DYNAMICAL BEHAVIOR IN A HARVESTED DIFFERENTIAL-ALGEBRAIC ALLELOPATHIC PHYTOPLANKTON MODEL

YIFU FENG¹,², QINGLING ZHANG², AND CHAO LIU²

Abstract. A differential-algebraic model is proposed, which considers an allelopathic two-species phytoplankton ecosystem with harvest effort on toxin liberating phytoplankton. By using the differential-algebraic system theory and bifurcation theory, local stability of the proposed model around the interior equilibrium is investigated. Furthermore, instability mechanism of the proposed model due to variation of economic interest of harvesting is studied. With the purpose of stabilizing the proposed model around the interior equilibrium and maintaining the economic interest of harvesting at an ideal level, state feedback controllers are designed. Finally, numerical simulations are carried out to show consistency with theoretical analysis.

Key Words. differential-algebraic model, bifurcation, allelopathic phytoplankton, harvesting, economic interest

1. Introduction

The regular change and abrupt fluctuation of phytoplankton density in the marine environment has become an essential subject of study in aquatic ecology. Some recent observations made in [1-2] reveal that the allelopathic toxins or stimulaters produced by one species can affect the growth of several other species within certain marine environment, which may further cause the seasonal change in population density of various phytoplankton species. According to [2], allelopathy is a part of chemical communications between various organisms with a community, and allelopathic toxins or stimulaters released by one species of phytoplankton are defined as allelochemicals, which have both positive and negative effects on the growth of another species lived in the surrounding environment.

In order to improve the understanding of allelopathy in phytoplankton community and the fact that each species produces allelochemicals into the surrounding environment, some mathematical models have been proposed in [3-13], which are utilized to investigate the allelopathic ecosystem. Based on the Lotka-Volterra competitive model, Maynard-Smith [10] and then Chattopadhyay [11] investigate the dynamical behavior of following model,

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[\alpha_1 - \beta_1 x_1(t) - v_1 x_2(t) - \gamma_1 x_1(t)x_2(t)], \\
\dot{x}_2(t) &= x_2(t)[\alpha_2 - \beta_2 x_2(t) - v_2 x_1(t) - \gamma_2 x_1(t)x_2(t)].
\end{align*}
\]

where \(x_1(t)\) and \(x_2(t)\) are the densities of two competing phytoplankton species (measured as the number of cells per liter); \(\alpha_1\) and \(\alpha_2\) denotes the cell proliferation rate per day for the first and second species, respectively; \(\beta_1\) and \(\beta_2\) represents the rate of intraspecific competition term for the first and second species, respectively; \(v_1\) and \(v_2\) stands for the rate of interspecific competition for the first and second species, respectively. Therefore, the
growth of either species follows the usual logistic law of growth with intrinsic growth rate $\alpha_1$, $\alpha_2$ and environmental carrying capacity $\alpha_1/\beta_1$ and $\alpha_2/\beta_2$, respectively. Interspecific competition for limited resources is modeled by the law of mass action. $\gamma_1$ and $\gamma_2$ are the toxic inhibition rates for the first species by the second species and vice versa.

Solé et al. [12] consider a Lotka-Volterra type of model for two competing phytoplankton species, and they have neglected the interspecific competition terms based upon the fact that the growth curve of toxic algae is similar to the monocultures and all the mixed cultured for various initial concentrations of two species. With the help of numerical simulations, they point out that the interspecific competition term is significant for long time behavior of the model system, and the allelopathic interaction term will be $\gamma x_1^2(t)x_2(t)$ instead of $\gamma x_1^2(t)x_2(t)$ where $\gamma$ denotes the rate of toxic inhibition for the species $x_1(t)$ released by the second species $x_2(t)$. Based on these modified allelopathic interaction term as suggested by Solé et al., Bandyopadhyay [13] extended the work done in [10-11], and the mathematical model studied in [13] consists of two competing phytoplankton species, where $x_1(t)$ and $x_2(t)$ denotes the density of nontoxic phytoplankton and toxin liberating phytoplankton, respectively. The mathematical model in [13] is expressed as follows:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[\alpha_1 - \beta_1 x_1(t) - v_1 x_2(t) - \gamma x_1(t)x_2(t)], \\
\dot{x}_2(t) &= x_2(t)[\alpha_2 - \beta_2 x_2(t) - v_2 x_1(t)].
\end{align*}
\]

where all parameters share the same biological interpretations as mentioned above.

It is well known that harvesting has a strong impact on the dynamic evolution of a population. Nowadays, the biological resource from the allelopathic phytoplankton ecosystem is commercially harvested and sold with the aim of achieving economic interest[14]. Furthermore, harvest effort is usually adjusted based on the variation of economic interest of harvesting. Hence, it is realistic to investigate the dynamical behavior of harvested ecosystem due to the variation of economic interest of harvesting. By considering the economic interest of harvesting, a differential-algebraic model is proposed in this paper. It is utilized to investigate the dynamical behavior of harvested allelopathic phytoplankton ecosystem, which extends the previous work done in [13] from an economic perspective. We aim to obtain some results which are theoretically beneficial to maintaining the sustainable development of species in the allelopathic phytoplankton ecosystem as well as the economic interest of harvesting at an ideal level.

