y = y_e - k_e \}.
\end{align*}

Assume

\begin{align*}
  V &= H^1(\Omega), \quad V_0 = H^1_{\Gamma_1}(\Omega) = \{ v \in H^1(\Omega), v = 0 \text{ on } \Gamma_1 \}, \\
  V_h &= \{ v_h \in C(\bar{\Omega}), v_h|_e \in Q_1(e), \forall e \in T_h \}, \quad \text{and} \quad V_{h0} = \{ v_h \in V_h, v_h|_{\Gamma_1} = 0 \}.
\end{align*}

Then the variational form of (2.1) is: find $u \in V$, such that

\begin{align*}
  \int_\Omega \left( \sum_{i,j=1}^2 \alpha_{ij} \partial_i u \partial_j v + \sum_{i=1}^2 \beta_i \partial_i u v + \gamma u v \right) \, dx \, dy &+ \int_{\Gamma_3} \sigma u v \, ds \\
  = \int_\Omega f v \, dx \, dy + \int_{\Gamma_2} g_2 v \, ds + \int_{\Gamma_3} g_3 v \, ds, \quad \forall \ v \in V_0, \tag{2.3}
\end{align*}

where $\partial_1 = \frac{\partial}{\partial x}$, and $\partial_2 = \frac{\partial}{\partial y}$.

The associated discrete problem is: find $u_h \in V_h$, such that

\begin{align*}
  \int_\Omega \left( \sum_{i,j=1}^2 \alpha_{ij} \partial_i u_h \partial_j v_h + \sum_{i=1}^2 \beta_i \partial_i u_h v_h + \gamma u_h v_h \right) \, dx \, dy &+ \int_{\Gamma_3} \sigma u_h v_h \, ds \\
  = \int_\Omega f v_h \, dx \, dy + \int_{\Gamma_2} g_2 v_h \, ds + \int_{\Gamma_3} g_3 v_h \, ds, \quad \forall \ v_h \in V_{h0}. \tag{2.4}
\end{align*}
We define the average function of \( \alpha_{ij} \) as

\[
\overline{\alpha_{ij}}_e = \frac{\int_e \alpha_{ij}}{|e|},
\]

where \(|e|\) is the area of the element \(e\), and

\[
|\alpha_{ij} - \overline{\alpha_{ij}}| \leq Ch \|\alpha_{ij}\|_{1,\infty}.
\]

Let \( h = \max_e \{k_e, h_e\} \). Further suppose the mesh satisfy the almost uniform rectangular mesh in the following sense:

\[
h_e - h = O(h^{\frac{3}{2}}), \quad k_e - h = O(h^{\frac{3}{2}}).
\]

Let \( u_I \) denote the bilinear interpolation of \( u \), and set \( w = u - u_I \). Based on the result of [12], there holds

\[
\int_e \partial_i w \partial_i v_h \ dx dy = O(h^2)\|u\|_{3,e}\|v_h\|_{1,e}, \quad i = 1, 2.
\]

So we have

\[
\int \Omega \alpha_{ij} \partial_i w \partial_i v_h \ dx dy = \sum_{e \in T_h} \int_e \alpha_{ij} \partial_i w \partial_i v_h \ dx dy + \int \Omega (\alpha_{ij} - \overline{\alpha_{ij}}) \partial_i w \partial_i v_h \ dx dy
\]

\[
= \sum_{e \in T_h} O(h^2)\|u\|_{3,e}\|v_h\|_{1,e} + O(h)\|w\|_{1,\Omega}\|v_h\|_{1,\Omega}
\]

\[
= O(h^2)\|u\|_{3,\Omega}\|v_h\|_{1,\Omega}, \quad i = 1, 2.
\]

**Theorem 1.** Assume \( V_h \) is the bilinear finite element space, and \( \alpha_{i,j} \ (i, j = 1, 2) \in W^{1,\infty}(\Omega) \), we have

\[
\int \Omega (\alpha_{12} w_x v_{hy} + \alpha_{21} w_y v_{hx}) \ dx dy = O(h^2)\|u\|_{3,\Omega}\|v_h\|_{1,\Omega}, \quad \forall \ v_h \in V_h.
\]

