THE PARAMETRIZATION FOR THE MATRIX PENCIL $A + BKC$ WITH CONSTANT RANK AND ITS APPLICATION

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Abstract. In this paper, parametrization of the variable $K$ in the matrix pencil $A + BKC$ with constant rank is presented. Furthermore, parametrization of the variable $K$ with the minimal Frobenius norm is investigated. An application of these parametrizations in dynamical order assignment for singular systems is also discussed. An illustrative example is given to show the effectiveness of the proposed results.

Key Words. Matrix pencil, Singular systems, Dynamical order assignment, Frobenius norm.

1. Introduction

In control theory and signal processing, many theoretical results are closely related to the rank issue for matrices with variables. For example, the criteria for controllability and observability for linear time-invariant systems are expressed explicitly with rank conditions [1, 2]. Also there are many results in the existing literature for the matrix theory for control systems, see [3] and references therein. Specially, in the theory of singular systems, some basic and important problems are related to the rank of the matrix $A + BKC$, for instance, the dynamical order assignment [4, 5], the elimination of impulsive modes [6, 7], and the eigenstructure assignment [8, 9, 10].

Formulas for the minimum and maximum rank of the matrix $A + BKC$ were presented in [4] and [11] respectively. However, in many cases, we not only need to know its rank value but also know how to parametrize $K$ for a given rank of $A + BKC$. In order to investigate this issue, the authors in [12] have proved that for all the ranks between the minimum and maximum ranks, there exists a matrix $K$ satisfying that the rank of $A + BKC$ is equal to the given rank. However, parametrization for all possible $K$ with a given rank of $A + BKC$ is still an open problem.

In this paper, the parametric form of the variable $K$ with a constant rank of $A + BKC$ is given when $K \in \mathbb{R}^{p \times q}$; Furthermore, the parametric form of the variable $K$ with the minimal Frobenius norm is presented, and an application of these parametrizations is also discussed.

This paper is organized as follows: section 2 will give the parametrization $K$ for a given rank of $A + BKC$. The parametrization with minimal Frobenius norm will be given in section 3. Its application in dynamic order assignment will be considered in section 4. An illustrative example will be presented in section 5 and some conclusions in section 6.

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2. Parametrization of $K$ for $A + BKC$ with constant rank

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{q \times n}$, $F \in \mathbb{R}^{q \times p}$, $K \in \mathbb{R}^{p \times q}$, where $A$, $B$, $C$ are constant matrices, $F$ and $K$ are variables changing freely. Also assume that all the matrices in the paper are compatible in dimensions. Then one major aim of this paper is to obtain the parametric form of the set

\[(1) \quad \mathcal{K} = \{K \mid \text{rank}[A + BKC] = h\}, \quad r \leq h \leq r^*,\]

where

\[(2) \quad r = \min_K \text{rank}[A + BKC], \quad r^* = \max_K \text{rank}[A + BKC].\]

It is clear from [12] that the set $\mathcal{K}$ in (1) is not empty. However, to parametrize all possible matrices $K$ in (1) is still an open problem and will be investigated in this paper. Before we proceed, we need the following result [12] from matrix theory:

**Lemma 1:** [12]

\[
\min_F \text{rank} \begin{bmatrix} 0 & B \\ C & F \end{bmatrix} = \text{rank}[B] + \text{rank}[C].
\]

**Theorem 1:** Let

\[r_{\text{min}} = \min_F \text{rank} \begin{bmatrix} 0 & B \\ C & F \end{bmatrix}, \quad r_{\text{max}} = \max_F \text{rank} \begin{bmatrix} 0 & B \\ C & F \end{bmatrix},\]

and

\[\mathcal{F} = \left\{ F \mid F \in \mathbb{R}^{q \times p}, \text{rank} \begin{bmatrix} 0 & B \\ C & F \end{bmatrix} = k \right\}, \quad r_{\text{min}} \leq k \leq r_{\text{max}}\]

Then

(i) $r_{\text{max}} = r_b + r_c + \min\{q - r_c, p - r_b\}$;

