CONTROL OF EPIDEMICS BY QUARANTINE AND ISOLATION STRATEGIES IN HIGHLY MOBILE POPULATIONS

XIEFEI YAN AND YUN ZOU

Abstract. In the absence of valid medicines or vaccine, quarantine and isolation strategies are the most important and effective measures against the outbreaks of epidemic diseases such as SARS. This paper discusses the application of the optimal quarantine and isolation strategies for SARS outbreak control via the Pontryagin’s Maximum Principle. We construct a multigroup SARS transmission model for traveling population and introduce pairs of control variables in terms of the quarantine and isolation strategies. A quadratic cost on the controls and a linear cost on the number of infected individuals are imposed. The simulation results illustrate how the disease spreads from region to region by means of travelers. And the results also demonstrate the importance of the early quarantine and isolation strategies and the necessity of the observation and quarantine of travelers to control the outbreaks of epidemics. This gives a theoretical interpretation to the practical experiences that the early quarantine and isolation strategies, as well as the observation and quarantine of travelers, are critically important to contain the epidemic.

Key Words. Severe acute respiratory syndrome (SARS), Optimization, Epidemiology, Quarantine, Isolation, Multigroup model.

1. Introduction

In the human history from ancient times to the present, quarantine and isolation strategies have been widely implemented against natural diseases, especially in the early stage of the epidemics outbreak without valid medicines or vaccine, such as the Black Death in the mid 14th century, influenza outbreak in 1918 and SARS (Severe acute respiratory syndrome) in late 2002 and 2003 [1]. The emergence of SARS challenged the global public health community to confront a novel epidemic that spread rapidly from its origins in southern China until it had reached more than 25 other countries within a matter of months. In addition to the number of patients infected with the SARS virus, totaling more than 8000 cases and 774 known deaths [2], the disease had profound economic and political repercussions in many of the affected regions. Since there are no valid medicines or vaccine for SARS, measures to control the spread of SARS had to take two major forms: isolation of symptomatic individuals and quarantine and close observation of asymptomatic individuals [1, 3, 4]. It had been shown that quarantine and isolation of the diseased individuals was a critically important strategy that can control SARS outbreaks because of the

Received by the editors May 9, 2008 and, in revised form, October 1, 2008.

2000 Mathematics Subject Classification. 49J15, 49K15, 92C50, 92D30.

This work is supported by the National Natural Science Foundation of China under Grant Nos. 60474078, 60574015, 60304001.
effective reduction in the contact rate between susceptible and diseased individuals [1,3,5–9].

Mathematical models have recently been used to examine the transmission dynamics and model the control of SARS in the literature [1,4–28]. For instance, Castillo-Chavez [1] has presented the mathematical models of isolation and quarantine for SARS (SEIJR). The quantitative assessment of the epidemic potential of SARS and the effectiveness of control measures have been discussed in Chowell et al [5,6], Gumel et al [7], Riley et al [11], Donnelly et al [12], Dye and Gay [13] and Lipsitch et al [14]. Chowell et al [5,6] attempted to obtain a threshold for the basic reproductive number $R_0$ for assessing the strategies of quarantine and isolation, and discussed control of SARS by looking at the role of disease transmission parameters in the reduction of $R_0$ and the prevalence of the disease. Gumel et al [7] examined mathematically the impact of isolation and quarantine on the control of SARS during the outbreaks in Toronto, Hong Kong, Singapore and Beijing. And the sensitivity and uncertainty analyses for the proposed SARS model with time-varying inputs and outputs have been discussed in [10]. Wang and Ruan [15] have proposed a mathematical model consists of six subpopulations, namely susceptible, exposed, quarantined, suspect, probable and removed, to simulate the SARS outbreak in Beijing. In addition, other mathematical models have also used to simulate the SARS outbreaks in China as well [15–21]. A stochastic dynamic model of SARS spreading has been given by Shi [17]. And the nosocomial spread of SARS has been studied using models introduced by Lloyd-Smith et al [22] (discrete, stochastic) and Webb et al [23] (continuous, deterministic). Small-world networks have been also introduced to study the propagation of the SARS by Masuda [24] and Small [25]. Furthermore, owing to the recent ongoing clinical trials of some candidate anti-SARS vaccines, the potential impact of SARS vaccination has been proposed, by Gumel [26] and Gjorgjieva et al [27], via mathematical model.

