STABILIZATION AND PERSISTENCE FOR AN N-SPECIES FOOD CHAIN FEEDBACK CONTROL SYSTEM IN A POLLUTED ENVIRONMENT

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Abstract. Stability and persistence of the populations for an n-species food chain system in a polluted environment have been studied. One of the main results presents sufficient conditions to guarantee the persistence of each population via static feedback control; on the other hand, via adaptive control, the global stability has been obtained.

Key Words. population, state feedback, adaptive control, persistence, asymptotically stable.

1. Introduction

As organisms are regularly exposed to some of the two million natural and synthetic chemicals, the probability of a toxic effect is huge. The fact that the potential consequences of exposure to these multitudes of pollutants cover a range from physiological changes to biospherical modifications, strongly motivate the area of ecological control systems.

In the early 1980s, T. G. Hallam and his colleagues put forward an idea to use the methods of dynamics to investigate the ecotoxicology[1, 2]. They proposed a simple toxicant-population model based on logistic equation and investigated the effect of toxicant on the population under the assumption that the capacity of the environment is very large relative to the population biomass, and that the limit of exogenous input of toxicant exists. The approach has been employed and developed to two-species, three-species and n-dimensional model (see, for example, [3-7]). Liu and Ma in [3] extended the approach proposed by Hallam to find the survival threshold of the Volterra model consisting of two populations in a polluted environment. The work of [4] investigated the effects of impulsive toxicant input with constant rate on naturally stable two-species Lotka-volterra competition system and the thresholds as the functions of model parameters for both population were derived. In [5], Ma et al. obtained sufficient conditions for persistence and extinction in three-dimensional communities. In a n-dimensional setting, Pan and Ma et al applied similar techniques to derive the threshold which governs persistence and extinction of populations ([6, 7]). These aforementioned research works, focusing on the threshold of population survival deal with toxicant effects passively. In the world today, however, exposure to a toxicant can result in changes in growth and maintenance of an individual. Modifications in population levels followed by rapid development of modern industry, are at the edge of the collapse of some
species. In such case, it makes sense to configure the control system to maintain the balance and achieve good performance.

In this paper a different scenario is considered, namely, the use of techniques of control theory. An n-dimensional food chain model coupled by a linear dose-response function is analyzed. Since the normal system is globally asymptotically stable (see \[8, 9\]) and the presence of exogenous toxicant can have a destabilizing effect, we will be concerned the following question: can we stabilize the food chain system exposed to toxicant or maintain the population persistent? In particular, we employ the techniques of static state feedback and adaptive control. Feedback control is very common concept. It is the process of measuring the controlled variable and using that information to influence the system variable to conform to some desired value. There are two main types: static feedback and dynamic feedback \[10\]. In this paper, we will study the food chain system under static feedback control method. We show that in the presence of exogenous toxicant, the populations can maintain persistent via our feedback control scheme. In the remaining of this paper, we focus on the stability analysis of the food chain system via adaptive control. Adaptive control is actually one type of nonlinear dynamic feedback. The underlying idea in the design is parameter estimation and the dynamic part of the controller is designed as a parameter update law with which the static part is continuously adapted to new parameter estimates. The relevant applications can be found in \[11\] and reference therein.

It is the purpose of this paper to present sufficient conditions for the persistence of each population or maintain the globally asymptotical stability via static state feedback control and adaptive control respectively. The paper is organized as follows. In section 2, we present model formulation and notations. In section 3 and 4, we state and prove the main results. And we illustrate the results by two examples in section 5. Finally, the paper is closed with conclusions in the last section.

2. Model formulation and notation

Consider the following food chain model with toxicant tress proposed in \[7\]:

\[
\begin{aligned}
\dot{x}_1 &= x_1(r_{10} - r_{11}c_{01}(t) - a_{11}x_1 - a_{12}x_2), \\
\dot{x}_2 &= x_2(-r_{20} - r_{21}c_{02}(t) + a_{21}x_1 - a_{22}x_2 - a_{23}x_3), \\
&\quad \ldots \\
\dot{x}_n &= x_n(-r_{n0} - r_{n1}c_{0n}(t) + a_{n,n-1}x_{n-1} - a_{n,n}x_n), \\
\dot{c}_{0j}(t) &= k_jc_e + d_j - (g_j + m_j)c_e, 1 \leq j \leq n \\
\dot{c}_e(t) &= \sum_{j=1}^{n} f_jc_{0j}x_j - \sum_{j=1}^{n} l_jc_e x_j - hc_e + \delta(t).
\end{aligned}
\]