The rest of this paper is organized as follows. In the second section, a differential-algebraic model is established, which is utilized to investigate dynamical behavior of harvested allelopathic phytoplankton ecosystem. The third section performs local stability analysis of the proposed model system around the interior equilibrium. In order to stabilize the proposed model system around the interior equilibrium and maintain the economic interest of harvesting at an ideal level, a state feedback controller is designed in the forth section, and numerical simulations are also given to support the theoretical analysis obtained in this paper. Finally, this paper ends with a conclusion.

2. Model Formulation

In 1954, Gordon proposes the economic theory of a common-property resource in [15], which studies the effect of harvest effort on the ecosystem from an economic perspective. In [15], an equation is proposed to investigate the economic interest of the yield of harvest effort, which takes form as follows:

\[
\text{Net Economic Revenue} = \text{Total Revenue (TR)} - \text{Total Cost (TC)},
\]

let $E(t)$ and $Y(t)$ represent the harvest effort and the density of harvested population, respectively, and then $\text{TR} = wE(t)Y(t)$, $\text{TC} = cE(t)$, $w$ is a constant that denotes the unit price
of harvested population, $c$ is a constant representing the unit cost of harvest effort. Associated with the model system (2), an algebraic equation is established, which considers the economic interest $m$ of the harvest effort $E(t)$ on toxin liberating phytoplankton $x_2(t)$ and can be expressed as follows:

$$E(t)(wx_2(t) - c) = m.$$  \hspace{1cm} (4)

Recently, based on the economic theory proposed by Gordon in [15], a class of differential-algebraic biological economic models are proposed in [16-18], which are established by several differential equations and an algebraic equation. These differential-algebraic biological economic models are used to investigate the dynamical behavior of harvested ecosystem from an economic perspective. Motivated by the work done in [16-18], a harvested differential-algebraic model is proposed in this paper. Compared with the models proposed in the previous references (see [3-13] and the references therein), advantages of the proposed model are obvious. It can not only study the interaction and coexistence mechanism of population in the allelopathic ecosystem, but also offer a simpler way to discuss the dynamical behavior of harvested ecosystem due to the variation of economic interest of harvesting.

**Remark 1.** Some introductions about differential-algebraic system and its recent applications in the real world can be found in Appendix A.

Based on the above analysis, (2) and (4), a differential-algebraic model which consists of two differential equations and an algebraic equation can be established as follows:

$$\begin{align*}
&x_1(t) = x_1(t)[a_1 - \beta_1 x_1(t) - v_1 x_2(t) - \gamma x_1(t)x_2^2(t)], \\
&x_2(t) = x_2(t)[a_2 - \beta_2 x_2(t) - v_2 x_1(t)] - E(t)x_2(t), \\
&0 = E(t)(wx_2(t) - c) - m.
\end{align*}$$  \hspace{1cm} (5)

where $x_1(t)$, $x_2(t)$, $E(t)$ and all parameters share the same biological interpretations mentioned in (2) and (4). The differential-algebraic model system (5) can be expressed in the following form,

$$AX(t) = G(x_1(t), x_2(t), E(t)),$$  \hspace{1cm} (6)

where $X(t) = (x_1(t), x_2(t), E(t))^T$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

$$G(x_1(t), x_2(t), E(t)) = \begin{bmatrix} x_1(t)[a_1 - \beta_1 x_1(t) - v_1 x_2(t) - \gamma x_1(t)x_2^2(t)] \\ x_2(t)[a_2 - \beta_2 x_2(t) - v_2 x_1(t)] - E(t)x_2(t) \\ E(t)(wx_2(t) - c) - m \end{bmatrix}.$$  

**Remark 2.** The algebraic equation in the differential-algebraic model system (6) contains no differentiated variables, hence the leading matrix $A$ in (6) has a corresponding zero row.

### 3. Local Stability Analysis

In this section, dynamical behavior of the differential-algebraic system (5) is investigated by using the differential-algebraic system theory and bifurcation theory. By performing local stability analysis of the model system around the interior equilibrium, bifurcation phenomenon which can reveal the instability mechanism of model system is studied. Furthermore, the biological interpretations of bifurcation phenomenon are also discussed. It should be noted that we only concentrate on the interior equilibrium of model system (5), since the biological interpretations of the interior equilibrium imply that two competing species and harvest effort on toxin liberating species exist, which are relevant to our study.

