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MINIMAL NORMAL AND COMMUTING COMPLETIONS

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Abstract. We study the minimal normal completion problem: given $A \in \mathbb{C}^{n\times n}$, how do we find an $(n+q)\times(n+q)$ normal matrix $A_{ext} := \begin{pmatrix} A & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ of smallest possible size? We will show that this smallest number q of rows and columns we need to add, called the *normal defect* of A, satisfies

$$\mathrm{nd}(A) \geq \max\{i_-(AA^* - A^*A), i_+(AA^* - A^*A)\},$$

where $i_\pm(M)$ denotes the number of positive and negative eigenvalues of the Hermitian matrix M counting multiplicities. Subsequently, we will show that for some matrices a minimal normal completion can be chosen to be a multiple of a unitary, addressing a conjecture from [H. J. Woerdeman, Linear and Multilinear Algebra 36 (1993), 59–68].

In addition, we study the related question where $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are given, and where we look for $A_{ext} := \begin{pmatrix} A & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B_{ext} := \begin{pmatrix} B & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ such that they commute and are of smallest possible size.

Key Words. commuting completions, commuting defect, normal completions, normal defect, inertia, inverse defect, unitary defect.

1. Introduction

The minimal normal completion problem was introduced in [6] and concerns the following. Given $A \in \mathbb{C}^{n\times n}$, we wish to find a smallest possible normal matrix with A as a principal submatrix. Recall that a matrix A is normal if and only if the *commutator* of A and its conjugate transpose A^*, denoted by $[A, A^*] := AA^* - A^*A$, equals 0. In other words, we would like to find a normal completion of

$$\begin{pmatrix} A & ? \\ ? & ? \end{pmatrix} : \begin{matrix} \mathbb{C}^n \\ \oplus \\ \mathbb{C}^q \end{matrix} \rightarrow \begin{matrix} \mathbb{C}^n \\ \oplus \\ \mathbb{C}^q \end{matrix}$$

of smallest possible size (thus smallest possible q). We shall call this smallest number q the *normal defect* of A, and denote it by $\mathrm{nd}(A)$. Clearly, $\mathrm{nd}(A) = 0$ if and only if A is normal. As observed in [2], the matrix $\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}$ is normal, so it follows that for an $n \times n$ matrix A we have that $\mathrm{nd}(A) \leq n$. As was observed in [6], and as we shall see further on, we have in fact that $\mathrm{nd}(A) \leq n - 1$. It is also not hard to come up with the lower bound $\mathrm{nd}(A) \geq \frac{1}{2}\mathrm{rank}(AA^* - A^*A)$. Indeed if $\begin{pmatrix} A & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is of size $(n+q)\times(n+q)$ and normal then $AA^* - A^*A = A_{21}^*A_{21} - A_{12}A_{12}^*$ and thus $\mathrm{rank}(AA^* - A^*A) \leq \mathrm{rank}\,(A_{21}^*A_{21}) + \mathrm{rank}\,(A_{12}A_{12}^*) \leq q + q = 2q$.

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In order to obtain sharper bounds for $\mathrm{nd}(A)$, the so-called *unitary defect* was introduced in [6]. It corresponds to the smallest number of rows and columns we need to add to A such that the completion is a multiple of a unitary matrix, and as it turns out we have that

$$\mathrm{ud}(A) := \mathrm{rank}\left(||A||^2 - A^*A\right), \tag{1}$$

where $||\ .\ ||$ denotes the spectral norm. As multiples of unitaries are normal we clearly have that $\mathrm{nd}(A) \leq \mathrm{ud}(A)$. Formula (1) implies that $\mathrm{ud}(A) \leq n-1$, which yields $\mathrm{nd}(A) \leq n-1$. In order to state a conjecture from [6], let us recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called *unitarily reducible* if $A = U^* \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} U$, with U unitary and A_1, A_2 of nontrivial size. Clearly, with A as above we have that

$$\mathrm{nd}(A) \leq \mathrm{nd}(A_1) + \mathrm{nd}(A_2) \leq \mathrm{ud}(A_1) + \mathrm{ud}(A_2) \leq \mathrm{ud}(A). \tag{2}$$

As soon as $\|A_1\| \neq \|A_2\|$ we have that the last inequality in (2) is strict, and thus $\mathrm{nd}(A) < \mathrm{ud}(A)$ in that case. So for a general statement for the case when $\mathrm{nd}(A) = \mathrm{ud}(A)$, it is natural to require that A is *unitarily irreducible*, which by definition means that A is not unitarily reducible. An open question from [6] is whether the following conjecture holds.