Most of above literatures [1,5–7,10–28] have been concerned with the simulative and predictive functions of models, thus providing insight into the dynamics of disease spread and, indirectly, leading to improved prevention of epidemics. However, these works did not consider the dynamical control strategies except our pervious work [8,9]. The optimal control theory has been widely discussed in the control of epidemics [29–52]. Moreover, the Pontryagin’s Maximum Principle [53] and the time dependent control strategies have been applied for the control of insect pests [29], pets and infectious disease [30], HIV/AIDS [31–38], cancer chemotherapy [39–44] mosquito-borne diseases [45], dengue models [46], tuberculosis models [47], influenza in Highly Mobile Populations [48] and others [49–52]. In our previous work [8,9], we have investigated the optimal control strategies associated with quarantining asymptomatic individuals and isolating symptomatic individuals for our SARS transmission model that is improved based on the models in Chowell et al [5] and Gumel et al [7]. The results have demonstrated that the early quarantine and isolation strategies are critically important to control the outbreaks of epidemics.

Additionally, in the epidemiological literature, the term “multigroup” or “compartment” usually refers to the division of a heterogeneous population into several homogeneous groups based on individual behaviors [54,55]. Each group is then subdivided into epidemiological compartments. The multigroup model has been extended in many ways to incorporate e.g. mortality and fertility or population structure by age, sex and spatial distribution [48]. The majority of multigroup models in the literature are used for sexually transmitted diseases, such as HIV/AIDS.
or gonorrhea, where behavior is an important factor in the probability of contracting the disease [54–57]. Recently, Zhang et al [21] have presented compartmental models for the analysis of SARS transmission patterns and outbreak control measures in China. And Ruan et al [28] have proposed a multiregional compartmental model using medical geography theory and regarding each outbreak zone as one region. The effect of the travel of individuals (especially the infected and exposed ones) between regions on the global spread of the disease has been discussed by calculating the basic reproduction number.

SARS, the first severe infectious disease to emerge in the twenty-first century, had taken advantage of opportunities for rapid international spread made possible by the unprecedented volume and speed of international travel [28]. This project was motivated by the need to understand how an epidemic progresses through a region or through the world by means of travelers, such as SARS. The situation we are mainly interested in is the outbreak control of epidemics in multiple regions.

In this paper, we first propose our multigroup SARS transmission model. The total population is divided into groups by different regions. Each group in the model monitors the dynamics of seven sub-populations (classes), namely susceptible, asymptomatic, quarantined, symptomatic, isolated, recovered and disease-induced dead individuals. Based on the work [8], we consider the optimal control strategies associated with quarantining asymptomatic individuals and isolating symptomatic individuals for this model. We introduce into each group two control variables representing the rate of quarantining of the asymptomatic individuals who have been exposed to the virus, but have not yet developed clinical symptoms and the rate of isolating of symptomatic individuals, respectively. Then our task is to find optimal quarantine and isolation strategies for our proposed multigroup model. Furthermore, numerical experiments about the influences of different numbers of travelers on optimal control are also carried out. This is important for the discussion of epidemics control because these numbers vary from place to place depending on many factors including time, economic relationship and capacity of transportation between regions.

The rest of the paper is organized as follows. Section 2 describes our multigroup SARS transmission model with control variables. The analysis of optimization problems are presented in section 3. In section 4, we give the simulation results and discussion. Finally, the conclusions are summarized in Section 5.

2. Multigroup Transmission Model For SARS

Our multigroup dynamic SARS transmission model is improved based on the models in Chowell et al [5], Gumel et al [7], Yan and Zou [8,9] and Ögren and Martin [48]. The host population in each group consists of seven sub populations: namely susceptible, asymptomatic, quarantined, symptomatic, isolated, recovered and disease-induced dead individuals. The detailed descriptions are in Table 1. The multigroup SARS transmission model with quarantine and isolation controls is given by the following nonlinear system of differential equations:
\[ \begin{align*}
\dot{S}_i &= -\beta_i S_i \left( I_i + q_i E_i + \varepsilon_i Q_i + l_i J_i \right) N_i - \sum_{j=1, j \neq i}^n \beta_{ij} S_i M_{ij} I_j N_i N_j, \\
\dot{E}_i &= \beta_i S_i \left( I_i + q_i E_i + \varepsilon_i Q_i + l_i J_i \right) N_i + \sum_{j=1, j \neq i}^n \beta_{ij} S_i M_{ij} I_j N_i N_j - (u_i(t) + k_i) E_i, \\
\dot{Q}_i &= u_i(t) E_i - \sigma_i Q_i, \\
\dot{I}_i &= k_i E_i - (v_i(t) + \mu_i + \delta_i) I_i, \\
\dot{J}_i &= v_i(t) I_i + \sigma_i Q_i - (\gamma_i + d_i) J_i, \\
\dot{R}_i &= \mu_i I_i + \gamma_i J_i, \\
\dot{D}_i &= \delta_i I_i + d_i J_i,
\end{align*}\]