**Theorem 1.** The differential-algebraic model system (5) has a singularity induced bifurcation around the interior equilibrium, and $m = 0$ is a bifurcation value. Furthermore, a stability switch occurs as $m$ increases through 0.
Proof. According to the economic theory of a common-property resource proposed in [15], there is a phenomenon of bio-economic equilibrium when the economic interest of harvesting is zero. Consequently, in the case of phenomenon of bio-economic equilibrium \( m = 0 \), an interior equilibrium \( P^*(x_1^*, x_2^*, E^*) = \left( \frac{w\alpha_1 - cv_1}{w^2\beta_1 + c^2}, \frac{w\alpha_2 - cv_2}{w^2\beta_1 + c^2}, \frac{w^2v_2(w\alpha_1 - cv_1)}{w^2\beta_1 + c^2} \right) \) can be obtained. In order to guarantee the existence of interior equilibrium, the following inequalities should be satisfied,

\[
\begin{align*}
\{ w\alpha_1 - cv_1 > 0, \\
(w\alpha_2 - cv_2)(w^2\beta_1 + c^2) - w^2v_2(w\alpha_1 - cv_1) > 0.
\end{align*}
\]

Let \( m \) be a bifurcation parameter, \( D \) be a differential operator and

\[
H(t) = (x_1(t), x_2(t))^T,
\]

\[
f(H(t), E(t), m) = \begin{bmatrix} x_1(t)[\alpha_1 - \beta_1x_1(t) - v_1x_2(t) - \gamma x_1(t)x_2(t)] \\
x_2(t)[\alpha_2 - \beta_2x_2(t) - v_2x_1(t) - E(t)x_2(t)] \end{bmatrix},
\]

\[
g(H(t), E(t), m) = E(t)[w_2x_2(t) - c] - m.
\]

It follows from (7) that,

\[
\text{trace}(D_{E,f} \text{adj}(D_{E,g})D_{H,g})|_{P^*} = \frac{cvv_2(w\alpha_1 - cv_1)}{w^2\beta_1 + c^2} - \frac{c(w\alpha_2 - cv_2)}{w} \neq 0.
\]

By virtue of (7), it also gives that

\[
|P^*| = \frac{c(vv_2(w\alpha_1 - cv_1))}{w^2\beta_1 + c^2} - \frac{w_2v_2(w\alpha_1 - cv_1)}{w^2\beta_1 + c^2} \neq 0.
\]

According to the Part A of Section IV[23], \( \Delta(H(t), E(t), m) \) can be defined as

\[
\Delta(H(t), E(t), m) = D_{E,g}(H(t), E(t), m) = w_2x_2(t) - c.
\]

By virtue of (7), it can be calculated that

\[
\begin{bmatrix} D_{H,f} & D_{E,f} & D_{m,f} \\
D_{H,g} & D_{E,g} & D_{m,g} \\
D_{H,\Delta} & D_{E,\Delta} & D_{m,\Delta} \end{bmatrix} |_{P^*} = \frac{c(vv_2(w\alpha_1))}{w} \neq 0.
\]

Based on the above analysis, four items can be obtained as follows,

i: It can be shown that \( f(H(0), E(0), 0) = 0, g(H(0), E(0), 0) = 0 \), and \( D_{E,g}(H(t), E(t), v)|_{P^*} \) has an algebraically simple zero eigenvalue. Furthermore, according to (8),

\[
\text{trace}(D_{E,f} \text{adj}(D_{E,g})D_{H,g})|_{P^*} \neq 0;
\]

ii: From (9), it can be shown that

\[
\begin{bmatrix} D_{H,f} & D_{E,f} & D_{m,f} \\
D_{H,g} & D_{E,g} & D_{m,g} \\
D_{H,\Delta} & D_{E,\Delta} & D_{m,\Delta} \end{bmatrix} \text{is nonsingular around } P^*;
\]

iii: By virtue of (10), it can be shown that

\[
\begin{bmatrix} D_{H,f} & D_{E,f} & D_{m,f} \\
D_{H,g} & D_{E,g} & D_{m,g} \\
D_{H,\Delta} & D_{E,\Delta} & D_{m,\Delta} \end{bmatrix} \text{is nonsingular around } P^*; \text{ so rank}
\]

\[
\begin{bmatrix} D_{H,f} & D_{E,f} & D_{m,f} \\
D_{H,g} & D_{E,g} & D_{m,g} \\
D_{H,\Delta} & D_{E,\Delta} & D_{m,\Delta} \end{bmatrix} = 4;
\]

iv: By using \( \text{rank}(f(H(t), E(t), m)) = 2 \) and \( \text{rank}(g(H(t), E(t), m)) = 1 \), it can be obtained that

\[
\text{rank}
\begin{bmatrix} D_{H,f} & D_{E,f} & D_{m,f} \\
D_{H,g} & D_{E,g} & D_{m,g} \\
D_{H,\Delta} & D_{E,\Delta} & D_{m,\Delta} \end{bmatrix}
= \text{rank}(f(H(t), E(t), m)) + \text{rank}(g(H(t), E(t), m)) + 1.
\]
According to the above items (i-iv), it follows that the condition for occurrence of singularity induced bifurcation introduced in Theorem 3 of [23] are all satisfied, hence the differential-algebraic model system (5) has a singularity induced bifurcation around the interior equilibrium $P^*$ and the bifurcation value is $m = 0$.