The initial conditions are \(x_j(0) > 0, c_{0j}(0) \geq 0(1 \leq j \leq n), c_e(0) \geq 0\), where \(x_j(t)\) represents the density of the \(j\)th population at time \(t\); \(c_{0j}(t)\) is the concentration of toxicant in the \(j\)th organism at time \(t\); \(c_e(t)\) is the concentration of toxicant in the environment at time \(t\). The toxicant and population equations are coupled by linear dose-response equation \(r_{j0} - r_{j1}c_{0j}(t)\), where \(r_{j0}\) is the intrinsic growth rate or death rate of the \(j\)th population and \(r_{j1}\) is the dose-response of species \(j\) to the organismal toxicant concentration. \(\alpha_{ii}(i = 1, 2, \ldots, n)\) is a positive constant, which represent the density dependence of the \(i\)th population and positive constant \(a_{ij}(i \neq j)\) is interspecific interaction coefficient. The parameters \(k_j, d_j, g_j, m_j, f_j, l_j\)
and \( h \) are positive constants, where \( k_j \) denotes the environmental toxicant uptake rate per unit mass organism; \( d_j \) denote the uptake of toxicant in food of population \( x_j \); \( g_j \) and \( m_j \) are the net egestion and depuration rates of toxicant per organism respectively; and \( h \) denotes the loss rate of toxicant from the environment. The term \( \sum_{j=1}^{n} l_j c_j x_j \) is the increase of the toxicant in the environment coming from the egestion of the total populations and the term \( - \sum_{j=1}^{n} l_j c_j x_j \) is the loss of toxicant in the environment due to the uptake of toxicant by all populations. The exogenous rate of toxicant input into the environment is represented by \( \delta(t) \), which is bounded by 0 ≤ \( \delta(t) \) ≤ \( b < +\infty \), \( b \geq 0 \).

Model (1) is \((2n + 1)\)-dimensional nonlinear system. Assume that the capacity of the environment is so large that the change of toxicant in the environment that comes from the uptake and egestion by the organisms can be neglected [2]. Then instead of (1), we consider the following simplified model:

\[
\begin{align*}
\dot{x}_1 &= x_1(r_{10} - r_{11} c_0_1(t) - a_{11} x_1 - a_{12} x_2), \\
\dot{x}_2 &= x_2(-r_{20} - r_{21} c_0_2(t) + a_{21} x_1 - a_{22} x_2 - a_{23} x_3), \\
&\vdots \\
\dot{x}_n &= x_n(-r_{n,0} - r_{n1} c_0_n(t) + a_{n,n-1} x_{n-1} - a_{n,n} x_n), \\
\dot{c}_0_j &= k_j c_e + d_j - (g_j + m_j) c_e, 1 \leq j \leq n \\
\dot{c}_e(t) &= -h c_e + \delta(t).
\end{align*}
\]

(2)

Since the last two equations of (2) are linear, \( c_e(t) \) and \( c_0_j(t) \) can be solved for successively. Therefore, the simplified model (2) is essentially a \( n \)-dimensional system in which \( c_0_j(t) \) is a function of \( \delta(t) \). We need actually only impose the control on the first \( n \) equations and investigate the asymptotic behavior of the solution of the control system below:

\[
\begin{align*}
\dot{x}_1 &= x_1(r_{10} - r_{11} c_0_1(t) - a_{11} x_1 - a_{12} x_2) + u_1, \\
\dot{x}_2 &= x_2(-r_{20} - r_{21} c_0_2(t) + a_{21} x_1 - a_{22} x_2 - a_{23} x_3) + u_2, \\
&\vdots \\
\dot{x}_n &= x_n(-r_{n,0} - r_{n1} c_0_n(t) + a_{n,n-1} x_{n-1} - a_{n,n} x_n) + u_n,
\end{align*}
\]

(3)

where \( u_i \) (\( i = 1, 2, \ldots, n \)) is control input.

3. Persistence of the food chain feedback control system

We first state some definitions and notations. The Euclidean norm of a vector is denoted as \( |\cdot| \) in the sequel.

