CHAOS CONTROL OF A RATIO-DEPENDENT FOOD CHAIN MODEL

LICHUN ZHAO 1 AND QINGLING ZHANG2, QICHANG YANG1

Abstract. The chaos control of a ratio-dependent food chain model is investigated by using nonlinear control theory, then the control law which makes the system stable at a positive equilibrium is obtained, and some simulation figures are given.

Key Words. Chaos control, Lyapunov exponent, food chain model, nonlinear control theory.

1. Introduction

Some control theories have been extensively applied to biological systems and many results have been reached. For example, optimal control is applied to some biological systems [1∼5], impulse control and optimal impulse control are found in some population systems [6∼9], dissipation control and induction control in some biological systems are investigated[10∼12].

As biological systems are complicated, chaos may occur. Although there are fewer researches concerning the control of chaos in biological systems, it has received an extensive amount of interest[13∼15]. In [16], authors study a three trophic level food chain model with ratio-dependent Michaelis-Mentes type functional response and show that its dynamical outcomes may depend on initial population levels, which indicates that chaos may occur in the system. In fact, authors have showed this result via simulation. In [17∼19], authors investigate chaos control in some biological systems.

Considering the following model[16]

\[
\begin{align*}
\dot{x}(t) &= r x \left(1 - \frac{x}{K}\right) - \frac{1}{\eta_1} \frac{m_1 xy}{a_1 y + x}, \quad x(0) = x_0 > 0, \\
\dot{y}(t) &= \frac{m_1 xy}{a_1 y + x} - d_1 y - \frac{1}{\eta_2} \frac{m_2 yz}{a_2 z + y}, \quad y(0) = y_0 > 0, \\
\dot{z}(t) &= \frac{m_2 yz}{a_2 z + y} - d_2 z, \quad z(0) = z_0 > 0,
\end{align*}
\]

where \(x, y, z\) stand for the population density of prey, predator and top predator, respectively. For \(i = 1, 2, \eta_i, m_i, a_i, d_i\) are the yield constants, maximal predator growth rates, half-saturation constants and predator’s death rates respectively. \(r\)
and $K$, as above, are the prey intrinsic growth rate and carrying capacity respectively.

For simplicity, we non-dimensionality system (1.1) with the following scaling:

$$ t \rightarrow rt, \quad x \rightarrow \frac{x}{K}, \quad y \rightarrow \frac{a_1}{K} y, \quad z \rightarrow \frac{a_2 a_1}{K} z, \quad m_1 \rightarrow \frac{m_1}{r}, \quad d_1 \rightarrow \frac{d_1}{r}, \quad m_2 \rightarrow \frac{m_2}{r}, \quad d_2 \rightarrow \frac{d_2}{r}, $$

then system (1.1) turns into the form

$$
\begin{align*}
\dot{x}(t) &= x(1 - x) - \frac{c_1 xy}{x + y}, \quad x(0) = x_0 > 0, \\
\dot{y}(t) &= \frac{m_1 xy}{x + y} - d_1 y - \frac{c_2 yz}{y + z}, \quad y(0) = y_0 > 0, \\
\dot{z}(t) &= \frac{m_2 yz}{y + z} - d_2 z, \quad z(0) = z_0 > 0,
\end{align*}
$$

where

$$ c_1 = \frac{m_1}{\eta_1 a_1 r}, \quad c_2 = \frac{m_2}{\eta_2 a_2 r}. $$

For system (1.2), if $m_1 = 10$, $m_2 = 2$, $d_1 = d_2 = 1$, $c_1 = 1$, $c_2 = 11$, then yields a model of the following form

$$
\begin{align*}
\dot{x} &= x(1 - x) - \frac{xy}{x + y}, \quad x(0) = x_0 > 0, \\
\dot{y} &= 10xy \quad y - \frac{11yz}{y + z}, \quad y(0) = y_0 > 0, \\
\dot{z} &= \frac{2yz}{y + z} - z, \quad z(0) = z_0 > 0,
\end{align*}
$$

If $x_0 = [0.3433; 0.0467; 0.0767]$, authors indicate that system (1.4) is chaotic by Figure.1.

![Figure 1. Time response graph and three dimensional graph in phase space of system (1.4)](image)

In this paper, we study model (1.4) with parameters which make the system generate chaos dynamics. Although chaos in model (1.4) may be found [16], further, we test the existence of chaos by the calculation of the Lyapunov exponent and investigate the control of chaos by using nonlinear control theory. We obtain the
control law which makes the system stable at a positive equilibrium and give some simulation figures.