**Remark 3.** Some preliminaries of singularity induced bifurcation are introduced, which can be found in Appendix B.

Along with the line of the above proof, for the differential-algebraic model system (5), by simple computing,

$$
M = -\text{trace}(D_E f \text{adj}(D_E g) D_H g)|_{P^*} = \frac{c(wa_2 - cf_2)}{w} - \frac{cv_2(wa_1 - cv_1)}{w^2b_1 + c^2\gamma},
$$

$$
N = (D_m \Delta - (D_H \Delta, D_E \Delta)) \left[ \begin{array}{cc} D_H f & D_E f \\ D_H g & D_E g \end{array} \right]^{-1} \left[ \begin{array}{c} D_m f \\ D_m g \end{array} \right]|_{P^*} = w/(w^2a_2 - c^2\gamma) - \frac{w^2v_2(wa_1 - cv_1)}{w^2b_1 + c^2\gamma},
$$

then it can be obtained,

$$
\frac{M}{N} = \left( \frac{w^2a_2 - c^2\gamma}{w^2b_1 + c^2\gamma} \right) > 0.
$$

It can be shown that the inequality (11) satisfies the Theorem 3 in [23]. According to the Theorem 3 in [23], when $m$ increases through 0, one eigenvalue (denoted by $\lambda_1$) of the differential-algebraic model system (5) moves from $\mathbb{C}^-$ to $\mathbb{C}^+$ along the real axis by diverging through infinity. Hence the movement behavior of this eigenvalue influences the stability of the differential-algebraic model system (5).

The other eigenvalue of the differential-algebraic model system (5) around $P^*$ can be calculated below. The Jacobian of the differential-algebraic model system (5) evaluated around $P^*$ takes the form as follows:

$$
J_{P^*} = \begin{bmatrix} -\beta_1 + \gamma x_2^2 & -v_1 + 2\gamma x_1 x_2 & 0 \\ -v_1 x_2 & -\beta_2 x_2 & -x_2 \\ 0 & wE^* & 0 \end{bmatrix},
$$

according to the leading matrix $A$ in the model system (6) and $J_{P^*}$, we obtain the characteristic equation of the differential-algebraic model system (5) around $P^*$ is $\det(A - J_{P^*}) = 0$, and then it can be calculated that the other eigenvalue of the differential-algebraic system (5) around $P^*$ is $\lambda_2 = \frac{cv_2}{w^2a_2 - c^2\gamma}$. From (7), it is easy to show that $\text{Re}\lambda_2 < 0$. It follows from the Theorem 3 in [23] that $\lambda_2$ is continuous, nonzero and can not jump from on half open complex plane to another one as $m$ increases through 0. Therefore, $\lambda_2$ is continuous and bounded in the $\mathbb{C}^-$ half plane as $m$ increases through 0 and its movement behavior has no influence on the stability of the differential-algebraic system (5) around the interior equilibrium $P^*$. Table 1 shows the change in signs of real parts of eigenvalues ($\lambda_1$ and $\lambda_2$) due to the variation of economic interest of harvesting.

**Table 1.** Signs of real parts of eigenvalues of model system (5) around $P^*$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\text{Re}\lambda_1$</th>
<th>$\text{Re}\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m &lt; 0$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$m &gt; 0$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

According to Table 1 and the stability theory, it can be concluded that the differential-algebraic model system (5) is stable around the interior equilibrium $P^*$ as $m < 0$ and the differential-algebraic model system (5) is unstable around the interior equilibrium $P^*$ as $m > 0$. Consequently, a stability switch occurs as $m$ increases through 0. This completes
According to the leading matrix takes the form, 

\[ \begin{align*}
\dot{\rho}_1(t) &= \rho_1(t)[\alpha_1 - \beta_1 \rho_1(t) - \gamma \rho_1(t) \rho_2(t)], \\
\dot{\rho}_2(t) &= \rho_2(t)[\alpha_2 - \beta_2 \rho_2(t) - \gamma \rho_1(t) \rho_2(t)] - E(t) \rho_2(t), \\
0 &= E(t)[\omega \rho_2(t) - c] - m + k(E(t) - E^*).
\end{align*} \]

**Theorem 2.** In the case of \( m = 0 \), if the feedback gain \( k \) satisfies

\[ k > \max\left\{ \frac{w \rho_2^2 E^*}{\beta_2 \rho_2 + \gamma \rho_2^2}, \frac{w E^*(\beta_1 + \gamma \rho_2^2)}{\beta_2 \rho_2 + \gamma \rho_2^2} \right\}, \]

then differential-algebraic model system (13) is stable around \( P^* \).