where \( i = 1, 2, \ldots, n \), \( S_i(0) \), \( E_i(0) \), \( Q_i(0) \), \( I_i(0) \), \( J_i(0) \) and \( R_i(0) \) are given and the definitions of above model parameters are listed in Table 2.

<table>
<thead>
<tr>
<th>Class</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_i )</td>
<td>Susceptible individuals in ( i )th group</td>
</tr>
<tr>
<td>( E_i )</td>
<td>Asymptomatic individuals who have been exposed to the virus but have not yet developed clinical symptoms of SARS in ( i )th group</td>
</tr>
<tr>
<td>( Q_i )</td>
<td>Quarantined individuals in ( i )th group</td>
</tr>
<tr>
<td>( I_i )</td>
<td>Symptomatic individuals (infected, infectious and undiagnosed) in ( i )th group</td>
</tr>
<tr>
<td>( J_i )</td>
<td>Isolated individuals (for special diagnosis and treatment) in ( i )th group</td>
</tr>
<tr>
<td>( R_i )</td>
<td>Recovered individuals in ( i )th group</td>
</tr>
<tr>
<td>( D_i )</td>
<td>SARS disease-induced death in ( i )th group</td>
</tr>
<tr>
<td>( N_i )</td>
<td>Total population in ( i )th group ( N_i = S_i + E_i + Q_i + I_i + J_i + R_i + D_i )</td>
</tr>
</tbody>
</table>

Table 1. Epidemiological classes definitions

<table>
<thead>
<tr>
<th>Class</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_i )</td>
<td>Transmission rate per day in ( i )th group</td>
</tr>
<tr>
<td>( \beta_{ij} )</td>
<td>Transmission rate per day by travelers from ( j )th group to ( i )th group</td>
</tr>
<tr>
<td>( q_i )</td>
<td>Relative measure of infectiousness for the asymptomatic class ( E ) in ( i )th group</td>
</tr>
<tr>
<td>( \varepsilon_i )</td>
<td>Relative measure of reduced risk among quarantined SARS cases in ( i )th group</td>
</tr>
<tr>
<td>( l_i )</td>
<td>Relative measure of reduced risk among isolated SARS case in ( i )th group</td>
</tr>
<tr>
<td>( M_{ij} )</td>
<td>Number of travelers from ( j )th group to ( i )th group</td>
</tr>
<tr>
<td>( k_i )</td>
<td>Rate of normal progression to the infectious state per day in ( i )th group</td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>Rate of progression to the isolated class for treatment per day in ( i )th group</td>
</tr>
<tr>
<td>( \mu_i )</td>
<td>Rate at which individuals in the infectious class recover per day in ( i )th group</td>
</tr>
<tr>
<td>( \gamma_i )</td>
<td>Rate at which diagnosed individuals recover per day in ( i )th group</td>
</tr>
<tr>
<td>( \delta_i )</td>
<td>SARS-induced mortality in the infectious class per day in ( i )th group</td>
</tr>
<tr>
<td>( d_i )</td>
<td>SARS-induced mortality in the isolated class per day in ( i )th group</td>
</tr>
</tbody>
</table>

Table 2. Parameter definitions

We assume that a susceptible individual may be infected not only through contacts with an asymptomatic individual, a quarantined individual, a symptomatic
individual or an isolated individual from its own group but also through contacts with asymptomatic and symptomatic individuals by means of travelers from other groups. When considering the force of infection by other groups, following the idea of Ögren [48], we take into account the sizes of both the influenced and the influencing populations disregarding the rest. This is a reasonable such force. Since we are aiming for a traveler type of contact rate we replace the ordinary and vague rate [54–57] with the actual number of travelers $M_{ij}$. In this way the influence of another population is dependent on the proportion of asymptomatic and symptomatic individuals in that population and the proportion of people from the first population that travels to the second [48].