2. Chaos Control

2.1. Preliminaries. In this section, we give some definitions and lemma for later use.

Definition 1 [20]

\[ < d\lambda(x), f(x) > = \frac{\partial\lambda}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial\lambda}{\partial x_i} f_i(x) \]

is called the derivative of \( \lambda \) along \( f(x) \) at point \( x \) and rewritten as \( L_f \lambda \).

Definition 2 [20] The single-input single-output nonlinear system

\[
\begin{cases}
\dot{x} = f(x) + g(x)u, \\
y = h(x)
\end{cases}
\]

is said to have relative degree \( r \) at point \( x_0 \), if

(i) \( L_g L_k^r h(x_0) = 0 \) for all \( x \) in a neighborhood of \( x_0 \) and all \( k < r - 1 \),

(ii) \( L_g L_r^{r-1} h(x_0) \neq 0 \).

Definition 3 [21] Considering the following system

\[
\begin{cases}
\dot{x} = f(x), \\
x(t_0) = x_0,
\end{cases}
\]

where \( f : R^n \rightarrow R^n \) is a continuously differentiable vector function, \( x(t) \) is the trajectory \( x(t) \) of system (2.3). Additional, the time evaluation of a tangent vector \( \delta x(t) \) at \( x(t) \) is represented by linearizing equation (2.4)

\[ \delta\dot{x} = F(t)\delta x \]

where \( F(t) = \frac{\partial\lambda}{\partial x(t)} |_{x=x(t)} \) is the Jacobi matrix of

\[ \lambda_i = \lim_{t \to \infty} \sup_{t} \frac{1}{t} \ln \left( \frac{\|\delta x(t)\|}{\|\delta x(t_0)\|} \right) \]

exists, then \( \lambda_i \) is called the ith Lyapunov exponent of the function \( f(x) \) at point \( x_0 \).

Theorem 1 [23] Let \( \Phi^t : R^n \rightarrow R^n \) is a continuous dynamical system. If there exists a set \( A \subset R^n \) such that the first Lyapunov exponent \( \lambda_1(x) > 0 \) for all \( x \in A \), then dynamical system \( \Phi^t \) exhibits sensitive dependence on initial conditions on set \( A \), i.e. \( \Phi^t \) is chaotic on \( A \). If Legesque measure \( L^n(A) > 0 \), then dynamical system \( \Phi^t \) is chaotic.

Theorem 2 [16] The State space exact linearization problem is solvable if and if there is a neighborhood \( U \) of \( x^0 \) and a real-valued function \( \lambda(x) \) defined on \( U \), such that system (2.2) has relative degree \( n \) at point \( x^0 \).
2.2. Lyapunov Exponent. In literature [17], authors indicate that system (1.2) with \( m_1 = 10, m_2 = 2, d_1 = 1, d_2 = 1, c_1 = 1, c_2 = 1 \) may generate chaotic dynamics by simulation. Further, we calculate the Lyapunov exponents of the system in some domain and then affirm that the system under consideration is chaotic according to theorem in literature [24].

First, We draw a rectangular parallelepiped with vertexes

\[
A = (0.6269, 1.3360, 0.2710), B = (0.6669, 1.3360, 0.2710),
C = (0.6669, 1.3760, 0.2710), D = (0.6269, 1.3760, 0.2710),
E = (0.6269, 1.3360, 0.2310), F = (0.6669, 1.3360, 0.2319),
G = (0.6669, 1.3760, 0.2310), G = (0.6269, 1.3760, 0.2310)
\]

and choose some points on each edges of the rectangular parallelepiped, and then we calculate the Lyapunov exponent of corresponding solution.

Second, we takes some points on each plane of rectangular paralleled piped and calculate the Lyapunov exponent of corresponding solution.

Finally, we choose some points randomly in the rectangular parallelepiped and calculate the Lyapunov exponent of corresponding solution, see table 1 and table 2.

From table 1 and table 2, we can know that the Lyapunov exponents of system (1.4) on rectangular parallelepiped ABCD-EFGH are positive everywhere. According to theorem in literature [23], we may reach the consult-system (1.4) is chaotic.