Conjecture 1. *For a unitarily irreducible matrix A we have that* $\mathrm{nd}(A) = \mathrm{ud}(A)$.

In this paper we refine some of the estimates for $\mathrm{nd}(A)$ and as a result obtain more evidence for this conjecture. Let us mention that the separability problem that appears in quantum computation can be seen as a normal completion problem where additional constraints need to be met; see [7] for details. In that context, minimizing the size of the matrix corresponds to minimizing the number of states in the separable representation. We will end this paper with considering the problem of completing two matrices to make them commute.

The paper is organized as follows. In Section 2, we obtain an improved lower bound for $\mathrm{nd}(A)$ by showing that $\mathrm{nd}(A) \geq \max\{i_+([A,A^*]), i_-([A,A^*])\}$, where $i_\pm(M)$ denotes the number of positive/negative eigenvalues of the Hermitian matrix M. Using this improved lower bound we are able to show that for some weighted Jordan blocks we have that $\mathrm{nd}(A) = \mathrm{ud}(A)$, providing new evidence for Conjecture 1. Next, in Section 3 we examine matrices for which $\mathrm{nd}(A) = 1$. Finally in Section 4 we explore the commuting completion problem.

2. Main result and normal defect conjecture

In this section we shall prove our main result and provide evidence for Conjecture 1.

Theorem 1. *Given $A \in \mathbb{C}^{n\times n}$. Then* $\mathrm{nd}(A) \geq \max\{i_+([A,A^*]), i_-([A,A^*])\}$.

Proof. Let $A_{ext} := \begin{pmatrix} A & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a normal completion of size $(n+q)\times(n+q)$. From the normality of A_{ext} we get

$$AA^* - A^*A = A_{21}^*A_{21} - A_{12}A_{12}^*. \tag{3}$$

Let us denote the eigenvalues of the Hermitian matrices $A_{21}^*A_{21} - A_{12}A_{12}^*$ and $A_{21}^*A_{21}$ by $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \leq \ldots \leq \mu_n$, respectively. By the Courant-Fischer theorem (see e.g., Theorem 4.2.11 in [3]), we get for $1 \le j \le n$

$$\lambda_j = \min_{w_1,w_2,\dots,w_{n-j}\in\mathbb{C}^n} \max_{\substack{x\neq 0,\ x\in\mathbb{C}^n \\ x\perp w_1,w_2,\dots,w_{n-j}}} \frac{x^*(A_{21}^*A_{21} - A_{12}A_{12}^*)x}{x^*x} \tag{4}$$

$$\le \min_{w_1,w_2,\dots,w_{n-j}\in\mathbb{C}^n} \max_{\substack{x\neq 0,\ x\in\mathbb{C}^n \\ x\perp w_1,w_2,\dots,w_{n-j}}} \frac{x^*A_{21}^*A_{21}x}{x^*x} = \mu_j. \tag{5}$$

Since $\mu_{n-q} = 0$, we get $\lambda_{n-q} \le 0$. Thus (3) gives $i_+([A,A^*]) \le q$. Notice that a similar argument can be carried out by looking at the eigenvalues of $A_{21}^*A_{21} - A_{12}A_{12}^*$ and $-A_{12}A_{12}^*$, which will give $i_-([A,A^*]) \le q$. This proves the result. □

Using the well-known connection between normal matrices N and pairs of commuting Hermitian matrices $(\mathrm{Re}N, \mathrm{Im}N)$, where $\mathrm{Re}N = \frac{1}{2}(N+N^*)$ and $\mathrm{Im}N = \frac{1}{2i}(N-N^*)$, one can easily deduce the following corollary.

Corollary 1. *Let Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ be given. If there exist Hermitian matrices $A_{ext} = \begin{pmatrix} A & * \\ * & * \end{pmatrix}$, $B_{ext} = \begin{pmatrix} B & * \\ * & * \end{pmatrix}$ of size $(n+q)\times(n+q)$ that commute, then $q \ge \max\{i_+(i(BA-AB)), i_-(i(BA-AB))\}$.*

Proof. Let $N = A + iB$. Calculating NN^* we get $NN^* = (A+iB)(A-iB) = A^2 - iAB + iBA + B^2$. Now if we calculate N^*N we get $N^*N = A^2 + iAB - iBA + B^2$. Thus $NN^* - N^*N = 2i(BA-AB)$. It then follows from Theorem 1 that $q \ge \max\{i_+(i(BA-AB)), i_-(i(BA-AB))\}$. □

Next we explore a class of matrices for which the equality $\mathrm{nd}(A) = \mathrm{ud}(A)$ is true.