**Proof.** It has been proved in [11, 12] that

\[
\int \Omega w_x v_{hy} \ dx dy = \frac{1}{3} \sum_{e \in T_h} h_e^2 \int_e u_{xx} v_{hxy} \ dx dy + O(h^2)\|u\|_{3,\Omega}\|v_h\|_{1,\Omega}, \quad \forall \ v_h \in V_h.
\]

An application of integrating by parts gives

\[
\int \Omega w_x v_{hy} \ dx dy = \frac{1}{3} \sum_{e \in T_h} h_e^2 \left\{ - \int_e u_{xx} v_{hxy} \ dx dy + \left( \int_{l_2} - \int_{l_4} \right) u_{xx} v_{hx} \ dx \right\} + O(h^2)\|u\|_{3,\Omega}\|v_h\|_{1,\Omega}.
\]

Noticing that for any two adjacent elements \( e_k \) and \( e_l \),

\[
|\alpha_{ij}|_{e_k} - |\alpha_{ij}|_{e_l} = O(h),
\]

and invoking the following inequality (cf. [12]),

\[
\|v_h\|_{1,\partial \Omega} \leq C h^{-\frac{1}{2}} \|v_h\|_{1,\Omega},
\]

we have
by (2.5) and (2.9), it can be derived that
\[ \int_\Omega \alpha_{12} w_x v_{hy} \, dxdy \]
\[ = \sum_{e \in T_h} \int_\Omega \alpha_{12}^e w_x v_{hy} \, dxdy + \int_\Omega (\alpha_{12} - \alpha_{12}^e) w_x v_{hy} \, dxdy \]
\[ = \sum_{e \in T_h} O(h_x^2) \|u_{3,e}\| v_{1,e} + h \sum_{e \in T_h} O(h_x^2) \|u_{2,e}\| v_{1,e} \]
\[ + O(h^2) \left( \int_{\partial\Omega_2} - \int_{\partial\Omega_4} \right) \alpha_{12} u_{xx} v_{hx} \, dx + O(h) \|u_{1,\Omega}\| v_{1,\Omega} \]  \hspace{1cm} (2.10)
\[ = O(h^2) \|u_{3,\Omega}\| v_{1,\Omega} + O(h^2) \left( \int_{\partial\Omega_2} - \int_{\partial\Omega_4} \right) (\alpha_{12} - \alpha_{12}) u_{xx} v_{hx} \, dx \]
\[ + O(h^2) \left( \int_{\partial\Omega_2} - \int_{\partial\Omega_4} \right) \alpha_{12} u_{xx} v_{hx} \, dx \]
\[ = O(h^2) \|u_{3,\Omega}\| v_{1,\Omega} + O(h^2) \left( \int_{\partial\Omega_2} - \int_{\partial\Omega_4} \right) \alpha_{12} u_{xx} v_{hx} \, dx. \]

By the same argument, we have
\[ \int_\Omega \alpha_{21} w_y v_{hx} \, dxdy = O(h^2) \|u_{3,\Omega}\| v_{1,\Omega} + O(h^2) \left( \int_{\partial\Omega_1} - \int_{\partial\Omega_3} \right) \alpha_{21} u_{yy} v_{hy} \, dy, \]  \hspace{1cm} (2.11)
which, together with (2.10), we can obtain
\[ \int_\Omega (\alpha_{12} w_x v_{hy} + \alpha_{21} w_y v_{hx}) \, dxdy \]
\[ = O(h^2) \|u_{3,\Omega}\| v_{1,\Omega} + O(h^2) \left( \int_{\partial\Omega_2} - \int_{\partial\Omega_4} \right) \alpha_{12} u_{xx} v_{hx} \, dx \]
\[ + O(h^2) \left( \int_{\partial\Omega_1} - \int_{\partial\Omega_3} \right) \alpha_{21} u_{yy} v_{hy} \, dy \]  \hspace{1cm} (2.12)
\[ = O(h^2) \|u_{3,\Omega}\| v_{1,\Omega} + O(h^2) \int_{\partial\Omega} \alpha_{12} u_{xx} v_{hs} \, ds, \]
where \( s \) is the unit tangential derivative along \( \partial\Omega \). Now, the key ingredient is to estimate the boundary term \( \int_{\partial\Omega} \alpha_{12} u_{xx} v_{hs} \, ds \) of (2.12). To this end, let’s consider the auxiliary equation:
\[ \begin{cases} 
-\Delta \phi = 0, \text{ in } \Omega, \\
\phi = \alpha_{12} u_{ss}, \text{ on } \partial\Omega. 
\end{cases} \]  \hspace{1cm} (2.13)
After some applications of Green’s formula, we have
\[ \int_{\partial\Omega_1} \phi v_{hs} \, ds = \int_{\partial\Omega_1} \phi \nabla v_h \cdot s \, ds \]
\[ = \int_{\partial\Omega} \phi \mathrm{curl} \, v_h \cdot n \, ds \]
\[ = \int_{\Omega} \nabla\phi \cdot v_h \, dxdy + \int_{\Omega} \phi \, \mathrm{div} \, (\mathrm{curl} \, v_h) \, dxdy \]
\[ = \int_{\Omega} \nabla\phi \cdot v_h \, dxdy, \]  \hspace{1cm} (2.14)
where
\[ \text{curl } v = \left( \frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right), \]
and \( n \) is the unit normal derivative of \( \partial \Omega \). It follows from the regularity result of problem (2.13) and the trace theorem that
\[ |\phi|_{1, \Omega} \leq C |\alpha_{12} u_{ss}|_{\frac{1}{2}, \partial \Omega} \leq C |u|_{3, \Omega}. \]  
(2.15)
Then the desired result can be obtained by a combination of (2.12) – (2.15), which completes the proof of Theorem 1.