(ii) $\mathcal{F}$ can be parameterized as follows:

\[
\mathcal{F}(F_1, F_2, F_3, F_4) = \left\{ P_c \begin{bmatrix} F_1 \\ F_3 \\ F_4 \end{bmatrix} Q_b \mid F_1 \in \mathbb{R}^{r_c \times r_b}, F_2 \in \mathbb{R}^{r_c \times (p-r_b)}; F_3 \in \mathbb{R}^{(q-r_c) \times r_b}, F_4 \in \mathbb{R}^{(q-r_c) \times (p-r_b)}, \right. \\
\left. \text{rank}[F_4] = k - r_b - r_c \right\}
\]

where $\text{rank}[B] = r_b$, $\text{rank}[C] = r_c$, and the singular value decompositions (SVD) of the matrices $B$ and $C$ respectively, are

\[
B = P_b \begin{bmatrix} B_{r_b} & 0 \\ 0 & 0 \end{bmatrix} Q_b, \quad C = P_c \begin{bmatrix} C_{r_c} & 0 \\ 0 & 0 \end{bmatrix} Q_c,
\]

in which the matrices $P_b$, $Q_b$, $P_c$ and $Q_c$ are all orthogonal, $B_{r_b}$ and $C_{r_c}$ are diagonal non-singular matrices.

**Proof:** It is easy to compute the following:

\[
\begin{bmatrix} P_b^{-1} & 0 \\ 0 & P_c^{-1} \end{bmatrix} \begin{bmatrix} 0 & B \\ C & F \end{bmatrix} \begin{bmatrix} Q_c^{-1} & 0 \\ 0 & Q_b^{-1} \end{bmatrix} = \begin{bmatrix} 0 & P_b^{-1}BQ_b^{-1} \\ P_c^{-1}Q_c^{-1} & P_c^{-1}FQ_b^{-1} \end{bmatrix} \begin{bmatrix} 0 & B_{r_b} & 0 \\ 0 & 0 & 0 \\ C_{r_c}^{-1} & F_1 & F_2 \\ 0 & 0 & F_3 & F_4 \end{bmatrix},
\]

where

\[
P_c^{-1}FQ_b^{-1} = \begin{bmatrix} F_1 \\ F_3 \\ F_4 \end{bmatrix}, \quad F_4 \in \mathbb{R}^{(q-r_c) \times (p-r_b)}.
\]
From (3), we get

\[ \text{rank } \begin{bmatrix} 0 & B \\ C & F \end{bmatrix} = r_b + r_c + \text{rank}[F_4], \]

which indicates that \( F_1, F_2 \) and \( F_3 \) have no effect on the rank of \( \begin{bmatrix} 0 & B \\ C & F \end{bmatrix} \).

Based on this fact, we can obtain that \( r_{\min} = r_b + r_c \), which is consistent with Lemma 1. Since \( F_4 \) is an arbitrary matrix in \( \mathbb{R}^{(q-r_c) \times (p-r_b)} \), the part (i) of the theorem is true and the parametrization (part (ii) of the theorem) follows from equations (3) and (4).

\[ \square \]

**Remark 1:** Note that if \( B \) is of full column rank, i.e., \( r_b = p \), or if \( C \) is of full row rank, i.e., \( r_c = q \), then from Lemma 1 and Theorem 1, we have \( r_{\min} = r_{\max} = r_b + r_c \). This indicates that \( k \) can only be \( r_b + r_c \). In other words, for arbitrary matrix \( F \), the following is always true

\[ \text{rank } \begin{bmatrix} 0 & B \\ C & F \end{bmatrix} = r_b + r_c. \]

Then the following corollary holds:

**Corollary 1:** If \( r_b = p \) or \( r_c = q \), then \( F = \mathbb{R}^{q \times p} \).

**Remark 2:** From the parametrization (Theorem 1, part (ii)), we can see that the parameters \( F_1, F_2 \) and \( F_3 \) are totally free, however, the parameter \( F_4 \) is constrained by its rank value.

We will now use Theorem 1, to parametrize the set \( \mathcal{K} \) in (1) which leads to our main results:

**Theorem 2:** The set \( \mathcal{K} \) in (1) can be parameterized as

\( \mathcal{K}(K_1, K_2, K_3, K_4) = \left\{ C_1 r_{br} + P_{cr} \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} Q_{br} \mid K_1 \in \mathbb{R}^{r_{br} \times r_{br}}, \right. \)

\( K_2 \in \mathbb{R}^{r_{cr} \times (q-r_{br})}, \; K_3 \in \mathbb{R}^{(p-r_{cr}) \times r_{br}}, \; K_4 \in \mathbb{R}^{(p-r_{cr}) \times (q-r_{br})}, \)

\[ \text{rank}[K_4] = h + p + q - r_a - r_{br} - r_{cr} \}

where

\[ \text{rank } \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r_a, \; T_1 \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} T_2 = \text{diag} \{ I_{r_a}, 0 \} \]

\[ T_1 \begin{bmatrix} 0 \\ I_q \end{bmatrix} = \begin{bmatrix} B_{1r} \\ B_{2r} \end{bmatrix}, \; \begin{bmatrix} 0 & I_r \end{bmatrix} T_2 = \begin{bmatrix} C_{1r} & C_{2r} \end{bmatrix} \]