**Definition 1.** A population \( x(t) \) is called persistent if \( \lim_{t \to +\infty} \inf x(t) > 0 \).

**Lemma 1.** ([12], LaSle’s) Let \( \Omega \subset D \) be a compact set that is positively invariant with respect to the system \( \dot{x} = f(x) \). Let \( V : D \to R \) be a continuously differentiable function such that \( \dot{V}(x) \leq 0 \) in \( \Omega \). Let \( E \) be the set of all points in \( \Omega \) where \( \dot{V}(x) = 0 \). Let \( M \) be the largest invariant set in \( E \). Then every solution starting in \( \Omega \) approaches \( M \) as \( t \to \infty \).
Without loss of generality, we assume the normal system of (3)

\[
\begin{align*}
\dot{x}_1 &= x_1(r_{10} - a_{11}x_1 - a_{12}x_2), \\
\dot{x}_2 &= x_2(-r_{20} + a_{21}x_1 - a_{22}x_2 - a_{23}x_3), \\
\vdots \\
\dot{x}_n &= x_n(-r_{n0} + a_{n,n-1}x_{n-1} - a_{n,n}x_n),
\end{align*}
\]

exists a unique positive equilibrium \(x^*(x_1^*, x_2^*, \ldots, x_n^*)\).

**Theorem 1.** Given any strictly positive numbers \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) with

\[
\sum_{i=1}^{n} \frac{\alpha_i r_{i1}^2}{\lambda_i} \leq \frac{4c(\min \{x_i^*\})^2}{b^2},
\]

where \(c = \min\{a_{11}, a_{22}, \ldots, a_{nn}\}\), there exists a state feedback control law: \(u_i = -\lambda_i x_i(x_i - x_i^*)\), \((i = 1, 2, \ldots, n)\) such that the solution of the closed-loop system (3) satisfies \(x(t) \to R = \{x| x - x^*| \leq b \sqrt{\sum_{i=1}^{n} \frac{\alpha_i r_{i1}^2}{\lambda_i c}}\}\), and each population is persistent.

**Proof:** Modifying system (3) by the term \(x_i - x_i^*\), we get

\[
\begin{align*}
\dot{x}_1 &= x_1(r_{10} - a_{11}x_1 - a_{12}x_2) + u_1, \\
\dot{x}_2 &= x_2(-r_{20} + a_{21}x_1 - a_{22}x_2 - a_{23}x_3) + u_2, \\
\vdots \\
\dot{x}_n &= x_n(-r_{n0} + a_{n,n-1}x_{n-1} - a_{n,n}x_n) + u_n.
\end{align*}
\]

Let

\[
W(x) = \sum_{i=1}^{n} \alpha_i (x_i - x_i^*)\ln x_i,
\]

where \(\alpha_1 = 1, \alpha_2 = \frac{a_{12}}{a_{21}}, \ldots, \alpha_n = \frac{a_{12}a_{23} \cdots a_{n,n-1,n}}{a_{21}a_{32} \cdots a_{n,n-1}}\). It is easy to see \(W(x)\) achieves the minimum at \(x = x^*\). Thus, we choose Lyapunov function \(V(x) = W(x) - W(x^*)\), then we have

\[
\dot{V} = \alpha_1 (x_1 - x_1^*)\dot{x}_1 + \alpha_2 (x_2 - x_2^*)\dot{x}_2 + \cdots + \alpha_n (x_n - x_n^*)\dot{x}_n.
\]

Choose the state feedback control law

\[
u_i = -\lambda_i x_i(x_i - x_i^*),
\]

then we get

\[
\dot{V} = -\sum_{i=1}^{n} \alpha_i a_{ii} (x_i - x_i^*)^2 - \sum_{i=1}^{n} \alpha_i \lambda_i [(x_i - x_i^*) + \frac{r_{i1}c_{0i}(t)}{\lambda_i}]^2 + \sum_{i=1}^{n} \frac{\alpha_i r_{i1}^2}{4\lambda_i} c_{0i}(t)^2
\]

\[
\leq -\sum_{i=1}^{n} \alpha_i a_{ii} (x_i - x_i^*)^2 + \sum_{i=1}^{n} \frac{\alpha_i r_{i1}^2}{4\lambda_i} b^2
\]

\[
\leq -c|x - x^*|^2 + \sum_{i=1}^{n} \frac{\alpha_i r_{i1}^2}{4\lambda_i} b^2,
\]

where \(c = \min\{a_{11}, a_{22}, \ldots, a_{nn}\}\). Let

\[
R = \{x| x - x^*| \leq b \sqrt{\sum_{i=1}^{n} \frac{\alpha_i r_{i1}^2}{\lambda_i c}}\}.
\]
From the condition of Theorem 1: \[ \sum_{i=1}^{n} \frac{\alpha_i r_i^2}{\lambda_i} \leq \frac{4c(min.x_i^*)^2}{b^2}, \] it follows that \( R \) is a compact set of positive quadrant. Note that outside \( R \), we have \( \dot{V} < 0 \). Therefore by Lemma 1, we have \( x(t) \to R \) and all the populations are persistent.