Now, we are in the position to bound other terms. Noticing that \( \beta_i \) (\( i = 1, 2 \)) \( \in W^{1, \infty}(\Omega) \) and \( \gamma, \sigma \in L^{\infty}(\Omega) \), and by the interpolation error estimates, it is easy to verify (c.f. [12]) that \( \forall v_h \in V_h \), the following estimates hold:
\[ \int_{\Omega} \sum_{i=1}^{2} \beta_i \partial_i w v_h \, dx \, dy = - \int_{\Omega} \sum_{i=1}^{2} w \partial_i (\beta_i v_h) \, dx \, dy + \int_{\partial \Omega} w v_h \beta \cdot n \, ds = O(h^2) \|u\|_{3, \Omega} \|v_h\|_{1, \Omega}, \]
(2.16)
\[ \int_{\Omega} \gamma w v_h \, dx \, dy = O(h^2) \|u\|_{2, \Omega} \|v_h\|_{1, \Omega}, \]
(2.17)
and
\[ \int_{\Gamma_3} \sigma w v_h \, ds = O(h^2) \|u\|_{2, \Gamma_3} \|v_h\|_{0, \Gamma_3} = O(h^2) \|u\|_{3, \Omega} \|v_h\|_{1, \Omega}. \]
(2.18)
Together with (2.7) and (2.8), we obtain
\[ \int_{\Omega} \left( \sum_{i,j=1}^{2} \alpha_{ij} \partial_i (u_h - u_I) \partial_j v_h + \sum_{i=1}^{2} \beta_i \partial_i (u_h - u_I) v_h + \gamma (u_h - u_I) v_h \right) \, dx \, dy \]
\[ + \int_{\Gamma_3} \sigma (u_h - u_I) v_h \, ds \]
\[ = \int_{\Omega} \left( \sum_{i,j=1}^{2} \alpha_{ij} \partial_i w \partial_j v_h + \sum_{i=1}^{2} \beta_i \partial_i w v_h + \gamma w v_h \right) \, dx \, dy + \int_{\Gamma_3} \sigma w v_h \, ds \]
\[ = O(h^2) \|u\|_{3, \Omega} \|v_h\|_{1, \Omega}, \quad \forall v_h \in V_{h0}. \]
(2.19)
By virtue of Garding’s Inequality (cf. [3]), we know that there exists a constant \( K < \infty \) such that
\[ a(v, v) + K \|v\|_{0, \Omega}^2 \geq \frac{\theta}{2} \|v\|_{1, \Omega}^2. \]
(2.20)
where
\[ a(u, v) := \int_{\Omega} \left( \sum_{i,j=1}^{2} \alpha_{ij} \partial_i u \partial_j v + \sum_{i=1}^{2} \beta_i \partial_i u v + \gamma u v \right) \, dx \, dy + \int_{\Gamma_3} \sigma u v \, ds, \]
and \( \theta \) is the coefficient in (2.2). By (2.19) and (2.20), we have
\[ \frac{\theta}{2} \|u_h - u_I\|_{1, \Omega}^2 \]
\[ \leq a(u_h - u_I, u_h - u_I) + K \|u_h - u_I\|_{0, \Omega}^2 \]
\[ = a(u - u_I, u_h - u_I) + K \|u_h - u + u - u_I\|_{0, \Omega}^2 \]
\[ \leq Ch^2 \|u\|_{3, \Omega} \|u_h - u_I\|_{0, \Omega}. \]
Therefore the following superapproximation estimate holds:
\[
\|u_h - u_I\|_{1, \Omega} \leq C h^2 \|u\|_{3, \Omega}.
\] (2.21)