\[ \text{rank}[B_{2r}] = r_{br}, \; \text{rank}[C_{2r}] = r_{cr}, \]

and

\[ B_{2r} = P_{br} \begin{bmatrix} B_{cr} & 0 \\ 0 & 0 \end{bmatrix} Q_{br}, \; C_{2r} = P_{cr} \begin{bmatrix} C_{cr} & 0 \\ 0 & 0 \end{bmatrix} Q_{cr} \]

are SVD of matrices \( B_{2r} \) and \( C_{2r} \) respectively, and the matrices \( P_{br}, Q_{br}, P_{cr} \) and \( Q_{cr} \) are all orthogonal, \( B_{r_{br}} \) and \( C_{r_{cr}} \) are diagonal non-singular matrices.
Proof: First, we have
\[
\text{rank} \begin{bmatrix} A & B & 0 \\ C & 0 & I_q \\ 0 & I_p & K \end{bmatrix} = \text{rank} \begin{bmatrix} A & B & 0 \\ C & 0 & I_q \\ 0 & I_p & K \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_p & -K \\ 0 & 0 & I_q \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_p & 0 \\ -C & 0 & I_q \end{bmatrix}
\]
\[
= \text{rank} \begin{bmatrix} A + BKC & B & -BK \\ 0 & 0 & I_q \\ 0 & I_p & 0 \end{bmatrix} = \text{rank} [A + BKC] + p + q,
\]
that is
\[
(9) \quad \text{rank} [A + BKC] = \text{rank} \begin{bmatrix} A & B & 0 \\ C & 0 & I_q \\ 0 & I_p & K \end{bmatrix} - p - q.
\]
So we have
\[
\mathcal{K} = \mathcal{K}_1,
\]
where
\[
\mathcal{K}_1 = \left\{ K \mid \text{rank} \begin{bmatrix} A & B & 0 \\ C & 0 & I_q \\ 0 & I_p & K \end{bmatrix} = h + p + q \right\},
\]
that is the parametrization problem for the set \( \mathcal{K} \) is equivalent to parametrization of the set \( \mathcal{K}_1 \).

From (5) and (6), we have
\[
(10) \quad \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \\ 0 & I_p & K \end{bmatrix} = \begin{bmatrix} I_{r_a} & 0 & B_{1r} \\ 0 & 0 & B_{2r} \\ C_{1r} & C_{2r} & K \end{bmatrix},
\]
where
\[
T_1 \begin{bmatrix} A \\ C \\ 0 \end{bmatrix} T_2 = \text{diag} \{ I_{r_a}, 0 \}, \quad T_1 \begin{bmatrix} 0 \\ I_q \end{bmatrix} = \begin{bmatrix} B_{1r} \\ B_{2r} \end{bmatrix}, \quad T_2 \begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} C_{1r} & C_{2r} \end{bmatrix}.
\]
Further, we simplify the right side of (10), to obtain
\[
\begin{bmatrix} I_{r_a} & 0 & 0 \\ 0 & I_{m+p+r_a} & 0 \\ -C_{1r} & 0 & I_p \end{bmatrix} \begin{bmatrix} I_{r_a} & 0 & B_{1r} \\ 0 & 0 & B_{2r} \\ C_{1r} & C_{1r} & K \end{bmatrix} \begin{bmatrix} I_{r_a} & 0 & 0 \\ 0 & I_{n+p-r_a} & 0 \\ 0 & 0 & I_q \end{bmatrix}
\]
\[
= \begin{bmatrix} I_{r_a} & 0 & 0 \\ 0 & 0 & B_{2r} \\ 0 & C_{2r} & K - C_{1r}B_{1r} \end{bmatrix}.
\]
From (10) and above equation, we have
\[
(11) \quad \text{rank} \begin{bmatrix} A & B & 0 \\ C & 0 & I_q \\ 0 & I_p & K \end{bmatrix} = r_a + \text{rank} \begin{bmatrix} 0 & B_{2r} \\ C_{2r} & K - C_{1r}B_{1r} \end{bmatrix}.
\]
Next, applying \textbf{Theorem 1} to the matrix
\[
(12) \quad \begin{bmatrix} 0 & B_{2r} \\ C_{2r} & K - C_{1r}B_{1r} \end{bmatrix},
\]
we get the parametrization result. \hfill \Box

The following corollary follows immediately.

\textbf{Corollary 2:} If \( r_{cr} = p \) or \( r_{br} = q \). Then \( \mathcal{K} = \mathbb{R}^{p \times q} \).