The control variables, $u_i(t)$ and $v_i(t)$, are bounded, Lebesgue integrable functions [47]. The control $u_i$ represents the rate of quarantining of people who have been in contact with an infected individual by a quarantine program and educational campaigns for close observation in $i$th group. The control $v_i$ represents the rate of isolating of symptomatic individuals by an isolation program for special medical treatment in $i$th group. Furthermore, from epidemiological modeling view, the transfer rate in $i$th group $u_i E_i$ corresponds to an exponential waiting time $e^{-u_i t}$ as the fraction that is still in the asymptomatic class $t$ units after entering this class and to $1/u_i$ as the mean waiting time [58]. And the interpretation of $v_i I_i$ is similar to $u_i E_i$.

3. The Optimization Problem

The problem is to minimize the cost function

$$J = \int_0^{t_f} \left[ \sum_{i=1}^{n} (E_i(t) + Q_i(t) + I_i(t) + J_i(t)) + \frac{1}{2} U^T C U + \frac{1}{2} V^T \overline{C} V \right] dt,$$

subjected to the differential equations (1)-(6), where $t_f$ is the final time, $U = (u_1, u_2, \ldots, u_n)^T$ and $V = (v_1, v_2, \ldots, v_n)^T$ are $n \times 1$ vectors and $C$ and $\overline{C}$ are positive semi definite $n \times n$ matrix. This performance specification involves the numbers of individuals of symptomatic, asymptomatic, quarantined, or isolated, respectively, as well as the cost of implementing the quarantine control ($u_i$) and the isolation control ($v_i$). A natural thing to minimize is $\int_0^{t_f} \sum_{i=1}^{n} (E_i + Q_i + I_i + J_i) dt$, that represents the total time spent infected in the population [48]. This is a good measure of the population inconvenience and it’s also proportional to the total number of people who endured the disease. But this approach would obviously lead to a maximal quarantine and isolation effort at all times. In fact, there is always a cost of implementing the controls. We must make a tradeoff between the two costs. This is realized in terms of adding a function of controls in the integral above. Here we choose a quadratic cost on the controls by reference to many literatures in epidemics control [31–39,42,43,45–49]. As we know, compared with a quadratic cost, a pure linear cost would lead to a more complicated analysis involving singular and bang-bang controls [29,30,39–41,44,50–52]. Moreover, the resulting continuous controls by applying a quadratic term are preferable in many cases when a sudden change in epidemics control is not advisable. Furthermore, by properly choosing the coefficients with controls, $C$ and $\overline{C}$, the behavior could be explored to close to bang-bang [48]. In this paper, for simplicity, we let $C$ and $\overline{C}$ be diagonal matrix: $C = \text{diag}\{c_1, c_2, \ldots, c_n\}$ and $\overline{C} = \text{diag}\{\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_n\}$, where $c_i$ and $\overline{c}_i$ ($i = 1, 2, \ldots, n$) are positive constants.

We seek to find an optimal control pair, $U^*$ and $V^*$, such that
Here $\Omega = \{(u_i, v_i) \in L^1(0, t_f) | a_i \leq u_i \leq b_i, \bar{\sigma}_i \leq v_i \leq \bar{b}_i, i = 1, 2, \ldots, n\}$ and $a_i$, $b_i$, $\bar{\sigma}_i$ and $\bar{b}_i$ $i = 1, 2, \ldots, n$ are fixed positive constants.

Pontryagin’s Maximum Principle \cite{53,59} provides the necessary conditions for an optimal control problem. This principle converts (1) - (6), (8) and (9) into a problem of minimizing a pointwise Hamiltonian $H_i$ with respect to $U$ and $V$:

$$H = \sum_{i=1}^{n}(E_i + Q_i + I_i + J_i + \frac{1}{2}c_iu_i^2 + \frac{1}{2}r_i v_i^2) + \sum_{i=1}^{n}\lambda_i^T f_i,$$

where $\lambda_i = (\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{6i})^T$, $f_i = (f_{1i}, f_{2i}, \ldots, f_{6i})^T$, $f_{ij}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, 6)$ is the right hand side of the differential equation of $j$th state variable in $i$th group. Let $X_i = (S_i, E_i, Q_i, I_i, J_i, R_i)^T$ be the states in $i$th group. By applying Pontryagin’s Maximum Principle \cite{53,59}, we have the adjoint equations

$$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial X_i}, \lambda_i(t_f) = 0, \; i = 1, 2, \ldots, n$$

i.e.