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$ be of the form $A := \begin{pmatrix} 0 & a_1 & & 0 \\ & \ddots & \ddots & \\ & & & a_{n-1} \\ 0 & & & 0 \end{pmatrix}$ with either $|a_1| = \ldots = |a_l| > \ldots > |a_{n-1}| > 0$ or $0 < |a_1| < \ldots < |a_{n-l}| = \ldots = |a_{n-1}|$, where $1 \le l \le n-1$. Then $\mathrm{nd}(A) = \mathrm{ud}(A) = n-l$.*

Proof. Let $\alpha := |a_1| = \ldots = |a_l| = ||A||$. By Proposition 5.4 in [6] we have that $\mathrm{nd}(A) \le \mathrm{ud}(A)$. Recall that $\mathrm{ud}(A) = \mathrm{rank}(||A||^2 - A^*A)$. Thus

$$\mathrm{ud}(A) = \mathrm{rank}\left\{ \begin{pmatrix} \alpha^2 & & 0 \\ & \ddots & \\ 0 & & \alpha^2 \end{pmatrix} - \begin{pmatrix} 0 & & & 0 \\ & |a_1|^2 & & \\ & & \ddots & \\ 0 & & & |a_{n-1}|^2 \end{pmatrix} \right\}$$

$$= \mathrm{rank} \begin{pmatrix} \alpha^2 & & & & & & 0 \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & \alpha^2 - |a_{l+1}|^2 & & \\ & & & & & \ddots & \\ 0 & & & & & & \alpha^2 - |a_{n-1}|^2 \end{pmatrix}$$

$$= n - l.$$

Thus we have $\mathrm{nd}(A) \leq \mathrm{ud}(A) = n - l$.

To achieve a lower bound for $\mathrm{nd}(A)$, we notice that Theorem 1 yields $\mathrm{nd}(A) \geq \max\{i_+([A, A^*]), i_-([A, A^*])\}$. Observe that

$$[A, A^*] = \begin{pmatrix} \alpha^2 & & & & & & 0 \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & |a_{l+1}|^2 - |a_l|^2 & & \\ & & & & & \ddots & \\ & & & & & |a_{n-1}|^2 - |a_{n-2}|^2 & \\ 0 & & & & & & -|a_{n-1}|^2 \end{pmatrix}.$$

Therefore $i_+([A, A^*]) = 1$ and $i_+([A, A^*]) = n - l$. Clearly, $n - l \geq 1$ and thus $\mathrm{nd}(A) \geq n - l$. We conclude that $\mathrm{nd}(A) = \mathrm{ud}(A) = n - l$. The proof for the second part of the statement is similar. □

We observe that the matrices in Proposition 1 are unitarily irreducible, as the following lemma shows.

Lemma 1. *Let $A \in \mathbb{C}^{n \times n}$ be strictly upper triangular with $a_{i,i+1} \neq 0$ for $1 \leq i \leq n - 1$. Then A is unitarily irreducible.*

Proof. Let $A = U^* \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} U$ with U unitary, $A_1 \in \mathbb{C}^{p \times p}$ and $A_2 \in \mathbb{C}^{q \times q}$ with $p + q = n$. We need to show that $p = 0$ or $q = 0$. Notice that $n - 1 = \mathrm{rank}(A) = \mathrm{rank}(A_1) + \mathrm{rank}(A_2) \leq p + q = n$. So there exist two possibilities, either $\mathrm{rank}(A_1) = p$ and $\mathrm{rank}(A_2) = q - 1$ or $\mathrm{rank}(A_1) = p - 1$ and $\mathrm{rank}(A_2) = q$. Without loss of generality we assume that $\mathrm{rank}(A_1) = p$ and $\mathrm{rank}(A_2) = q - 1$. Thus A_1 is invertible and $\mathrm{rank}(A_1^k) = p$ for all k. As $0 = \mathrm{rank}(A^n) \geq \mathrm{rank}(A_1^n) = p$, it follows that $p = 0$. □

3. When the normal defect equals one

Inspired by [4] we consider the case when the normal defect is equal to one. For the class of matrices considered before we have the following observation.