\textbf{Theorem 3:}
\[
(13) \quad \min_K \text{rank}[A + BKC] = r_a + r_{br} + r_{cr} - p - q;
\]
and
\[ \max_K \operatorname{rank}[A + BKC] = r_a + r_{br} + r_{cr} - p - q + \min\{p - r_{cr}, q - r_{br}\}. \]

**Proof:** Applying Lemma 1 to the matrix (12), we can get equation (13); and applying Theorem 1 (i) to the matrix (12), we can get equation (14). □

Using the results derived here, we present more elegant and alternative proofs for the results in [12].

**Lemma 2:** [12]
\[ \min_K \operatorname{rank}[A + BKC] = \operatorname{rank}[A B] + \operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} A \\ C \\ 0 \end{bmatrix}; \]
and
\[ \max_K \operatorname{rank}[A + BKC] = \min\{\operatorname{rank}[A B], \operatorname{rank}[A^T C^T]\}. \]

**Proof:** On one hand, we have
\[ \operatorname{rank}\begin{bmatrix} A & B & 0 \\ C & 0 & I_q \end{bmatrix} = \operatorname{rank}[A B] + q, \]
and on the other hand, from (10), we know that
\[ \operatorname{rank}\begin{bmatrix} A & B & 0 \\ C & 0 & I_q \end{bmatrix} = r_a + r_{br}, \]
therefore, one can obtain
\[ \operatorname{rank}[A B] = r_a + r_{br} - q; \]
Similarly, one can obtain
\[ \operatorname{rank}[A^T C^T] = r_a + r_{cr} - p. \]
Then, from Theorem 3, we can obtain
\[ \min_K \operatorname{rank}[A + BKC] = \max_K \operatorname{rank}[A + BKC] = r_a + r_{br} + r_{cr} - p - q + \min\{p - r_{cr}, q - r_{br}\} \]
\[ = r_a + r_{br} + r_{cr} - p - q + \min\{r_{cr} - p, r_{br} - q\} \]
\[ = \operatorname{rank}[A B] + \operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} A \\ C \\ 0 \end{bmatrix}. \]
Similarly,
\[ \max_K \operatorname{rank}[A + BKC] = \min\{\operatorname{rank}[A B], \operatorname{rank}[A^T C^T]\}, \]
which indicates that the results here are equivalent to ones in Theorem 3. □

**Remark 3:** From (15) and (16), the rank of $K_4$ in parametrization of Theorem 2 can be easily shown as follows:
\[ \operatorname{rank}[K_4] = h + p + q - r_a - r_{br} - r_{cr} \]
\[ = h - \operatorname{rank}[A B] - \operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} + \operatorname{rank}\begin{bmatrix} A \\ C \\ 0 \end{bmatrix} \]
\[ = h - \min_K \operatorname{rank}[A + BKC] \]
\[ = h - r. \]
Remark 4: Let
\[ F_f = \{ F \mid F \in \mathbb{R}^{q \times p}, \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix} = k \}, \]
where
\[ r_{f \min} = \min_F \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix}, \quad r_{f \max} = \max_F \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix}, \]
and \( r_{f \min} \leq k \leq r_{f \max} \), then it is easy to get the parametric form of the set \( F_f \) via a similar procedure as shown above for the matrix pencil \( A+BKC \), and therefore, it is omitted here for brevity.

3. Parametrization of the set \( K \) with minimal Frobenius norm

Frobenius norm of a matrix has a lot of applications. For example, in modern signal and image processing fields, Frobenius matrix norm is related to the concept of energy of an image. For instance, if we want to compress an image which is represented by a matrix, we usually want to remove as much data as possible without losing too much energy of the image, in this case, we can use the Frobenius norm to measure the difference in energy between the original image and the compressed image. Therefore, in this section, we are motivated to provide the parametric form of all the possible matrices \( K \) in \( K \) with minimum Frobenius norm.

Assuming \( \gamma = \min_{K \in K} \| K \|_F^2 \), we derive the parametric form of the set \( K_F \), where
\[ K_F = \{ K \mid K \in K, \| K \|_F^2 = \gamma \}. \]

We need the following lemma for this derivation.