$$\lambda_{1i} = -\frac{\partial H}{\partial S_i} = (\lambda_{1i} - \lambda_{2i})[\beta_i (I_i + q_i E_i + \varepsilon_i Q_i + l_i J_i) - \sum_{j=1, j \neq i}^{n} \beta_{ij} \frac{M_{ij}}{N_i} I_j + q_i E_j],$$

$$\lambda_{2i} = -\frac{\partial H}{\partial E_i} = -1 + \lambda_{1i} \frac{\beta_i S_i q_i}{N_i} - \lambda_{2i} (\frac{\beta_i S_i q_i}{N_i} - u_i - k_i) - \lambda_{1i} u_i - \lambda_{1i} k_i + \frac{\sum_{k=1, k \neq i}^{n} (\lambda_{k1} - \lambda_{k2}) (\beta_{ki} S_k (\frac{M_{ki} q_k}{N_i})}{N_i},$$

$$\lambda_{3i} = -\frac{\partial H}{\partial Q_i} = -1 + \lambda_{1i} \frac{\beta_i S_i \varepsilon_i}{N_i} - \lambda_{2i} \frac{\beta_i S_i \varepsilon_i}{N_i} + \lambda_{1i} \sigma_i - \lambda_{1i} \sigma_i,$$

$$\lambda_{4i} = -\frac{\partial H}{\partial I_i} = -1 + \lambda_{1i} \frac{\beta_i S_i}{N_i} - \lambda_{2i} \frac{\beta_i S_i}{N_i} + \lambda_{1i} (v_i + \mu_i + \delta_i) - \lambda_{1i} v_i - \lambda_{1i} \mu_i + \frac{\sum_{k=1, k \neq i}^{n} (\lambda_{k1} - \lambda_{k2}) (\beta_{ki} S_k (\frac{M_{ki} 1}{N_i})}{N_i},$$

$$\lambda_{5i} = -\frac{\partial H}{\partial J_i} = -1 + \lambda_{1i} \frac{\beta_i S_i l_i}{N_i} - \lambda_{2i} \frac{\beta_i S_i l_i}{N_i} + \lambda_{1i} (\gamma_i + d_i) - \lambda_{1i} \gamma_i,$$

$$\lambda_{6i} = -\frac{\partial H}{\partial R_i} = 0$$

with transversality conditions.
\[ \lambda_{ij}(t_f) = 0, \ i = 1, 2, \ldots, n, j = 1, 2, \ldots, 6 \]

By the bounds in \( \Omega \), the optimal controls are given by

\[ u_i(t) = \min \{ \max \{ a_i, \frac{1}{c_i} (\lambda_{i3} - \lambda_{i2}) E_i \}, b_i \} \]

and

\[ v_i(t) = \min \{ \max \{ a_i, \frac{1}{c_i} (\lambda_{i4} - \lambda_{i5}) I_i \}, b_i \} \]

which are derived from the condition

\[ \frac{\partial H}{\partial U} = 0, \ \frac{\partial H}{\partial V} = 0. \]

**Remark 1:** Due to the a priori boundedness of the state and adjoint functions and the resulting Lipschitz structure of the ODEs, we obtain the uniqueness of the optimal control for small \( t_f \) \([47, 59]\). The uniqueness of the optimal control pair follows from the uniqueness of the optimality system, which consists of (1)-(6) and (8), (9) with characterizations (19) and (20). There is a restriction on the length of the time interval in order to guarantee the uniqueness of the optimality system. This smallness restriction on the length on the time interval is due to the opposite time orientations of (1)-(6), (8), and (9); the state problem has initial values and the adjoint problem has final values. This restriction is very common in control problems \([31, 33, 47]\).

**Remark 2:** The problem described above is a Two Point Boundary Value Problem (TPBVP), with specified initial conditions for state equations (1)-(6) and terminal boundary conditions (18) for adjoint equations (12)-(17).