Proposition 2. *Given a matrix A of the form* $A := \begin{pmatrix} 0 & a_1 & & 0 \\ & \ddots & \ddots & \\ & & & a_{n-1} \\ 0 & & & 0 \end{pmatrix}$, *with* $a_1, \ldots, a_{n-1} \in \mathbb{C} \setminus \{0\}$. *Then $\mathrm{nd}(A) = 1$ if and only if $|a_1| = \ldots = |a_{n-1}| =: \alpha$. Furthermore, when $n \geq 4$ all minimal normal completions have the form*

$$A_{ext} := \begin{pmatrix} 0 & a_1 & & 0 & 0 \\ & \ddots & \ddots & & \vdots \\ & & & a_{n-1} & 0 \\ 0 & & & 0 & a_n \\ a_{n+1} & 0 & \cdots & & 0 \end{pmatrix},$$

with $\alpha = |a_n| = |a_{n+1}|$.

Proof. Observe that

$$[A,A^*] = \begin{pmatrix} |a_1|^2 & & & & 0 \\ & |a_2|^2-|a_1|^2 & & & \\ & & \ddots & & \\ & & & |a_{n-1}|^2-|a_{n-2}|^2 & \\ 0 & & & & -|a_{n-1}|^2 \end{pmatrix},$$

which yields that $\max\{i_+([A,A^*]), i_-([A,A^*])\} \geq 1$, and equality holds if and only if $|a_1| = \ldots = |a_{n-1}|$. Using Theorem 1 it follows that for $\mathrm{nd}(A) = 1$ it is necessary that $|a_1| = \ldots = |a_{n-1}|$.

Let now $|a_1| = \ldots = |a_{n-1}|$, and put

$$A_{ext} := \begin{pmatrix} 0 & a_1 & & 0 & \beta_1 \\ & \ddots & \ddots & & \beta_2 \\ & & & a_{n-1} & \vdots \\ 0 & & & 0 & \beta_n \\ \gamma_1 & \gamma_2 & \ldots & \gamma_n & \delta \end{pmatrix} = \begin{pmatrix} A & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\beta := \begin{pmatrix} \beta_1 & \cdots & \beta_n \end{pmatrix}^T$ and $\gamma := \begin{pmatrix} \gamma_1 & \ldots & \gamma_n \end{pmatrix}$. Let us assume that A_{ext} is normal. Then we see that $AA^* + \beta\beta^* = A^*A + \gamma^*\gamma$, or equivalently $[A,A^*] = \gamma^*\gamma - \beta\beta^*$. Recall that

$$[A,A^*] = \begin{pmatrix} |a_1|^2 & & & & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & -|a_{n-1}|^2 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\gamma}_1 \\ \vdots \\ \overline{\gamma}_n \end{pmatrix} \begin{pmatrix} \gamma_1 & \ldots & \gamma_n \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \begin{pmatrix} \overline{\beta}_1 & \ldots & \overline{\beta}_n \end{pmatrix}$$

$$= \begin{pmatrix} |\gamma_1|^2 - |\beta_1|^2 & \overline{\gamma}_1\gamma_2 - \beta_1\overline{\beta}_2 & \ldots & \overline{\gamma}_1\gamma_n - \beta_1\overline{\beta}_n \\ \overline{\gamma}_2\gamma_1 - \beta_2\overline{\beta}_1 & |\gamma_2|^2 - |\beta_2|^2 & \ldots & \overline{\gamma}_2\gamma_n - \beta_2\overline{\beta}_n \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\gamma}_n\gamma_1 - \beta_n\overline{\beta}_1 & \overline{\gamma}_n\gamma_2 - \beta_n\overline{\beta}_2 & \ldots & |\gamma_n|^2 - |\beta_n|^2 \end{pmatrix}$$