Lemma 3: [13] Let a matrix \( M \in \mathbb{R}^{r \times n} \), has the following structure:
\[ M = \text{diag} \{ D, 0 \}, \]
where
\[ D = \text{diag} \{ d_1, d_2, \cdots, d_m \} = \text{diag} \{ \lambda_1 I_{m_1}, \lambda_2 I_{m_2}, \cdots, \lambda_s I_{m_s} \}, \]
and \( \sum_{i=1}^{s} m_i = m, d_1 \leq d_2 \leq \cdots \leq d_m, \lambda_1 < \lambda_2 < \cdots < \lambda_s \). Furthermore, let \( l_0 = 0, l_k = \sum_{i=1}^{k} m_i, k = 1, 2, \cdots, s \). Then for arbitrary matrices \( X \in \mathbb{R}^{r \times n}_p \), where \( \mathbb{R}^{r \times n}_p \) denote all real matrices whose dimension is \( r \times n \) with rank \( p < m \), the following inequality holds:
\[ \| X - M \|_F^2 \geq d_1^2 + d_2^2 + \cdots + d_{m-p}^2. \]
Furthermore, the equality holds if and only if
\[ X = \begin{bmatrix} 0 \\ D + P \hat{X} D \end{bmatrix}, \]
where
\[ \hat{X} = \begin{bmatrix} 0 \\ \hat{Y} \end{bmatrix}, \]
and \( \hat{Y} = \text{diag} \{ Y_1, Y_2, \cdots, Y_{k_{m-p}} \}, 1 \leq k_{m-p} \leq m - p \leq l_{m-p}, P \in \mathbb{R}^{m \times m} \) is an arbitrary orthogonal matrix, \( Y_i \in \mathbb{R}^{m_i \times m}, i = 1, 2, \cdots, k_{m-p} - 1 \), and \( Y_{k_{m-p}} \in \mathbb{R}^{(m - p - k_{m-p} + 1) \times m_{k_{m-p}}} \) are full-row rank upper triangular matrices satisfying
\[ Y_i Y_i^T = I, i = 1, 2, \cdots, k_{m-p}. \]

With this lemma, we can have the following theorems.
**Theorem 4:** If one of the following conditions is satisfied, then \( \gamma = 0 \), and \( K_F = \{0\} \):
1) \( r_{cr} = p \);
2) \( r_{br} = q \);
3) \( \text{rank}[P_4] = h - r \),
where the matrix \( P_4 \) is obtained from partitioning:

\[
P^{-1}_{cr} C_{1r} B_{1r} Q^{-1}_{br} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}.
\]

**Proof:** Proofs of 1) and 2) follows immediately from Corollary 2.

Proof of 3): From the parametrization (16), for arbitrary matrices \( K \in \overline{K} \), there holds

\[
\|K\|_F^2 = \left\| P_{cr} \begin{bmatrix} P_1 + K_1 \\ P_3 + K_3 \\ P_4 + K_4 \end{bmatrix} Q_{br} \right\|_F^2.
\]

Moreover, since the matrices \( P_{cr} \) and \( Q_{br} \) are orthogonal, the following equality holds:

\[
\|K\|_F^2 = \left\| \begin{bmatrix} P_1 + K_1 \\ P_3 + K_3 \\ P_4 + K_4 \end{bmatrix} \right\|_F^2
= \|P_1 + K_1\|_F^2 + \|P_3 + K_3\|_F^2 + \|P_4 + K_4\|_F^2.
\]

Since \( K_1, K_2 \) and \( K_3 \) are arbitrarily free matrices, we have

\[
\min_{K_i} \|P_i + K_i\|_F^2 = 0, \ i = 1, 2, 3,
\]
and

\[
\gamma = \|P_4 + K_4\|_F^2.
\]

Then since \( \text{rank}[P_4] = h - r = \text{rank}[K_4] \), we can let \( K_4 = -P_4 \), in this case, \( \gamma = 0 \), and furthermore, \( K_F = \{0\} \).

**Theorem 5:** If none of the conditions in Theorem 4 (conditions 1-3) are satisfied, then
i) If \( \text{rank}[P_4] < h - r \), then the minimum \( \gamma \) does not exist, but it can be as small as possible;
ii) If \( \text{rank}[P_4] > h - r \), then

\[
\gamma = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_t^2.
\]

Furthermore, in this case, the set \( K_F \) can be parameterized as follows:

\[
K_F = \left\{ P_{cr} \begin{bmatrix} 0 \\ 0 \\ P_4 + P \lambda P_4 + P \bar{X} \lambda P_4 \alpha \end{bmatrix} Q_{br} \right\},
\]

where \( Q_{br} \) and \( P_{cr} \) are same as in (8), and the SVD of the matrix \( P_4 \) is given by:

\[
P_4 = P \begin{bmatrix} \lambda P_4 \\ 0 \end{bmatrix} Q,
\]

where \( P \) and \( Q \) are both orthogonal matrices, \( \lambda P_4 \) is diagonal non-singular matrix having the following form