4. Numerical Simulation

In the simulation, we study the cases of SARS outbreaks among Beijing, Guangdong and Hong Kong in China. Figure 1 illustrates the statistics of travelers among these three places. As we know, SARS emerged in Guangdong province late in 2002 and quickly spread over China, especially in Beijing and Hong Kong. First, let us look at the model without controls applied to the traveling and population data. Compared with the person stay in the group, normally, a traveler between groups contacts much more people. So the contact rate associated with infectious travelers is much higher. Here we assume \( \beta_{ij} \) equals 4 times \( \beta_i \). The other epidemiological parameters we used, in Table 3, are mainly from the published data \([7]\). Figure 2 shows that SARS starting in Guangdong spreads to Hong Kong without any quarantine and isolation control with initial values \( E_2(0) = 10, Q_2(0) = 0, I_2(0) = 1, J_2(0) = 0, R_2(0) = 0 \) in Guangdong. Without effective controls there is only a question of time until the majority of the population is contaminated. Furthermore, the case in Beijing from Figure 2 shows that after about 150 to 160 days, the outbreak of SARS in Beijing will reach the peak period. This case is validated by the fact that the peak season of SARS outbreak in Beijing was during April to May in 2003 \([21]\).

Then we study the impact of optimal quarantine and isolation control on SARS outbreaks among these three places. The above TPBVP problem is numerically solved by the improved multiple shooting method \([8, 46, 60]\). We set the upper
Figure 1. Travelers among Beijing, Guangdong and Hong Kong. The average numbers of travelers per day are $M_{12} = M_{21} = 30000$, $M_{13} = M_{31} = 30000$ and $M_{23} = M_{32} = 50000$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Beijing</th>
<th>Guangdong</th>
<th>Hongkong</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.23</td>
<td>0.23</td>
<td>0.20</td>
</tr>
<tr>
<td>$q$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$l$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$k$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.413</td>
<td>0.413</td>
<td>0.337</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.413</td>
<td>0.413</td>
<td>0.386</td>
</tr>
<tr>
<td>$\delta$</td>
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<td>0.0055</td>
<td>0.0079</td>
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<tr>
<td>$d$</td>
<td>0.0041</td>
<td>0.0041</td>
<td>0.0068</td>
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<tr>
<td>$N$</td>
<td>12 million</td>
<td>11 million</td>
<td>6.9 million</td>
</tr>
<tr>
<td>$E(0)$</td>
<td>1565</td>
<td>1850</td>
<td>1047</td>
</tr>
<tr>
<td>$Q(0)$</td>
<td>292</td>
<td>420</td>
<td>236</td>
</tr>
<tr>
<td>$I(0)$</td>
<td>695</td>
<td>802</td>
<td>325</td>
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<tr>
<td>$J(0)$</td>
<td>326</td>
<td>314</td>
<td>298</td>
</tr>
<tr>
<td>$R(0)$</td>
<td>20</td>
<td>45</td>
<td>35</td>
</tr>
</tbody>
</table>

Table 3. Baseline values of the SARS model parameters in Beijing, Guangdong and Hong Kong

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final time $t_f$</td>
<td>1 year</td>
</tr>
<tr>
<td>Time step duration</td>
<td>1 day</td>
</tr>
<tr>
<td>$u_i$ upper bound</td>
<td>0.50</td>
</tr>
<tr>
<td>$u_i$ lower bound</td>
<td>0.05</td>
</tr>
<tr>
<td>$v_i$ upper bound</td>
<td>0.50</td>
</tr>
<tr>
<td>$v_i$ lower bound</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 4. Values of the computational parameters

bound of $u_i$ equals 0.50 according to the reasonable case in China that it took at
least average 2 days to quarantine the asymptomatic individuals who have been
exposed to the virus, but have not yet developed clinical symptoms. And the
choice of upper bound of $v_i$ is similar to $u_i$. Considering the cost by implementing
the controls, we choose $C = \text{diag}(300, 300, 300)$ and $\overline{C} = \text{diag}(600, 600, 600)$ to
illustrate the optimal strategies. The other parameters are listed in Table 4.