4. Adaptive control of the food chain system

In this section we employ a form of nonlinear integral feedback. The underlying idea in the design of this dynamic part of feedback is parameter estimation. Then dynamic part of the controller is designed as a parameter update law with which the static part is continuously adapted to new parameter estimates, hence its name: adaptive control law. In using this traditional terminology, however, we should keep in mind that so conceived adaptive controllers are but one type of nonlinear dynamic feedback.

In this section, we assume that the concentration of toxicant in the organism is an uncertain constant. This ecological setting might be reasonable when the concentration of toxicant in the organism is saturable and achieves the limiting point, that is, the threshold value of terminal organismal concentration, which determines the populations that go to extinction. Hence, we consider the model

\[
\begin{align*}
\dot{x}_1 &= x_1(r_{10} - r_{11}\theta - a_{11}x_1 - a_{12}x_2) + u_1, \\
\dot{x}_2 &= x_2(-r_{20} + a_{21}x_1 - a_{22}x_2 - a_{23}x_3) + u_2, \\
&\quad \vdots \\
\dot{x}_n &= x_n(-r_{n,0} - a_{n,n-1}x_{n-1} - a_{n,n}x_n) + u_n,
\end{align*}
\]

where the concentration of toxicant in the organism, denoted by \( \theta \), is unknown parameter that enters the system through the control channel. We assume the normal system of (8)

\[
\begin{align*}
\dot{x}_1 &= x_1(r_{10} - a_{11}x_1 - a_{12}x_2), \\
\dot{x}_2 &= x_2(-r_{20} + a_{21}x_1 - a_{22}x_2 - a_{23}x_3), \\
&\quad \vdots \\
\dot{x}_n &= x_n(-r_{n,0} + a_{n,n-1}x_{n-1} - a_{n,n}x_n),
\end{align*}
\]

exists a unique positive equilibrium \( x^*(x_1^*, x_2^*, \ldots, x_n^*) \).

**Theorem 2.** There exists an adaptive control law:

\[
\begin{align*}
u_i &= r_{i1}x_i\hat{\theta}, \\
\dot{\hat{\theta}} &= -\gamma \sum_{i=1}^{n} r_{i1}\alpha_i(x_i - x_i^*),
\end{align*}
\]