\[
\lambda P_4 = \text{diag} \{\alpha_1, \alpha_2, \cdots, \alpha_{p_4}\} = \text{diag} \{\beta_1 I_{m_1}, \beta_2 I_{m_2}, \cdots, \beta_s I_{m_s}\}.
\]
where \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{p_4}, \sum_{i=1}^{s} m_i = p_4, p_4 = \text{rank}[P_4] \). And let \( l_0 = 0, l_k = \sum_{i=1}^{k} m_i, k = 1, 2, \cdots, s \). And
\[
(24) \quad \tilde{X} = \begin{bmatrix} 0 & 0 \\ \tilde{\Gamma} & 0 \end{bmatrix},
\]
here \( \tilde{\Gamma} = \text{diag} \{ Y_1, Y_2, \cdots, Y_{k_t} \} \), \( l_{k_t-1} < t \leq l_{k_t}, \tilde{P} \in \mathbb{R}^{p_4 \times p_4} \) are arbitrary orthogonal matrices, \( Y_i \in \mathbb{R}^{m_i \times m_i}, i = 1, 2, \cdots, k_t - 1, Y_{k_t} \in \mathbb{R}^{(t-l_{k_t-1}) \times m_{k_t}} \) are full-row rank upper triangular matrices satisfying
\[
(25) \quad Y_i Y_i^T = I, i = 1, 2, \cdots, k_t.
\]
And \( t = \text{rank}[P_4] + r - h \).

**Proof:** From the decomposition (22), we have
\[
(26) \quad \| P_4 + K_4 \|_F = \left\| \begin{bmatrix} \Lambda P_4 & 0 \\ 0 & 0 \end{bmatrix} + P_4^{-1} K_4 Q^{-1} \right\|_F.
\]
Now, we discuss two cases respectively, according to the rank relation between \( \text{rank}[P_4] \) and \( \text{rank}[K_4] = h - r \).

Proof of i) In this case, \( \text{rank}[P_4] < \text{rank}[K_4] \), then \( \gamma \) must not be zero, that is \( \gamma > 0 \). Then we choose
\[
K_4 = P \text{diag} \{ -\Lambda P_4, \varepsilon I_{\text{rank}[K_4] - \text{rank}[P_4]} \} Q,
\]
where \( \varepsilon \) is arbitrary small nonzero positive real number. Then there holds:
\[
\| P_4 + K_4 \|_F^2 = \left\| \text{diag} \{ 0, \varepsilon I_{\text{rank}[K_4] - \text{rank}[P_4]} \} \right\|_F^2 = (\text{rank}[K_4] - \text{rank}[P_4]) \varepsilon^2.
\]
Together with (19), if one chooses \( K_1 = -P_1, K_2 = -P_2, K_3 = -P_3 \), i.e., if we choose
\[
K = P_{cr} \begin{bmatrix} 0 & 0 \\ 0 & \text{diag} \{ 0, \varepsilon I_{\text{rank}[K_4] - \text{rank}[P_4]} \} \end{bmatrix} Q_{br},
\]
then
\[
\gamma = (\text{rank}[K_4] - \text{rank}[P_4]) \varepsilon^2,
\]
which indicates that i) is true.

Proof of ii) Applying **Lemma 3** to equation (26), one can easily get the parametric form (21) for the set \( \mathbb{K}_F \).

From the above two theorems, we can see that in some cases, \( \gamma = 0 \) and there is a unique zero element in the set \( \mathbb{K}_F \). In some cases, the minimum \( \gamma \) does not exist, but we can make it as small as possible; in other cases, the minimum \( \gamma \) is an exact value and the parametric form is given in (21).

To summarize the above procedure, we provide the following algorithm to get the parametrization of the set \( \mathbb{K}_F \):

**Algorithm:**

**Step 1.** For given matrices \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times p}, C \in \mathbb{R}^{q \times n} \) and a given integer \( h \), compute \( r \) and \( r^* \) in (2) using **Lemma 2**;

**Step 2.** If \( r \leq h \leq r^* \), continue; Otherwise, stop;

**Step 3.** Compute \( r_a \) in (5) and the matrices \( T_1, T_2 \) satisfying (10);

**Step 4.** From (10), get the matrices \( B_{1r}, B_{2r} \) and \( C_{1r}, C_{2r} \);

**Step 5.** Compute \( r_{br} \) and \( r_{cr} \) in (7);

**Step 6.** If \( r_{br} = q \) or \( r_{cr} = p \), then \( \mathbb{K} = \mathbb{R}^{p \times q} \), and \( \gamma = 0 \), \( \mathbb{K}_F = \{0\} \), stop; Otherwise, continue;