2.3. Chaos Control. For system (1.4), we easily know that the interior equilibrium \((13/20, 7/20, 7/20)\) of system (1.4) is unstable. And we introduce the following variables

\[
(6) \quad x_1 = x - \frac{13}{20}, \quad x_2 = y - \frac{7}{20}, \quad x_3 = z - \frac{7}{20}.
\]

Substitute (2.6) into system (1.4), we get the following equations

\[
(7) \quad \begin{cases} \dot{x} = f_1(x) + g_1(x)u, \\ y = h_1(x) \end{cases}
\]

where

\[
x = (x_1, x_2, x_3)^T, \quad h_1(x) = x_1, \quad f_1(x) = (f_{11}(x), f_{12}(x), f_{13}(x))^T,
\]
Table 1. Point and the Corresponding Lyapunov Exponent

<table>
<thead>
<tr>
<th>Point on prism AB</th>
<th>LE</th>
<th>Point on prism BC</th>
<th>LE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.6269, 1.3360, 0.2710)</td>
<td>0.0054</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0090</td>
</tr>
<tr>
<td>(0.6369, 1.3360, 0.2710)</td>
<td>0.0063</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0106</td>
</tr>
<tr>
<td>(0.6469, 1.3360, 0.2710)</td>
<td>0.0072</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0076</td>
</tr>
<tr>
<td>(0.6569, 1.3360, 0.2710)</td>
<td>0.0100</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0025</td>
</tr>
<tr>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0079</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0034</td>
</tr>
<tr>
<td>Point on prism CD</td>
<td>LE</td>
<td>Point on prism DA</td>
<td>LE</td>
</tr>
<tr>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0034</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0094</td>
</tr>
<tr>
<td>(0.6569, 1.3360, 0.2710)</td>
<td>0.0060</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0068</td>
</tr>
<tr>
<td>(0.6469, 1.3360, 0.2710)</td>
<td>0.0099</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0065</td>
</tr>
<tr>
<td>(0.6369, 1.3360, 0.2710)</td>
<td>0.0093</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0061</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2710)</td>
<td>0.0094</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0054</td>
</tr>
<tr>
<td>Point on prism EF</td>
<td>LE</td>
<td>Point on prism FG</td>
<td>LE</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2310)</td>
<td>0.0089</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0047</td>
</tr>
<tr>
<td>(0.6369, 1.3360, 0.2310)</td>
<td>0.0104</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0049</td>
</tr>
<tr>
<td>(0.6469, 1.3360, 0.2310)</td>
<td>0.0023</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0052</td>
</tr>
<tr>
<td>(0.6569, 1.3360, 0.2310)</td>
<td>0.0040</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0070</td>
</tr>
<tr>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0047</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0082</td>
</tr>
<tr>
<td>Point on prism GH</td>
<td>LE</td>
<td>Point on prism HE</td>
<td>LE</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2310)</td>
<td>0.0087</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0082</td>
</tr>
<tr>
<td>(0.6369, 1.3360, 0.2310)</td>
<td>0.0066</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0050</td>
</tr>
<tr>
<td>(0.6469, 1.3360, 0.2310)</td>
<td>0.0101</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0047</td>
</tr>
<tr>
<td>(0.6569, 1.3360, 0.2310)</td>
<td>0.0097</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0036</td>
</tr>
<tr>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0037</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0037</td>
</tr>
<tr>
<td>Point on prism AE</td>
<td>LE</td>
<td>Point on prism BF</td>
<td>LE</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2710)</td>
<td>0.0054</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0079</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2610)</td>
<td>0.0061</td>
<td>(0.6669, 1.3360, 0.2610)</td>
<td>0.0102</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2510)</td>
<td>0.0064</td>
<td>(0.6669, 1.3360, 0.2510)</td>
<td>0.0025</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2410)</td>
<td>0.0089</td>
<td>(0.6669, 1.3360, 0.2410)</td>
<td>0.0040</td>
</tr>
<tr>
<td>(0.6269, 1.3360, 0.2310)</td>
<td>0.0087</td>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0047</td>
</tr>
<tr>
<td>Point on prism CG</td>
<td>LE</td>
<td>Point on prism DH</td>
<td>LE</td>
</tr>
<tr>
<td>(0.6669, 1.3760, 0.2710)</td>
<td>0.0034</td>
<td>(0.6269, 1.3760, 0.2710)</td>
<td>0.0094</td>
</tr>
<tr>
<td>(0.6669, 1.3760, 0.2610)</td>
<td>0.0043</td>
<td>(0.6269, 1.3760, 0.2610)</td>
<td>0.0100</td>
</tr>
<tr>
<td>(0.6669, 1.3760, 0.2510)</td>
<td>0.0049</td>
<td>(0.6269, 1.3760, 0.2510)</td>
<td>0.0074</td>
</tr>
<tr>
<td>(0.6669, 1.3760, 0.2410)</td>
<td>0.0052</td>
<td>(0.6269, 1.3760, 0.2410)</td>
<td>0.0047</td>
</tr>
<tr>
<td>(0.6669, 1.3760, 0.2310)</td>
<td>0.0082</td>
<td>(0.6269, 1.3760, 0.2310)</td>
<td>0.0037</td>
</tr>
</tbody>
</table>