**Proof.** The Jacobian of the differential-algebraic model system (13) evaluated around \( P^* \) takes the form,

\[ J_p = \begin{bmatrix}
-x_1'(\beta_1 + \gamma x_2^2) & -x_1'(v_1 + 2\gamma x_1 x_2) & 0 \\
-x_2'(v_1 + 2\gamma x_1 x_2) & -x_2'(\beta_2 x_2 + \gamma x_2^3) & -x_2' \\
0 & w E^* & k
\end{bmatrix}, \]

according to the leading matrix \( A \) in the model system (6) and \( J_p \). We obtain the characteristic equation of model system (13) around \( P^* \) is \( \det(\lambda A - J_p) = 0 \), which can be expressed as follows:

\[ \lambda^2 + \Delta_1 \lambda + \Delta_2 = 0, \]

where \( \Delta_1 = x_1'(\beta_1 + \gamma x_2^2) + \beta_2 x_2^2 - \frac{w E^*}{k} \),

\[ \Delta_2 = x_1'(v_1 + 2\gamma x_1 x_2) + v_2(v_1 + 2\gamma x_1 x_2) \] 

The proof of Theorem 1.

**Remark 4.** As stated in Appendix B, the singularity induced bifurcation can result in impulse phenomenon, which may lead to the collapse of the proposed model system. In the two competing phytoplankton ecosystem, the impulse phenomenon of model system is usually connected with the rapid growth of phytoplankton species. If this phenomenon lasts for period of time, the density of species in marine ecosystem will be beyond the carrying capacity of surrounding environment, which is disastrous for ecological balance of marine ecosystem; On the other hand, people always try to make profit of harvesting, so they are usually interested in the case of positive economic interest. However, it follows from Theorem 1 of this paper that model system becomes unstable in the case of positive economic interest of harvesting, as is disadvantageous to persistent property of harvesting. With the purpose of dealing with such dilemma, some related measures should be applied to eliminate stability switch of model system and maintain economic interest of harvesting at an ideal level.

4. State Feedback Control for Model System

In this section, state feedback controllers are designed to eliminate the singularity induced bifurcation of model system (5) and stabilize the model system (5) around the interior equilibrium.

4.1. Design of state feedback control. According to the leading matrix \( A \) in the model system (6) and \( J_p \) in (12), it can be calculated that \( \text{rank}(J_p, AJ_p, A^2J_p) = 3 \). By using Theorem 2-2.1 in the reference [19], it is easy to show the differential-algebraic model system (5) is locally controllable around \( P^* \). Based on Theorem 3.1.2 in the reference [19], a state feedback controller \( u(t) = k(E(t) - E^*) \) (\( k \) is a feedback gain and \( E^* \) is a component of \( P^* \)) can be applied to stabilize the differential-algebraic model system (5) around \( P^* \). Apply the controller \( u(t) = k(E(t) - E^*) \) into the differential algebraic model system (5), and then a controlled differential-algebraic model system is as follows:

\[ \begin{align*}
\dot{\rho}_1(t) &= \rho_1(t)[\alpha_1 - \beta_1 \rho_1(t) - \gamma \rho_1(t) \rho_2(t)], \\
\dot{\rho}_2(t) &= \rho_2(t)[\alpha_2 - \beta_2 \rho_2(t) - \gamma \rho_1(t) \rho_2(t)] - E(t) \rho_2(t), \\
0 &= E(t)[\omega \rho_2(t) - c] - m + k(E(t) - E^*).
\end{align*} \]
By using the Routh-Hurwitz criteria\cite{27}, if the feedback gain $k$ satisfies,

$$
k > \max \left\{ \frac{w_1 \bar{E}^*}{\beta_2 \bar{x}_1 + \bar{x}_1 (\beta_1 + \gamma \bar{x}_2^2)}, \frac{w_1 \bar{E}^* (\beta_1 + \gamma \bar{x}_2^2)}{\beta_2 \bar{x}_1 + \bar{x}_1 (\beta_1 + \gamma \bar{x}_2^2)}, \beta_2 (\beta_1 + \gamma \bar{x}_2^2) - v_2 (v_1 + 2 \gamma \bar{x}_1^2) \right\},
$$

then differential-algebraic model system (13) is stable around $P^*$. This completes proof of Theorem 2.

In case of positive economic interest, i.e., $m > 0$. The interior equilibrium of model system (5) takes the form of $\bar{P}^*(\bar{x}_1^*, \bar{x}_2^*, \bar{E}^*)$, $\bar{x}_1^* = \frac{a_1 - v_1 \bar{x}_1^*}{\beta_1 + \gamma \bar{x}_2^2}$, $\bar{E}^* = \frac{m}{x_{12}^* - \bar{x}_1^*}$ and $\bar{x}_2^*$ satisfies the following equation

$$
y^4 + b_1 y^3 + b_2 y^2 + b_3 y + b_4 = 0,
$$

where $b_1 = -\frac{\beta_1 + \gamma \bar{x}_2^2}{w_1}$, $b_2 = \frac{m v_1 \beta_1 + \gamma \bar{x}_2^2}{w_1}$, $b_3 = \frac{v_2 (v_1 + \gamma \bar{x}_2^2)}{w_2}$, $b_4 = \frac{m \beta_1 + \gamma \bar{x}_2^2}{w_1} - \gamma v_1 \bar{x}_2^2$.