**Step 7.** Compute the SVD of \( B_{2r} \) and \( C_{2r} \) (equation (8)) to get the matrices \( Q_{br} \) and \( P_{cr} \);
Step 8. Put the obtained matrices $B_{1r}$, $C_{1r}$, $Q_{br}$ and $P_{cr}$ into the parametric form to get the set $\overline{K}$;

Step 9. Compute $\text{rank}[K_4]$ in (17);

Step 10. Compute (18) to get the matrix $P_4$ and $\text{rank}[P_4]$;

Step 11. If $\text{rank}[P_4] < \text{rank}[K_4]$, then the minimum $\gamma$ does not exist, but it can be as small as possible, stop; Otherwise, continue;

Step 12. If $\text{rank}[P_4] = \text{rank}[K_4]$, then the minimum $\gamma = 0$, and $\overline{K}_F = 0$, stop; Otherwise, continue;

Step 13. Compute the value $\gamma$ in (20);

Step 14. Compute the SVD $P_4$ (equation (22)) to obtain $P$, $Q$ and $\Lambda_{P_4}$ (equation (23)) ;

Step 15. Choose $Y_i$ satisfying (25), $i = 1, 2, \cdots, k_t$ to get the matrix $\tilde{X}$ in (24);

Step 16. Using matrices obtained construct the parametric form (21) of the set $\overline{K}_F$, stop.

In the next section, we will apply the proposed results to solve the dynamical order assignment problem for singular systems using output derivative feedback.

4. Dynamic order assignment for singular systems via output derivative feedback with minimum Frobenius norm

Consider the linear singular systems in the following form:

\[
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
\]

(27)

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$ are the state vector, the control input vector and output vector, respectively; $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$ are constant matrices; And without loss of generality, assume $B$ is of full column rank.

The dynamical order of the system (27) is $\text{rank}[E] = r < n$. Assume that the system is regular, i.e., there exists a constant scalar $\lambda \in \mathbb{C}$, such that $|\lambda E - A| \neq 0$. In this section, the realization quadruple $(E, A, B, C)$ is used to represent system (27), which is assumed to be minimal.

The problem of dynamical order assignment is fundamental to singular systems theory. It can be stated as follows:

*Given a singular system in the typical form and an appropriate integer $r_0$, find a feedback gain in one class of feedback gains for the given system (27) such that the closed-loop system has dynamical order $r_0$.*

In this section, we will discuss this assignment problem via output derivative feedback. From [12], the dynamical order of system (27) can be assigned to be between the minimum and maximum of the rank of the matrix pencil $E + BKC$. However, the gain with minimum Frobenius norm is not provided there.

We will apply the derivative output controller:

\[ u = -K \dot{y} + v \]

to system (27) and a closed-loop system is obtained as follows:

\[ (E + BKC)\dot{x}(t) = Ax(t) + Bu. \]

The dynamical order of the above closed-loop system is given by $\text{rank}[E + BKC]$, which is dependent on the feedback gain matrix $K$. After applying Theorem 2 to $E + BKC$, we can get the parametric form of $K$, such that $\text{rank}[E + BKC] = r_0$, and the assignment problem will be solved. In addition, we can apply the results in
section 3 to get the gain with minimum Frobenius norm. A numerical example will be used to illustrate the effectiveness of the proposed approach in the next section.

5. Numerical example

Consider the dynamical order assignment problem for the system \((E, A, B, C)\) in [13], which has the following parameters:

\[
E = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 3 & 6 & -3 & -12 \\ -1 & -1 & 0 & 2 \\ 2 & 3 & -1 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 3 & 9 & 3 \\ 0 & -6 & 9 & 18 \\ 0 & 4 & -9 & -15 \\ 0 & -4 & 9 & 15 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 2 & 1 \\ -3 & 3 \\ 1 & -2 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}^T.
\]

In this example, \(m = n = 4, p = q = 2, \text{rank}[E] = 2\).

Now, we apply the algorithm in section 3 to the above matrices \(E, B\) and \(C\). First, from Lemma 2, we can obtain that \(r = 2, r^* = 3\) and \(r_a = 5\). Then we want to get the matrix \(K\) such that \(\text{rank}[E + BKC] = r_0, 2 \leq r_0 \leq 3\).

In [13], the state derivative feedback was used to solve the assignment problem. Next, we will use the output derivative feedback and apply the algorithm in section 3 to the above example.