$$=: W = (w_{ij})_{i,j=1}^n.$$

As $|a_1|, |a_{n-1}| > 0$ we get that $|\gamma_1|^2 = |a_1|^2 + |\beta_1|^2 \neq 0$, $|\beta_n|^2 = |a_{n-1}|^2 + |\gamma_n|^2 \neq 0$. From $w_{ii} = 0$ for $2 \leq i \leq n-1$, we see that $|\gamma_i| = |\beta_i|$ for $i = 2, \ldots, n-1$. From $w_{i1} = 0$ for $i = 2, \ldots, n$, we see that $\overline{\gamma}_i\gamma_1 - \beta_i\overline{\beta}_1 = 0$. This implies that $|\gamma_i||\gamma_1| - |\beta_i||\beta_1| = 0$. Since $|\gamma_i| = |\beta_i|$ for $i = 2, \ldots, n-1$, we get $|\gamma_i|(|\gamma_1| - |\beta_1|) = 0$. So either $|\gamma_i| = |\beta_i| = 0$ or $|\gamma_1| = |\beta_1|$. But $|\gamma_1| \neq |\beta_1|$ since $|\gamma_1| = |a_1|^2 + |\beta_1|^2 > |\beta_1|^2$. Therefore $|\gamma_i| = |\beta_i| = 0$ for $i = 2, \ldots, n-1$.

To find $\beta_1, \beta_n, \gamma_1$ and γ_n, we observe the following equation that results from A_{ext} being normal, $A\gamma^* + \beta\overline{\delta} = A^*\beta + \gamma^*\delta$. Rewriting we see that

$$\begin{pmatrix} 0 & a_1 & & 0 \\ & \ddots & \ddots & \\ & & & a_{n-1} \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} \overline{\gamma}_1 \\ 0 \\ \vdots \\ 0 \\ \overline{\gamma}_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ 0 \\ \vdots \\ 0 \\ \beta_n \end{pmatrix} \overline{\delta}$$

$$= \begin{pmatrix} 0 & & & 0 \\ \overline{a}_1 & \ddots & & \\ & \ddots & & \\ 0 & & \overline{a}_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ 0 \\ \vdots \\ 0 \\ \beta_n \end{pmatrix} + \begin{pmatrix} \overline{\gamma}_1 \\ 0 \\ \vdots \\ 0 \\ \overline{\gamma}_n \end{pmatrix} \delta.$$

Simplifying we get that $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n-1}\overline{\gamma}_n \\ 0 \end{pmatrix} + \begin{pmatrix} \beta_1\overline{\delta} \\ 0 \\ \vdots \\ 0 \\ \beta_n\overline{\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{a}_1\beta_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \overline{\gamma}_1\delta \\ 0 \\ \vdots \\ 0 \\ \overline{\gamma}_n\delta \end{pmatrix}$. As $n \geq 4$ we see that

$$\beta_1\overline{\delta} = \overline{\gamma}_1\delta, \ \beta_n\overline{\delta} = \overline{\gamma}_n\delta, \tag{6}$$

$$\overline{a}_1\beta_1 = 0, \tag{7}$$

$$a_{n-1}\overline{\gamma}_n = 0. \tag{8}$$

From (7) we get $|\beta_1| = 0$ since $|a_1| \neq 0$. From (8) we get $|\gamma_n| = 0$. From (6) we get $|\delta| = 0$ since $\beta_1 = 0$ and $\gamma_1 \neq 0$. Finally from the equations for w_{11} and w_{nn} we see that $|\gamma_1| = |\beta_n| = \alpha$. In conclusion, for $n \geq 4$ the normality of A_{ext} implies that it is of the form

$$A_{ext} := \begin{pmatrix} 0 & a_1 & & & 0 & 0 \\ & \ddots & \ddots & & & \vdots \\ & & & & a_{n-1} & 0 \\ 0 & & & & 0 & a_n \\ a_{n+1} & 0 & \cdots & & & 0 \end{pmatrix}, |a_1| = \ldots = |a_{n+1}| \neq 0. \tag{9}$$

Clearly, for any n, if A_{ext} is as in (9), then A_{ext} is normal. This proves the proposition. □

It is worth mentioning that for $n = 2$ or $n = 3$ the above proposition does not describe the full situation. For example, consider $A = \begin{pmatrix} 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$ and $\widetilde{A} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A_{ext} := \begin{pmatrix} 0 & \sqrt{3} & 1 \\ 0 & 0 & 2 \\ 2 & 1 & \sqrt{3} \end{pmatrix}$ and $\widetilde{A}_{ext} := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 1 & 0 \end{pmatrix}$ also yield minimal normal completions for A and $\widetilde{A}$, respectively.

Next we determine the eigenvalues of the completed normal matrix A_{ext} from Proposition 2.