\[ f_1(x) = \begin{pmatrix}
    (x_1 + \frac{13}{20})(\frac{7}{20}) - x_1 - \left(\frac{13}{20}\frac{x_2 + \frac{7}{20}}{x_1 + x_2 + 1}\right) \\
    10(x_1 + \frac{13}{20})(x_2 + \frac{7}{20}) - (x_2 + \frac{7}{20}) - \left(\frac{11(x_2 + \frac{7}{20})(x_3 + \frac{7}{20})}{x_2 + x_3 + \frac{7}{10}}\right) \\
    2(x_2 + \frac{7}{20})(x_3 + \frac{7}{20}) - (x_3 + \frac{7}{20}) \\
    \frac{7}{10} - \frac{7}{10}
\end{pmatrix}, \]
Table 2. Point and the Corresponding Lyapunov Exponent

<table>
<thead>
<tr>
<th>Point on plane AE</th>
<th>LE</th>
<th>Point on plane BG</th>
<th>LE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.6269, 1.3360, 0.2710)</td>
<td>0.0054</td>
<td>(0.6669, 1.3360, 0.2710)</td>
<td>0.0079</td>
</tr>
<tr>
<td>(0.6369, 1.3360, 0.2610)</td>
<td>0.0066</td>
<td>(0.6669, 1.3360, 0.2610)</td>
<td>0.0041</td>
</tr>
<tr>
<td>(0.6469, 1.3360, 0.2510)</td>
<td>0.0079</td>
<td>(0.6669, 1.3550, 0.2510)</td>
<td>0.0041</td>
</tr>
<tr>
<td>(0.6569, 1.3360, 0.2410)</td>
<td>0.0024</td>
<td>(0.6669, 1.3660, 0.2410)</td>
<td>0.0049</td>
</tr>
<tr>
<td>(0.6669, 1.3360, 0.2310)</td>
<td>0.0047</td>
<td>(0.6669, 1.3760, 0.2310)</td>
<td>0.0082</td>
</tr>
</tbody>
</table>

In order to check whether system (2.7) can be transformed into a linear and controllable system via state feedback and coordinates transformation, we have to compute the relative degree at point \( x = (0, 0, 0) \) in system (2.7) and test the conditions of Theorem (2.2).

\[
g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

\[
L_g h(x) = (1, 0, 0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0,
\]

\[
L_f h(x) = (x_1 + \frac{13}{20})(\frac{7}{20} - x_2) - \frac{(x_1 + \frac{13}{20})(x_2 + \frac{7}{20})}{x_1 + x_2 + 1},
\]

\[
L_g L_f h(x) = 0,
\]
Let
\[
\begin{aligned}
A(x) &= \frac{\partial L_x h(x)}{\partial x_1} \\
&= -\frac{3}{10} - 2x_1 - \frac{(x_2 + \frac{7}{20})^2}{(x_1 + x_2 + 1)^2},
\end{aligned}
\]
\[
B(x) = \frac{\partial L_x h(x)}{\partial x_2} \\
&= \frac{(x_1 + \frac{13}{20})^2}{(x_1 + x_2 + 1)^2},
\]
\[
C(x) = -\frac{\partial L_x h(x)}{\partial x_3} = 0.
\]
(10)

Thus
\[
L^2_x h(x) = A(x)f_{11}(x) + B(x)f_{12}(x),
\]
and
\[
\frac{\partial L^2_x h(x)}{\partial x_3} = B(x)\frac{\partial f_{12}(x)}{\partial x_3},
\]
where
\[
\frac{\partial f_{12}(x)}{\partial x_3} = -11 \frac{(x_2 + \frac{7}{20})(x_2 + x_3 + \frac{7}{10}) - (x_2 + \frac{7}{20})(x_3 + \frac{7}{20})}{(x_2 + x_3 + \frac{7}{10})^2},
\]
(13)

So, we have
\[
L_x L^2_x h(0) = B(x)\frac{\partial f_{12}(x)}{\partial x_3} \big|_{x=0} = \frac{11}{4} \left( \frac{13}{20} \right)^2 \neq 0,
\]
(14)