After running some Matlab functions, we can get the following matrices

\[
C_{1r} = \begin{bmatrix} 0.1755 & 0.6795 & 0.5348 & 0.2257 & 0.3435 \\ -0.2351 & 0.2230 & -0.7256 & 0.1961 & 0.5267 \end{bmatrix}, \quad C_{2r} = \begin{bmatrix} 0.2294 \\ 0.2294 \end{bmatrix},
\]

\[
B_{1r} = \begin{bmatrix} -0.0008 & 0.0445 & -0.0401 & -1.1379 & -0.3503 \\ 0.0035 & 0.0393 & -0.2000 & 0.3365 & -1.2153 \end{bmatrix}^T, \quad B_{2r} = \begin{bmatrix} 0 & 0 \end{bmatrix},
\]

\[
C_{1r} B_{1r} = \begin{bmatrix} -0.3684 & -0.4211 \\ -0.3684 & -0.4211 \end{bmatrix}, \quad P_{cr} = P_{cr}^{-1} = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix},
\]

\[
Q_{br} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad r_{cr} = \text{rank}[C_{2r}] = 1, \quad r_{br} = \text{rank}[B_{2r}] = 0.
\]

And after computing, we can get \(P_4 = \begin{bmatrix} 0 & 0 \end{bmatrix}\).

Applying Theorem 2, we will obtain parametric form of the set \(\overline{K}\) as follows:

\[
\overline{K} (K_2, K_4) = \left\{ C_{1r} B_{1r} + P_{cr} \begin{bmatrix} K_2 \\ K_4 \end{bmatrix} \mid K_2 \in \mathbb{R}^{1 \times 2}, K_4 \in \mathbb{R}^{1 \times 2}, \text{rank}[K_4] = r_0 - 2 \right\}.
\]

If we choose \(r_0 = 2\), then

\[
\overline{K} (K_2) = \left\{ C_{1r} B_{1r} + P_{cr} \begin{bmatrix} K_2 \\ 0 \end{bmatrix} \mid K_2 \in \mathbb{R}^{1 \times 2} \right\}.
\]

But in this case, \(\text{rank}[P_4] = \text{rank}[K_4] = 0\), then from Theorem 4, \(\gamma = 0\) and

\[
\overline{K}_F = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},
\]

that is, in this case, there is unique zero solution to the dynamical order assignment problem with minimum Frobenius norm. If we choose \(r_0 = 3\), then we have

\[
\overline{K} (K_2, K_4) = \left\{ C_{1r} B_{1r} + P_{cr} \begin{bmatrix} K_2 \\ K_4 \end{bmatrix} \mid K_2 \in \mathbb{R}^{1 \times 2}, K_4 \in \mathbb{R}^{1 \times 2}, \text{rank}[K_4] = 1 \right\}.
\]
In this case, \( \text{rank}[P_4] < \text{rank}[K_4] = 1 \), therefore, from Theorem 4, the minimum \( \gamma \) does not exist, but it can be arbitrarily as small as possible. Given \( \varepsilon > 0 \), we can get the following parametric form for the set \( \mathcal{K}_{\varepsilon} \) with the Frobenius norm \( \varepsilon \):

\[
\mathcal{K}_{\varepsilon}(k_1, k_2) = \left\{ K \mid K \in \mathcal{K}(K_2, K_4), \|K\|_F^2 = \varepsilon \right\}
\]

\[
= \left\{ \begin{bmatrix} k_1 - k_2 & k_3 - dk_2 \\ k_1 + k_2 & k_3 + dk_2 \end{bmatrix} ight| k_1, k_2, k_3, d \in \mathbb{R}, k_2 \neq 0
\]

\[
\text{and } k_1^2 + (1 + d^2)k_2^2 + k_3^2 = \varepsilon^2/2 \}.
\]

So far, we have solved the dynamical order assignment problem with minimum Frobenius norm for the above example.

6. Conclusions

In this paper, the parametric form of the variable \( K \) in the matrix pencil \( A + BKC \) with constant rank is derived when \( K \) varies in \( \mathbb{R}^{p \times q} \); Furthermore, the parametric form of the variable \( K \) with the minimum Frobenius norm is also presented. Based on these parametrizations, we have completely solved the problems of dynamical order assignment with the minimum Frobenius norm via output derivative feedback in singular systems.

The advantage of dynamic order assignment proposed in this paper is twofold. We not only give all possible feedback gains but also provide all possible gains with minimum Frobenius norm. This will provide flexibility in controller system design.

Theoretically, the parametrization presented in this paper is very important contribution to algebraic theory. The results obtained will have potential applications in robust control, pole assignment, etc since it provides a full freedom in controller system design. The proposed results also have applications in signal processing in particular Karhunen-Loeve Transform [14] which is left as a suggestion for future work.

References


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