It is easy to show that the simplest condition which can guarantee Equation (14) has at least a positive root is $b_4 < 0$. Furthermore, $\bar{x}_1^*$, $\bar{x}_2^*$ and $\bar{E}^*$ are components of the interior equilibrium $\bar{P}^*$, which follows $\bar{x}_1^* > 0$ and $\bar{E}^* = \frac{m}{x_{12}^* - \bar{x}_1^*} > 0$.

Based on the above analysis, some following inequalities should be satisfied with the purpose of guaranteeing the existence of interior equilibrium $\bar{P}^*$,

$$
\begin{aligned}
& m \beta_1 + \gamma \bar{x}_2^2 - \gamma v_1 \bar{x}_2^2 < 0, \\
& \frac{\xi}{w} < \bar{x}_2^* < \frac{\xi}{v_1}.
\end{aligned}
$$

Similarly, it is easy to show that differential-algebraic model system (5) is locally controllable around $\bar{P}^*$. Based on Theorem 3.1.2 in the reference \cite{19}, a state feedback controller $u(t) = k (E(t) - \bar{E}^*)$ ($k$ is a feedback gain and $\bar{E}^*$ is a component of $\bar{P}^*$) can be applied to stabilize the differential-algebraic model system (5) around $\bar{P}^*$. Apply the controller $u(t) = k (E(t) - \bar{E}^*)$ into the differential algebraic model system (5), and then a controlled differential-algebraic model system is as follows:

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)[a_1 - \beta_1 x_1(t) - v_1 x_2(t) - \gamma x_1(t) \bar{x}_2^2(t)], \\
\dot{x}_2(t) &= x_2(t)[a_2 - \beta_2 x_2(t) - v_2 x_1(t) - E(t) \bar{x}_2^2(t)], \\
0 &= E(t)[wx_2(t) - c] - m + k(E(t) - \bar{E}^*).
\end{aligned}
$$

**Theorem 3.** In the case of $m > 0$, if the feedback gain $k$ satisfies

$$
k > \max \left\{ \frac{w_1 \bar{E}^*}{\beta_2 \bar{x}_1 + \bar{x}_1 (\beta_1 + \gamma \bar{x}_2^2)}, \frac{w_1 \bar{E}^* (\beta_1 + \gamma \bar{x}_2^2)}{\beta_2 \bar{x}_1 + \bar{x}_1 (\beta_1 + \gamma \bar{x}_2^2)}, \beta_2 (\beta_1 + \gamma \bar{x}_2^2) - v_2 (v_1 + 2 \gamma \bar{x}_1^2) \right\},
$$

then differential-algebraic model system (16) is stable around $\bar{P}^*$.

**Proof.** Similarly to the proof for Theorem 2, it is easy to show Theorem 3.

### 4.2. Numerical simulation of feedback control

A numerical simulation work with hypothetical set of parameters is performed to illustrate the effectiveness of the designed state feedback controllers, and the parametric values are given in Table 2.

#### 4.2.1. Simulation for the case $m = 0$

In the case of zero economic interest, i.e., $m = 0$, the differential-algebraic model system (5) takes the following form by virtue of given parameter values,

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)[2 - 0.02 x_1(t) - 0.05 x_2(t) - 0.008 x_1(t) \bar{x}_2^2(t)], \\
\dot{x}_2(t) &= x_2(t)[1 - 0.08 x_2(t) - 0.015 x_2(t)] - E(t) x_2(t), \\
0 &= E(t)(0.1667 x_2(t) - 1).
\end{aligned}
$$
Based on the analysis in Section 4.1, a state feedback controller can be applied to stabilize the differential-algebraic model system (17) with the state feedback controller takes the form as follows:

\[ \begin{align*}
\dot{x}_1(t) &= x_1(t)[2 - 0.02x_1(t) - 0.05x_2(t) - 0.008x_1(t)x_2^3(t)], \\
\dot{x}_2(t) &= x_2(t)[1 - 0.08x_2(t) - 0.015x_1(t)] - E(t)x_2(t), \\
0 &= E(t)(0.1667x_2(t) - 1) + k(E(t) - 0.4373).
\end{align*} \] (18)

By using Theorem 2 of this paper, if feedback gain \( k \) satisfies \( k > 1.4088 \), then singularity induced bifurcation is eliminated and differential-algebraic model system (18) is stable around \( P^* \). Dynamical responses of the differential-algebraic model system (18) can be shown in Figure 1.

### 4.2.2. Simulation for the case \( m > 0 \)

Since it is difficult to obtain analytical computation of (15), it can be numerically computed that positive economic interest \( m \) should be restricted within the interval, \( m \in (0, 0.1231) \), which guarantees that model system (5) has at least an interior equilibrium in the case of positive economic interest of harvesting.

In this section, \( m = 0.0879 \) is arbitrarily selected in the interval \( (0, 0.1231) \) which is enough to merit the above mathematical analysis. In this case, two positive roots of Equation (14) can be calculated, and then corresponding interior equilibria are \( P^*_1(3.4389, 7.5189, 0.3469) \) and \( P^*_2(1.5119, 10.8609, 0.1084) \).