Our next goal is to construct a postprocessing operator in order to get global superconvergence. Let \( \tilde{e} = \bigcup_{i=1}^{4} e_i \) (see Figure 2.2), and \( \Pi_{2h}^{2} : C(\tilde{e}) \to Q_2(\tilde{e}) \) be defined as
\[
\Pi_{2h}^{2} w(Z_i) = w(Z_i), \quad i = 1, \cdots, 9,
\] (2.22)
where \( Z_i (i = 1, \cdots, 9) \) are the nodes of the small elements.

It can be checked that the interpolation defined as (2.22) is well-posed. Furthermore, it has the following properties (cf. [12]):
\[
\begin{align*}
\Pi_{2h}^{2} v_I & = \Pi_{2h}^{2} v, \\
\|\Pi_{2h}^{2} v_h\|_{1, \Omega} & \leq C \|v_h\|_{1, \Omega}, \quad \forall v_h \in V_h, \\
\|\Pi_{2h}^{2} v - v\|_{1, \Omega} & \leq C h^2 \|v\|_{3, \Omega},
\end{align*}
\] (2.23)
where \( v_I \in V_h \) is the bilinear interpolation of \( v \).

Then we can get the following global superconvergence theorem.

**Theorem 2.** Assume \( u \in H^3(\Omega) \cap V \), \( u_h \in V_h \) be the solution of (2.2) and (2.3), respectively. \( T_h \) is an partition of \( \Omega \) which satisfies (2.4), then the superconvergence estimate holds
\[
\|\Pi_{2h}^{2} u_h - u\|_{1, \Omega} \leq C h^2 \|u\|_{3, \Omega}.
\] (2.24)

**Proof.** A combination of superapproximation result (2.21) and the properties of \( \Pi_{2h}^{2} \) gives
\[
\begin{align*}
\|\Pi_{2h}^{2} u_h - u\|_{1, \Omega} &= \|\Pi_{2h}^{2} u_h - \Pi_{2h}^{2} u_I + \Pi_{2h}^{2} u_I - u\|_{1, \Omega} \\
&\leq \|\Pi_{2h}^{2} u_h - \Pi_{2h}^{2} u_I\|_{1, \Omega} + \|\Pi_{2h}^{2} u_I - u\|_{1, \Omega} \\
&\leq \|\Pi_{2h}^{2} (u_h - u_I)\|_{1, \Omega} + \|\Pi_{2h}^{2} u - u\|_{1, \Omega} \\
&\leq C \|u_h - u_I\|_{1, \Omega} + C h^2 \|u\|_{3, \Omega} \\
&\leq C h^2 \|u\|_{3, \Omega}.
\end{align*}
\] (2.25)
3. Numerical test

In this numerical test, we take $\Omega = [0, 1] \times [0, 1]$, $\Gamma_1 = [0, 1] \times \{0\}$ is the Dirichlet boundary, $\Gamma_2 = \{0\} \times [0, 1] \cup \{1\} \times [0, 1]$ is the Neumann boundary, and $\Gamma_3 = [0, 1] \times \{1\}$ is the Robin boundary. Let $u = \sin \pi x \cos \pi y$, 

$$A = \begin{pmatrix} 1 + e^x & 0.5 \\ 0.5 & 1 + e^y \end{pmatrix},$$

which is symmetric and positive definite, $\beta = (0, 0)$, $\gamma = e^x + e^y$ and $\sigma = e^{x+y}$.

Let $T_h$ be a uniform rectangular partition of $\Omega$ which satisfies our assumptions, then the errors of convergence, superapproximation and superconvergence (in $H^1$ semi-norm) are laid out below, which are used to validate our result.

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Figure 3.1

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