Lemma 2. *For the normal matrix*

$$A_{ext} := \begin{pmatrix} 0 & |\alpha|e^{i\theta_1} & & 0 & 0 \\ & \ddots & \ddots & & \vdots \\ & & & |\alpha|e^{i\theta_{n-1}} & 0 \\ 0 & & & 0 & |\alpha|e^{i\theta_n} \\ |\alpha|e^{i\theta_{n+1}} & 0 & \cdots & & 0 \end{pmatrix}$$

all eigenvalues of A_{ext} are exactly $\lambda_k = |\alpha|e^{i\frac{(\psi+2\pi k)}{n+1}}$ where $\psi = \theta_1 + \ldots + \theta_{n+1}$ and $k = 0, \ldots, n$.

Proof. Computing the characteristic polynomial of A_{ext} we see that $\lambda^{n+1} = |\alpha|^{n+1}e^{i\psi}$, where $\psi = \theta_1 + \ldots + \theta_{n+1}$. Thus $\lambda_k = |\alpha|e^{i\frac{(\psi+2\pi k)}{n+1}}$ with $k = 0, \ldots, n$. □

In [4] the following problem was considered. Let $\lambda_0, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be two sequences of complex numbers. When can one find a $(n+1) \times (n+1)$ normal matrix with eigenvalues $\lambda_0, \ldots, \lambda_n$ whose $n \times n$ principal submatrix has eigenvalues $\mu_1, \ldots, \mu_n$? In [4] the author obtained the following result. Define $\Delta(\lambda) := \frac{\prod_{j=1}^{n}(\lambda-\mu_j)}{\prod_{k=0}^{n}(\lambda-\lambda_k)}$. Then there exists a normal matrix A with spectrum $\lambda_0, \ldots, \lambda_n$ and with an $n \times n$ principal submatrix with spectrum $\mu_1, \ldots, \mu_n$ if and only if the rational function Δ has only simple poles and $\mathrm{Res}_{\lambda_k}(\Delta(\lambda)) \geq 0$, $k = 0, \ldots, n$. If we take A_{ext} in Lemma 2 we observe that A_{ext} has spectrum $\lambda_k = |\alpha|e^{i\frac{\psi+2\pi k}{n+1}}$, where $\psi = \theta_1 + \ldots + \theta_{n+1}$ and $k = 0, \ldots, n$. When we remove the last row and last column in A_{ext} the remaining matrix has eigenvalues $\mu_i = 0$, $i = 1, \ldots, n$. Thus by the result in [4] we must have that $\mathrm{Res}_{\lambda_k}(\frac{\lambda^n}{\prod_{k=0}^{n}(\lambda-\lambda_k)}) \geq 0$ is satisfied. This is indeed true, since $\mathrm{Res}_{\lambda_k}(\frac{\lambda^n}{\lambda^{n+1}-|\alpha|^{n+1}e^{i\psi}}) = \lim_{\lambda\to\lambda_k} \frac{(\lambda-\lambda_k)\lambda^n}{\lambda^{n+1}-|\alpha|^{n+1}e^{i\psi}} = \lim_{\lambda\to\lambda_k} \frac{\lambda^n+(\lambda-\lambda_k)n\lambda^{n-1}}{(n+1)\lambda^n} = \frac{1}{n+1} \geq 0$, where we used L'Hôpital's rule in the second equality.

4. Commuting defect

In [1] the following minimal commuting completion problem was introduced. Given $A_1, \ldots, A_d \in \mathbb{C}^{n\times n}$, how do we find $(A_1)_{ext}, \ldots, (A_d)_{ext} \in \mathbb{C}^{(n+q)\times(n+q)}$ of smallest possible size with

$$(A_i)_{ext} = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix}$$

such that $[(A_i)_{ext}, (A_j)_{ext}] = 0$, $i \neq j$. We will restrict our investigation to completing only two matrices: given $A, B \in \mathbb{C}^{n\times n}$ how do we find A_{ext}, B_{ext} of smallest possible size, with

$$A_{ext} = \begin{pmatrix} A & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B_{ext} = \begin{pmatrix} B & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{10}$$

such that $[A_{ext}, B_{ext}] = 0$. We shall call the smallest possible number q the *commuting defect*, and denote it $\mathrm{cd}(A, B)$. Clearly, $\mathrm{cd}(A, B) = 0$ if and only if $[A, B] = 0$. As shown in [1], $\mathrm{cd}(A, B) \geq \frac{1}{2}\mathrm{rank}([A, B])$. Indeed if $[A_{ext}, B_{ext}] = 0$ then $AB - BA = B_{12}A_{21} - A_{12}B_{21}$ and thus $\mathrm{rank}([A, B]) \leq \mathrm{rank}(B_{12}A_{21}) + \mathrm{rank}(A_{12}B_{21}) \leq q + q = 2q$. The results in [1] also show that $\mathrm{cd}(A, B) \leq n$. One easily sees this by taking $A_{ext} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ and $B_{ext} = \begin{pmatrix} B & A \\ A & B \end{pmatrix}$.