By utilizing Theorem 1 of this paper, the differential-algebraic model system (5) is unstable around \( P^*_1 \) and \( P^*_2 \), respectively. It follows from the analysis in Section 4.1 that a state feedback controller can be applied to stabilize the differential-algebraic model system (16) around \( P^*_1 \), and then the model system (16) with the state feedback controller \( u(t) = k(E(t) - 0.3469) \) takes the form as follows:

\[ \begin{align*}
\dot{x}_1(t) &= x_1(t)[2 - 0.02x_1(t) - 0.05x_2(t) - 0.008x_1(t)x_2^3(t)], \\
\dot{x}_2(t) &= x_2(t)[1 - 0.08x_2(t) - 0.015x_1(t)] - E(t)x_2(t), \\
0 &= E(t)(0.1667x_2(t) - 1) - 0.0879 + k(E(t) - 0.3469).
\end{align*} \] (19)
By using Theorem 3 of this paper, if the feedback gain $k$ satisfies $k > 0.886$, then differential-algebraic model system (19) is stable around $\tilde{P}_1^*$. The dynamical responses of model system (19) can be shown in Figure 2.

Similarly, a state feedback controller can be applied to stabilize the differential-algebraic model system (16) around $\tilde{P}_2^*$, and then model system (16) with state feedback controller $u(t) = k(E(t) - 0.1084)$ takes the following form:

\begin{align}
\dot{x}_1(t) &= x_1(t)[2 - 0.02x_1(t) - 0.05x_2(t) - 0.008x_1(t)x_2^2(t)], \\
\dot{x}_2(t) &= x_2(t)[1 - 0.08x_2(t) - 0.015x_1(t)] - E(t)x_2(t), \\
0 &= E(t)[0.1667x_2(t) - 1] - 0.0879 + k(E(t) - 0.1084).
\end{align}

By using Theorem 3 of this paper, if the feedback gain $k$ satisfies $k > 0.2405$, then differential-algebraic model system (20) is stable around $\tilde{P}_2^*$. The dynamical responses of model system (20) can be shown in Figure 3.

Remark 5. Biological interpretations of the above analysis are summarized as follows.

i: As analyzed above, there exists at least an interior equilibrium when economic interest of harvesting is restricted within a certain interval. It is of inspiration for governmental agencies to formulate related policies to regulate commercial harvesting and control the economic interest of harvesting within certain range. These measures may, to some extent, guarantee that all the species in the harvested allelopathic ecosystem survive under harvesting, and constructively reduce the risk of population extinction caused by over-harvesting.

ii: According to the design of state feedback controller, it reveals that both abrupt fluctuation of species density can be avoided and the ideal economic interest of harvesting can be also maintained by adopting appropriate harvest effort on the toxin liberating phytoplankton.

iii: With the introduction of state feedback controller, model system can be stabilized around the interior equilibrium and the singularity induced bifurcation is also eliminated, which relates to the restoration of ecological balance and sustainable development of species in the harvested marine ecosystem.

5. Conclusion

It is well known that harvesting has a strong impact on the dynamic evolution of a population. Nowadays, the biological resource from the allelopathic phytoplankton ecosystem is commercially harvested and sold with the aim of achieving economic interest, and harvest effort is usually adjusted based on the variation of economic interesting of harvesting. Consequently, it is realistic to investigate the dynamical behavior of harvested ecosystem due to the variation of economic interest of harvesting. In this paper, a harvested differential-algebraic model system is proposed, which is governed by two-dimension differential equations and an algebraic equation. The model study gives the basic and essential features of a harvested allelopathic two-species phytoplankton ecosystem. The differential equations investigate the allelopathic interactions between two competing phytoplankton species. Specially, the algebraic equation in the proposed model system studies the harvest effort on toxin liberating phytoplankton species from an economic perspective. Further analysis focuses on stability switch of model system with the change of economic interest of harvesting, which extends the work done in [13] and the references therein. Some corresponding biological interpretations are given in Remark 4 of this paper.

By analyzing the local stability of model system around the interior equilibrium, mechanism of sustainable development of species in the harvested allelopathic ecosystem is studied. Furthermore, state feedback controllers are designed to eliminate the singularity
Dynamical responses of model system (18) around $P^\ast$.

Dynamical responses of model system (19) around $\tilde{P}_1^\ast$. 
induced bifurcation and stabilize the differential-algebraic model system around the interior equilibrium. Numerical simulations are carried out to illustrate effectiveness of the designed controllers, which can be shown in Figure 1, Figure 2 and Figure 3.

As introduced in Section 2 of this paper, some differential-algebraic models are also proposed in [16-18]. However, they are only used to investigate some biological system with a single species or prey-predator ecosystem. Compared with these work, the model proposed in this paper investigates dynamical behavior of marine ecosystem with two competing species and corresponding harvesting issues, which extends the application of differential-algebraic model in dealing with problems from the real world.