According to (2.7) and (2.14), we can see that system (2.7) has relative degree \( r = 3 \) at point \( (0,0,0) \). By means of the following transformation and feedback
\[
\Phi(x) = \begin{pmatrix}
    h(x) \\
    L_x h(x) \\
    L^2_x h(x)
\end{pmatrix}
\]
(15)

\[
x_1 = \left( x_1 + \frac{13}{20} \right) \frac{7}{20} - \frac{(x_1 + \frac{13}{20})(x_2 + \frac{7}{20})}{x_1 + x_2 + 1},
\]
\[
A(x)f_{11}(x) + B(x)f_{12}(x)
\]
(16)

\[
u = -\frac{L^3_x z_1}{L_x L^2_x z_1} + \frac{1}{L_x L^2_x z_1} v.
\]
where \( L_1^2 z_1 = L(L_1^2 h(x)) \). Around \( x^* = (0, 0, 0) \), system (2.7) can be transformed into the following linear and controllable normal form

\[
\begin{cases}
  \dot{z}_1 = z_2, \\
  \dot{z}_2 = z_3, \\
  \dot{z}_3 = v.
\end{cases}
\]

(17)

Its matrix form is

\[
\dot{z} = A_1 z + B_1 v
\]

where

\[
A_1 = \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
  0 \\
  0 \\
  1
\end{pmatrix}.
\]

(18)

Let \( v = Kz \), \( K = \text{diag} (K_1, K_2, K_3) \), we get closed-loop system of system (2.18)

\[
\dot{z} = (A_1 + B_1 K) z.
\]

(19)

For the system, we have the following theorem

**Theorem 3** If \( K \) is chosen such that all eigenvalues of matrix \( A_1 + B_1 K \) have negative real part, then system (2.18) is asymptotically stable at point \( x^* = (0, 0, 0) \).

Further, for system (2.7), we obtain the following main result

**Theorem 4** If \( K \) is chosen such that all eigenvalues of matrix \( (A_1 + B_1 K) \) have negative real part, and control \( u \) is as follows

\[
u = -\frac{L_1^3 z_1}{L_2 L_1^2 z_1} + \frac{1}{L_2 L_1^2 z_1} K z.
\]

then system (2.18) is asymptotically stable at point \( x^* = (0, 0, 0) \).

Remark 1: For system (2.20), \( z_1 \) and \( z \) are function of \( x \), thus \( u \) is represented by \( x \).

Remark 2: According to (2.6), the stability at point \( x^* = (0, 0, 0) \) in system (2.4) is equivalence to that at positive equilibrium \( x^0 = (13/20, 7/20, 7/20) \) in system (1.4) with the same control.

### 3. Simulation and Discussion

If \( X^1_0 = (0.5, 0.5, 0.5) \) and control is exerted on system (2.7), we get Figure 2.1

![Figure 2.1](image-url)
If $X_0^1 = (0.5, 0.3, -0.2)$ control is exerted on system (2.7), we get Figure 2.2

![Figure 2.2. Three dimensional graph in phase space and time response graph of system (2.7) with $X_0^1 = (0.5, 0.3, -0.2)$](image)

If $X_0^1 = (0.01, -0.004, 0.1)$ control is exerted on system (2.7), we get Figure 2.3

![Figure 2.3. Three dimensional graph in phase space and time response graph of system (2.7) with $X_0^1 = (0.01, -0.004, 0.1)$](image)

If $X_0^1 = (-0.04, 0.03, 0.02)$ control is exerted on system (2.7), we get Figure 2.4

![Figure 2.4. Three dimensional graph in phase space and time response graph of system (2.7) with $X_0^1 = (-0.04, 0.03, 0.02)$](image)

If $X_0^1 = (-0.04, -0.03, 0.02)$ control is exerted on system (2.7), we get Figure 2.5
Figure 2.5. Three dimensional graph in phase space and time response graph of system (2.7) with $X^1_0 = (0.04, -0.03, 0.02)$

If $X^1_0 = (0.04, -0.03, -0.02)$ control is exerted on system (2.7), we get Figure 2.6

Figure 2.6. Three dimensional graph in phase space and time response graph of system (2.7) with $X^1_0 = (0.04, -0.03, -0.02)$

If $X^1_0 = (-0.04, 0.03, -0.02)$ control is exerted on system (2.7), we get Figure 2.7

Figure 2.7. Three dimensional graph in phase space and time response graph of system (2.7) with $X^1_0 = (-0.04, 0.03, -0.02)$

If $X^1_0 = (-0.04, -0.03, -0.02)$ control is exerted on system (2.7), we get Figure 2.8
We take different points around $x^* = (0, 0, 0)$. From figures above, we can see that if control is exerted on system (2.7), then it is asymptotically stable at $x^* = (0, 0, 0)$.

Acknowledgement

This work is supported by the Natural Science Fund of China (70271066) and the Natural Science Fund of China (40372111)

References