One useful observation for this problem is that if two square matrices C and D satisfy $CD = \alpha I$ for some $\alpha \neq 0$, then automatically $CD = DC$. With this in mind we introduce the minimal inverse completion problem: Given $A, B \in \mathbb{C}^{n\times n}$, how do we find $A_{ext}, B_{ext} \in \mathbb{C}^{(n+q)\times(n+q)}$ as in (10) of smallest possible size such that $A_{ext}B_{ext} = \alpha I_{n+q}$, for some $\alpha \neq 0$. We shall call this smallest number q the *inverse defect* and denote it by $\mathrm{id}(A, B)$. The inverse defect of a pair of matrices is easily determined as the following theorem shows.

Theorem 2. *For $A, B \in \mathbb{C}^{n\times n}$, suppose α is the nonzero eigenvalue of AB with the highest geometric multiplicity. Then* $\mathrm{id}(A, B) = \mathrm{rank}(\alpha I_n - AB)$.

Proof. Let $A_{ext} := \begin{pmatrix} A & ? \\ ? & ? \end{pmatrix}$ and $B_{ext} := \begin{pmatrix} B & ? \\ ? & ? \end{pmatrix}$ exist such that $A_{ext}B_{ext} = \alpha I_{n+q}$, where $\alpha \neq 0$. We notice that $A_{ext}B_{ext} = \alpha I_{n+q}$ if and only if

$$\mathrm{rank}\begin{pmatrix} \alpha I_{n+q} & A_{ext} \\ B_{ext} & I_{n+q} \end{pmatrix} = n + q,$$

which is obviously bigger than or equal to $\mathrm{rank}\begin{pmatrix} \alpha I_n & A \\ B & I_n \end{pmatrix}$. As the latter equals $n + \mathrm{rank}(\alpha I_n - AB)$ by a Schur complement argument, we conclude that $n + q \geq n + \mathrm{rank}(\alpha I_n - AB)$ or $q \geq \mathrm{rank}(\alpha I_n - AB)$ as required.

Now let $\alpha \neq 0$ be a nonzero eigenvalue of AB with highest geometric multiplicity (or equivalently, the $\alpha \neq 0$ so that $\mathrm{rank}(\alpha I_n - AB)$ is minimal), and set $q = \mathrm{rank}(\alpha I_n - AB)$. We define $W := \begin{pmatrix} \alpha I_n & 0 & A & ? \\ 0 & \alpha I_q & ? & ? \\ B & ? & I_n & 0 \\ ? & ? & 0 & I_q \end{pmatrix}$. After a permutation similarity we arrive at $\tilde{W} := \begin{pmatrix} \alpha I_q & 0 & ? & ? \\ 0 & \alpha I_n & A & ? \\ ? & B & I_n & 0 \\ ? & ? & 0 & I_q \end{pmatrix}$. By inspection we see that $\tilde{W}$ is a partial banded matrix with a pattern J, say. Therefore we can apply Theorem 1.1 from [5]. We now have that $\min \mathrm{rank}(\tilde{W}) = \max_{T\subset J}(\min \mathrm{rank}\, W_T)$, where W_T is the partial matrix obtained from $\tilde{W}$ by only keeping the known entries that lie in the triangular subpattern T. This gives us that $\min \mathrm{rank}\, \tilde{W} = \max\left\{ n + q, \mathrm{rank}\begin{pmatrix} \alpha I_n & A \\ B & I_n \end{pmatrix} \right\} = n + q$, by the choice of q. Thus $\tilde{W}$ has a completion of rank $n + q$ and consequently we can find A_{ext} and B_{ext} such that $A_{ext}B_{ext} = \alpha I_{n+q}$. Thus $\mathrm{id}(A, B) \leq \mathrm{rank}(\alpha I_n - AB)$. This proves the proposition. □

Let us outline how to find a minimal inverse completion.