where \( \gamma > 0 \) is an arbitrary constant and \( \hat{\theta} \) is an arbitrary estimate value of \( \theta \), such that the closed-loop system (8) is globally asymptotically stable.
Proof: Modifying system (8) by the term $x_i - x_i^*$, we get
\[
\begin{align*}
\dot{x}_1 &= x_1(-r_{11}\dot{\theta} - a_{11}(x_1 - x_1^*) - a_{12}(x_2 - x_2^*)) + u_1, \\
\dot{x}_2 &= x_2(-r_{21}\dot{\theta} + a_{21}(x_1 - x_1^*) - a_{22}(x_2 - x_2^*) - a_{23}(x_3 - x_3^*)) + u_2, \\
\vdots \\
\dot{x}_n &= x_n(-r_{n1}\dot{\theta} + a_{n,n-1}(x_{n-1} - x_{n-1}^*) - a_{n,n}(x_n - x_n^*)) + u_n.
\end{align*}
\]
(11)
Choose Lyapunov function
\[
V(x) = W(x) - W(x^*),
\]
(12)
then we have
\[
\dot{V} = \alpha_1(x_1 - x_1^*)\dot{x}_1 + \alpha_2(x_2 - x_2^*)\dot{x}_2 + \cdots + \alpha_n(x_n - x_n^*)\dot{x}_n.
\]
(13)
Thus, the derivative is rendered negative definite
\[
\dot{V} = -\sum_{i=1}^{n} \alpha_i a_{ii}(x_i - x_i^*)^2 \leq 0
\]
by the control
\[
u_i = r_{i1}x_i\dot{\theta}, \quad (i = 1, 2, \ldots, n).
\]
Since $\theta$ is unknown, the control law (13) cannot be implemented. It is reasonable to replace $\theta$ with its arbitrary estimate $\hat{\theta}$ to obtain the adaptive law:
\[
u_i = r_{i1}x_i\dot{\hat{\theta}}.
\]
(14)
This results in the error system:
\[
\begin{align*}
\dot{\hat{\theta}} &= \dot{\theta} - \dot{\hat{\theta}}, \\
\dot{x}_1 &= x_1(-r_{11}\hat{\theta} - a_{11}(x_1 - x_1^*) - a_{12}(x_2 - x_2^*)), \\
\dot{x}_2 &= x_2(-r_{21}\hat{\theta} + a_{21}(x_1 - x_1^*) - a_{22}(x_2 - x_2^*) - a_{23}(x_3 - x_3^*)), \\
\vdots \\
\dot{x}_n &= x_n(-r_{n1}\hat{\theta} + a_{n,n-1}(x_{n-1} - x_{n-1}^*) - a_{n,n}(x_n - x_n^*)),
\end{align*}
\]
(15)
where $\hat{\theta} = \theta - \dot{\theta}$. Then we argument (12) with a quadratic term in the parameter error $\hat{\theta}$ to obtain the Lyapunov function:
\[
V_1(x, \hat{\theta}) = V(x) + \frac{1}{2\gamma} \hat{\theta}^2,
\]
(16)
where $\gamma > 0$ is so-called adaptation gain. Its derivative is
\[
\dot{V} = -\sum_{i=1}^{n} \alpha_i a_{ii}(x_i - x_i^*)^2 + \hat{\theta} \left[ \frac{\dot{\hat{\theta}}}{\gamma} - \sum_{i=1}^{n} \alpha_i r_{i1}(x_i - x_i^*) \right].
\]
(17)
The choice of update law
\[
\dot{\hat{\theta}} = -\gamma \sum_{i=1}^{n} \alpha_i r_{i1}(x_i - x_i^*)
\]
(18)
estimates the $\hat{\theta}$-term in (17) and renders the derivative of the Lyapunov function (16) non-positive:
\[
\dot{V} = -\sum_{i=1}^{n} \alpha_i a_{ii}(x_i - x_i^*)^2.
\]
This implies that the $x = x^*, \dot{\theta} = 0$ equilibrium point of the closed-loop adaptive system consisting of (8) and (10) is globally asymptotically stable and in addition, $x(t) \to \infty$ as $t \to \infty$.

5. Simulations

Example 1: consider two-population food chain model in a polluted environment

$$
\begin{aligned}
\dot{x}_1 &= x_1(2 - c_{01}(t) - x_1 - x_2) + u_1, \\
\dot{x}_2 &= x_2(-1 - c_{02}(t) + 2x_1 - x_2) + u_2.
\end{aligned}
$$

The exogenous rate of toxicant input into the environment, represented by $\delta(t)$, is bounded by $0 \leq \delta(t) \leq 1$. In the environment without toxicant, the normal system of (19) has the unique positive equilibrium point $x^* = (1, 1)$, which is globally asymptotically stable. Assume $x_0 = (1.5, 1.5)$, our simulation shows that the static feedback control law

$$
\begin{aligned}
u_1 &= -8x_1(x_1 - 1), \\
u_2 &= -4x_2(x_2 - 1),
\end{aligned}
$$

guarantee the closed-loop state remains bounded, $x(t) \to R = \{x| |x - x^*| \leq \frac{1}{8}\}$, and maintain the populations of the food chain system exposed to toxicant persistent(Figure 1).

![Figure 1. The performance of the closed-loop system in phase space.](image)

Example 2: assume the concentration of toxicant in the organism is a uncertain constant and consider the following model:

$$
\begin{aligned}
\dot{x}_1 &= x_1(2 - \theta - x_1 - x_2) + u_1 \\
\dot{x}_2 &= x_2(-1 - \theta + 2x_1 - x_2) + u_2
\end{aligned}
$$
with the adaptive controller

\[
\begin{cases}
    u_i = x_i \hat{\theta}, \\
    \dot{\hat{\theta}} = \frac{3}{2} - x_1 - \frac{1}{2} x_2.
\end{cases} \quad (i = 1, 2)
\]

Let \( x_0 = (1.5, 1.5) \), and \( \hat{\theta} = 2 \). Figure 2 shows that the stability is maintained.

![Graph showing the performance of the closed-loop adaptive system.](image)

**Figure 2.** The performance of the closed-loop adaptive system.

### 6. Conclusion

A theoretical treatment of toxicant-population interactions that employs an n-species food chain system and linear dose response function is presented. Our main results show that in the presence of exogenous toxicant, which can have a destabilizing effect, the population can maintain persistent or original asymptotical behavior via the technique of control, in particular, static state feedback and adaptive control.

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**References**


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