**Remark 3:** We assume that the weight factor $c_i$ associated with control $v_i$ is much
larger than $c_i$ which is associated with control $u_i$. This assumption is based on the
following facts: The cost associated with $u_i$ mainly includes the cost of monitoring
and quarantining programs, while the cost associated with $v_i$ includes the cost of
monitoring and isolating programs, and the hospital treatment resource. The ideal
weights are very hard to obtain in practice. It needs a lot of work on data mining,
analyzing, and fitting. Hence, the determination of appropriate practical weights
is a difficult problem and further investigation is needed. It should be pointed out
that the weights in the simulations here are of only theoretical interest to illustrate the control strategies proposed in this paper.

![Graphs of optimal control laws for Beijing, Guangdong, and Hong Kong](image)

**Figure 3.** Optimal control laws

Figure 3 shows the optimal quarantine and isolation control laws. The controls $u$ and $v$ are plotted as a function of time. In order to minimize the total infected individuals ($E + Q + I + J$), the optimal control $u_1$ in Beijing is at its upper bound for 242 days and then is steadily decreasing to its lower bound value, while the optimal control $v_1$ stays at its upper bound about 93 days and then also steadily decreases to its lower bound over the rest simulation time. The optimal control laws in Guangdong and Hong Kong are similar to Beijing. The control laws $u_2$, $v_2$, $u_3$ and $v_3$ stay at their upper bound for 252 days, 105 days, 152 days and 20 days, respectively.
In fact, at the beginning of simulation time, both controls are staying at their upper bound in order to quarantine and isolate as many as asymptomatic individuals \((E)\) and symptomatic individuals \((I)\) to prevent the increasing of the number of the infected individuals. The steadily decreasing to lower values is determined by the balance between the cost of the infected individuals and the controls.

A comparison of the number of infected individuals under the optimal control, constant control \((u \equiv 0.2 \text{ and } v \equiv 0.2)\) and lower bound control \((u \equiv 0.05 \text{ and } v \equiv 0.05)\) are also implemented (Figure 4). It is easy to see that the optimal control is much more effective for reducing the number of infected individuals and decreasing the duration of the outbreak. As normally expected, in the early phase of the epidemic breakouts, keeping the quarantine and isolation controls at their upper bounds will directly leads to the decreasing of the number of the infected people. Moreover, if the lower bound values are implemented for each control throughout the simulation time, the number of infected individuals would reach about \(4.17 \times 10^6\) (nearly 34.8% of the total population), \(3.84 \times 10^6\) (nearly 34.9% of the total population) and \(2.20 \times 10^6\) (nearly 31.9% of the total population) in
Beijing, Guangdong and Hong Kong, respectively. This illustrates that the quarantine and isolation strategies in the early stage is critically important to control the outbreak of SARS.

Figure 5 illustrates how the optimal control strategies depend on the numbers of travelers. These numbers vary from place to place depending on many factors including time, economic relationship and capacity of transportation between regions. The controls are plotted for three different values of $M' = 0$, $M' = M$ and $M' = 2M$. It shows that both $u$ and $v$ play increasing roles as $M$ increases. This is an expected result because as $M$ increases, the new cases of infections caused by travelers increase too. Figure 6 illustrates that, under constant control, as the number of travelers increases, the amount of infected individuals increases, too. Compared with the results of single isolated region (the case with $M_{ij} = 0$) in [8],
from Figure 5 and 6, it can be seen that when dealing with the epidemics control we should give more attention to the influence of highly mobile population such as travelers. In other words, the high level of observation and quarantine of travelers is important during the outbreaks of highly contagious epidemic.

5. Conclusions

As we know, highly contagious and viral diseases are significant threats to the future of human beings. SARS represents the most recent challenge to the well being of our species posed by microbes and viruses. In this paper, we have constructed a multigroup SARS transmission model to show how an epidemic progresses through
regions by means of travelers. In order to better prepare us against future cata-

strophic epidemics, the optimal quarantine and isolation strategies have been stud-

ied. The numerical results demonstrate the importance of the early quarantine and 

isolation strategies and the necessity of the observation and quarantine of travelers 

to control the outbreaks of epidemics. This also gives a theoretical interpretation to 

the practical experiences that the early quarantine and isolation strategies, as well 

as the observation and quarantine of travelers, are critically important to contain 

the epidemic. It should be pointed out that, as was mentioned in [8], the ideal 

time-varying optimal strategy might not be applied in practice easily. Neverthe-

less, it does provide a reference basis on which to design the practical quasi-optimal 

control strategies or policies and assess their effectiveness.

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