LEGENDRE SCALING FUNCTION FOR SOLVING GENERALIZED EMDEN-FOWLER EQUATIONS

S. A. YOUSEFI

Abstract. A numerical solution of the generalized Emden-Fowler equations as singular initial value problems is presented. We first rewrite Emden-Fowler equation in the form of integral equation by using especial integral operator and then applying Legendre scaling function approximation. The properties of Legendre scaling function are first presented. These properties together with the Gaussian integration method are then utilized to reduce the integral equations to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Key Words. Legendre scaling function, Emden-Fowler equation, Integral equations, Gaussian integration.

1. Introduction

There exists sufficiently large number of particular basic second-order singular nonlinear ordinary differential equations in mathematical physics and nonlinear mechanics that they do not admit exact analytic solution in terms of known functions [1,2].

One of the equations describing this type is the Emden-Fowler equations formulated as

\[
y''(x) + \frac{r}{x} y'(x) + af(x)g(y) = 0, \quad r \geq 0, \quad 0 < x \leq 1, \quad \alpha \geq 0, \quad (1)
\]

with initial conditions

\[
y(0) = \alpha, \quad y'(0) = 0, \quad (2)
\]

where \( f(x) \) and \( g(y) \) are some given functions of \( x \) and \( y \) respectively.

This equation was originally proposed by Chandrasekhar [3] to model stars. equation (1) also arises to model several phenomena in mathematical physics and astrophysics such as the thermal behavior of a spherical cloud of gas, isothermal gas sphere, nuclear physics, theory of thermionic currents, and the study of chemically reacting systems [1–4]. Several methods for the solution of Emden-Fowler equations are known. A discussion of the formulation of these models and the physical structure of the solutions can be found in [1-4]. Wazwaz [5,6] has given a general study to construct exact and series solutions to Emden-Fowler equations by employing the Adomian decomposition method.

In recent years there is much interest in the use of scaling functions and wavelets for the solution of nonlinear Physical and engineering problems. Wavelets theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [7]. Wavelets
permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [8].

In the present article, we are concerned with the application of Legendre scaling function to the numerical solution of (1). The method consists of convert of Emden-Fowler equations to integral equations and expanding the solution by Legendre scaling function with unknown coefficients. The properties of Legendre scaling function together with the Gaussian integration formula [9] are then utilized to evaluate the unknown coefficients and find an approximate solution to equation (1).

The article is organized as follows. In section 2, we describe the basic formulation of Legendre scaling function required for our subsequent development. Section 3 is devoted to the solution of equation (1) by using integral operator and Legendre scaling function. In section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Properties of Legendre scaling function

2.1. Legendre scaling function. Legendre scaling functions $\psi_{nm}^k(t)$ are defined by dilation and translation $\phi_m(t)$ as [10]:

\[
\phi_m(t) = \begin{cases} 
\sqrt{2m+1}L_m(2t-1), & \text{for } 0 \leq t \leq 1 \\
0, & \text{otherwise},
\end{cases}
\]

\[
\psi_{nm}^k(t) = 2^k \phi_m(2^kt-n),
\]

where $k$ is any fixed nonnegative integer number and $m = 0, 1, \ldots, M$, $n = 0, 1, 2, \ldots, 2^k - 1$. $\psi_{nm}^k(t)$ forms an orthonormal basis for $L^2[0, 1]$ [11].

The coefficient $\sqrt{2m+1}$ is for orthonormality. Here, $L_m(t)$ are the well-known Legendre polynomials of order $m$ which are defined on the interval $[-1, 1]$, and can be determined with the aid of the following recurrence formulae:

\[
L_0(t) = 1, \quad L_1(t) = t, \quad L_{m+1}(t) = \frac{2m+1}{m+1}tL_m(t) - \frac{m}{m+1}L_{m-1}(t), \quad m = 1, 2, 3, \ldots.
\]

2.2. Function Approximation. A function $f(t)$ defined over $[0, 1)$ may be expanded as

\[
f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t),
\]

where $c_{nm} = (f(t), \psi_{nm}^k(t))$, in which $(\ldots)$ denotes the inner product. If the infinite series in equation (3) is truncated, then equation (3) can be written as

\[
f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm} \psi_{nm}(t) = CT\Psi(t),
\]

where $C$ and $\Psi(t)$ are $(2^k)(M+1) \times 1$ matrices given by

\[
C = [c_{00}, c_{01}, \ldots, c_{0M}, c_{10}, \ldots, c_{1M}, \ldots, c_{(2^k-1)0}, \ldots, c_{(2^k-1)M}]^T,
\]

\[
\Psi(t) = [\psi_{00}(t), \psi_{01}(t), \ldots, \psi_{0M}(t), \psi_{10}(t), \ldots, \psi_{1M}(t), \ldots, \psi_{(2^k-1)0}(t), \ldots, \psi_{(2^k-1)M}(t)]^T.
\]
2.3. Convergence of the Legendre scaling basis. For a function \( f \in L^2[0,1] \), a nonnegative integer \( k \) and \( m = 0, 1, 2, \cdots, n = 0, 1, 2, \cdots, 2^k - 1 \), we can bound the error, as established by the following lemma.