Algorithm 1. Let A, B and α be as in Theorem 4.1 and put $q = \mathrm{id}(A, B)$ and $p = n - q$. First determine an invertible matrix S so that $SABS^{-1} = \begin{pmatrix} \alpha I_p & P \\ 0 & Q \end{pmatrix}$ for some P and Q of sizes $p \times q$ and $q \times q$, respectively. Write now $SA = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, $BS^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ with C_{11}, D_{11} of size $p\times p$ and C_{22}, D_{22} of size $q\times q$. Choose X and Y to be invertible matrices of size $q \times q$ and let

$$D_{ext} = \begin{pmatrix} D_{11} & D_{12} & -(D_{11}C_{12} + D_{12}C_{22})Y^{-1} \\ D_{21} & D_{22} & -(D_{21}C_{12} + D_{22}C_{22} - \alpha I)Y^{-1} \\ 0 & X & -XC_{22}Y^{-1} \end{pmatrix}$$

and put $B_{ext} = D_{ext}(S \oplus I_q)$, $A_{ext} = \alpha B_{ext}^{-1}$. Then one can check that A_{ext} and B_{ext} have the required form and (obviously) $A_{ext}B_{ext} = \alpha I$.

Let us try the algorithm out on an example.

Example 1. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ has eigenvalues $\pm\sqrt{2}$. Let us choose $\alpha = \sqrt{2}$ and $S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{2}} \end{pmatrix}$. Then $BS^{-1} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}$, $SA = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}$. Thus we get $D_{ext} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & -\frac{2}{y} \\ 1 & 1 & \frac{\sqrt{2}}{y} \\ 0 & x & \frac{x}{\sqrt{2}y} \end{pmatrix}$ and

$$A_{ext} = \begin{pmatrix} 1 & 0 & \frac{2\sqrt{2}}{x} \\ 0 & 2 & -\frac{4}{x} \\ -\frac{y}{\sqrt{2}} & y & -\frac{2y}{x} \end{pmatrix}, \; B_{ext} = \begin{pmatrix} 0 & 1 & -\frac{2}{y} \\ 1 & 0 & \frac{\sqrt{2}}{y} \\ \frac{x}{2} & -\frac{x}{2\sqrt{2}} & \frac{x}{\sqrt{2}y} \end{pmatrix}.$$

As observed before, we have that $\operatorname{cd}(A,B) \le \operatorname{id}(A,B)$. In general there is no equality. For instance, when $A = (0)$ and $B = (1)$ we have $\operatorname{cd}(A,B) = 0$ and $\operatorname{id}(A,B) = 1$. That such a simple example exists seems to be due to the fact that we are excluding the possibility $\alpha = 0$ in the definition of $\operatorname{id}(A,B)$; we do this since $CD = 0$ does not imply $DC = 0$. But now one can ask what happens when A and B are nonsingular. Even in that case one can in general improve upon the estimate $\operatorname{cd}(A,B) \le \operatorname{id}(A,B)$ by doing the following. Suppose there is an invertible matrix S so that

$$SAS^{-1} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_d \end{pmatrix}, SBS^{-1} = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_d \end{pmatrix}, \tag{11}$$

where A_i and B_i, $i = 1, \ldots, d$, are square matrices of the same nontrivial size. Then completing A_i and B_i to $(A_i)_{ext}$ and $(B_i)_{ext}$ that commute for all $i \in \{1, \ldots, d\}$ yields completions A_{ext} and B_{ext} for A and B, respectively, that commute. Thus $\operatorname{cd}(A,B) \le \sum_{i=1}^d \operatorname{cd}(A_i, B_i) \le \sum_{i=1}^d \operatorname{id}(A_i, B_i)$. We are now led to the following question: Let A and B be nonsingular matrices so that for no invertible S we have that (11) holds with $d \ge 2$. Is it then true that $\operatorname{cd}(A,B) = \operatorname{id}(A,B)$?

The questions in this section may also be pursued in the class of real symmetric matrices. In other words, let A and B be real symmetric and look for A_{ext} and B_{ext} that are also real symmetric. As a complex symmetric matrix N is normal if and only if the real symmetric matrices $A = \operatorname{Re} N$ and $B = \operatorname{Im} N$ commute, Corollary 1 applies. The real symmetric case is of interest in deriving multivariable quadrature formulas; see [1]. In their setting A and B have a tridiagonal block form and A_{ext} and B_{ext} are required to have this form as well. For this reason it may not be optimal to look for A_{ext} and B_{ext} with $A_{ext}B_{ext} = \alpha I_{n+q}$. We hope to further pursue the real symmetric case in a future publication.

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