It should be noted that the theoretical analysis obtained in Theorem 3 of this paper provides us a bio-economic way to maintain sustainable development of each species in harvested ecosystem in the case of positive economic interest of harvesting, whose biological interpretations can be found in Remark 5 of this paper. Since there are global concerns on maintaining the sustainable development of biological resource and persistent property of commercial harvesting on such resource, it makes the work studied here have some new and positive features.

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Appendices

Appendix A. Differential-algebraic systems (singular systems, descriptor systems, degenerate systems, constrained systems, etc.), which have been investigated over the last decades, are rather general kind of equations. They are established according to relationships among the variables. Naturally, it is usually differential or algebraic equations that form the mathematical model of the system, or the descriptor equation. The general form of differential-algebraic system is as follows:

\[
\begin{align*}
A(t)x(t) &= G(x(t), u(t), t), \\
y(t) &= K(x(t), u(t), t).
\end{align*}
\]

where \(x(t)\) is the state of the system composed of state variables; \(u(t)\) is the control input; \(y(t)\) is the measure output, \(G(\cdot), K(\cdot)\) are appropriate dimensional vector functions in \(x(t), u(t)\) and \(t\). The matrix \(A(t)\) may be singular. Differential-algebraic systems are suitable for describing systems which evolve over time. Especially, nonlinear differential-algebraic equations are the natural outcome of component-based modeling of complex dynamic systems.

Compared with the ordinary differential systems, the advantage they offer over the more often used ordinary differential equations is that they are generally easier to formulate. The price paid is that they are more difficult to deal with (see [19] and the references therein). In general, differential-algebraic model systems exhibit more complicated dynamics than ordinary differential models. With the help of the differential-algebraic model for the power systems and bifurcation theory, complex dynamical behavior of the power systems, especially the bifurcation phenomena which can reveal the instability mechanism of power systems have been extensively studied (see Ref. [20-22] and the references therein). However, as far as the biological systems are concerned, the related research results are few.

Appendix B. Parameter dependent differential-algebraic system of the form,

\[
\begin{align*}
\dot{x}(t) &= f(x, y, \lambda), f : R^n \times R^q \times R^p \rightarrow R^n, \\
0 &= g(x, y, \lambda), g : R^n \times R^q \times R^p \rightarrow R^q.
\end{align*}
\]

It is widely used to model the dynamics of differential-algebraic system, where assuming that \(x, y\) and \(\lambda\) have appropriate dimensions. The singularity induced bifurcation is firstly introduced and analyzed by Venkatasubramanian et al. [23-24]. It is a new type of bifurcation and does not occur in usual ordinary differential equation system, which has been characterized for differential-algebraic system, and later improved by Beardmore and Yang et al. [25-26].

Theorem 3 (Singularity Induced Bifurcation Theorem) in [23] is introduced as follows.

Consider the system (22) with a one-dimensional parameter space. Suppose the following conditions are satisfied at \((0, 0, \mu_0)\), where \(D\) is a differential operator:

SI1: \(f(0, 0, \mu_0) = 0, g(0, 0, \mu_0) = 0, D_g f\) has an algebraically simple zero eigenvalue and \(\text{trace}[D \dot{f}\, \text{adj}(D_g D_\lambda g)]\) is nonzero.

SI2: \(\begin{bmatrix} D_x f & D_x \dot{f} \\ D_g f & D_g \dot{f} \end{bmatrix}\) is nonsingular.

SI3: \(\begin{bmatrix} D_x f & D_x \dot{f} & D_x \ddot{f} \\ D_g f & D_g \dot{f} & D_g \ddot{f} \\ D_\Delta f & D_\Delta \dot{f} & D_\Delta \ddot{f} \end{bmatrix}\) is nonsingular.

Then there exists a smooth curve of equilibria in \(R^{n+m+1}\) which passes through \((0, 0, \mu_0)\) and is transversal to the singular surface at \((0, 0, \mu_0)\). When \(\mu\) increases through \(\mu_0\), one eigenvalue of the system moves from \(-\infty\) to \(+\infty\) if \(\frac{\partial}{\partial \mu} > 0\) (respectively, from \(-\infty\) to \(+\infty\) if \(\frac{\partial}{\partial \mu} < 0\).
\[ M = -\text{trace}[D_y f \text{adj}(D_y g) D_x g], \]
\[ N = D_\mu \Delta - (D_\Delta D_x \Delta, D_\Delta D_y \Delta) \left( D_y f D_x f - D_y g D_x g \right)^{-1} \left( D_y f D_x g \right), \]

at \((0, 0, \mu_0)\).

Roughly speaking, the singularity induced bifurcation refers to a stability change of the differential-algebraic system (22) owing to some eigenvalues of a related linearization diverging to infinity when the Jacobian is singular. One of the important consequences of the singularity induced bifurcation is that it leads to an impulse phenomenon of the differential-algebraic system, which may result in the collapse of the differential-algebraic system [25]. More detailed introductions of the singularity induced bifurcation can be found in [23-26].

References


1. Jilin Normal University, Siping, Jilin, P.R. China;
2. Institute of Systems Science, Northeastern University, Shenyang Liaoning, P.R. China.
E-mail: qlzhang@mail.neu.edu.cn