**Lemma 2.1.** Suppose that the function \( f : [0,1] \rightarrow \mathbb{R} \) is \( m \) times continuously differentiable, \( f \in C^m[0,1] \). Then \( C^T \Psi \) approximate \( f \) with mean error bounded as follows:

\[
\| f - C^T \Psi \| \leq \frac{1}{m! 2^m k} \sup_{x \in [0,1]} |f^{(m)}(x)|.
\]

**Proof.** We divide the interval \([0,1]\) into subintervals \([\frac{n}{2^k}, \frac{n+1}{2^k}]\) which the restriction of \( C^T \Psi \) is a polynomial of degree \( m \) that approximate \( f \) with minimum mean error. We then use the maximum error estimate for the polynomial which interpolates \( f \) of order \( m \) on \([\frac{n}{2^k}, \frac{n+1}{2^k}]\). We have

\[
\| f - C^T \Psi \|^2 = \int_0^1 [f(x) - C^T \Psi(x)]^2 dx
\]

\[
= \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} [f(x) - C^T \Psi(x)]^2 dx
\]

\[
\leq \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f(x) - f^*(x)|^2 dx
\]

\[
\leq \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \frac{1}{m! 2^m k} \sup_{x \in [0,1]} |f^{(m)}(x)|^2 dx
\]

\[
\leq \int_0^1 \frac{1}{m! 2^m k} \sup_{x \in [0,1]} |f^{(m)}(x)|^2 dx = \left[ \frac{1}{m! 2^m k} \sup_{x \in [0,1]} |f^{(m)}(x)| \right]^2,
\]

and by taking square roots we have bound (6). Here \( f^*(x) \) denotes the interpolating polynomial of \( f(x) \) and we have used the well-known maximum error bound for interpolation [9]. \( \Box \)

3. Solution of Emden-fowler Equations

Consider the Emden-fowler equations given in equation (1). Define integral operator

\[
L_r(.) = \int_0^x x^{-r} \int_0^x t^r(.) dt dx.
\]

Applying \( L_r \) to both sides of (1) yields

\[
y(x) = \alpha - aL_r(f(x)y(y))
\]

Let

\[
F(x,y(x)) = x^{-r} \int_0^x t^r f(t)g(y(t)) dt
\]

Thus we have

\[
y(x) = \alpha - \int_0^x F(t,y(t)) dt
\]

which is nonlinear volterra integral equation.

In order to use Legendre scaling function, we first approximate \( y(x) \) as

\[
y(x) = C^T \Psi(x),
\]

\[
\end{align*}
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2.3. \textbf{Convergence of the Legendre scaling basis.} \text{ For a function } f \in L^2[0,1], \text{ a nonnegative integer } k \text{ and } m = 0, 1, \cdots, \ n = 0, 1, 2, \cdots, 2^k - 1, \text{ we can bound the error, as established by the following lemma.}

\textbf{Lemma 2.1.} \text{ Suppose that the function } f : [0,1] \rightarrow \mathbb{R} \text{ is } m \text{ times continuously differentiable, } f \in C^m[0,1]. \text{ Then } C^T \Psi \text{ approximate } f \text{ with mean error bounded as follows:}

\[
\| f - C^T \Psi \| \leq \frac{1}{m! 2^m k} \sup_{x \in [0,1]} |f^{(m)}(x)|.
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\textbf{Proof.} \text{ We divide the interval } [0,1] \text{ into subintervals } [\frac{n}{2^k}, \frac{n+1}{2^k}] \text{ which the restriction of } C^T \Psi \text{ is a polynomial of degree } m \text{ that approximate } f \text{ with minimum mean error. We then use the maximum error estimate for the polynomial which interpolates } f \text{ of order } m \text{ on } [\frac{n}{2^k}, \frac{n+1}{2^k}]. \text{ We have}

\[
\| f - C^T \Psi \|^2 = \int_0^1 [f(x) - C^T \Psi(x)]^2 dx
\]

\[
= \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} [f(x) - C^T \Psi(x)]^2 dx
\]

\[
\leq \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f(x) - f^*(x)|^2 dx
\]

\[
\leq \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \frac{1}{m! 2^m k} \sup_{x \in [0,1]} |f^{(m)}(x)|^2 dx
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Thus we have

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which is nonlinear volterra integral equation.

In order to use Legendre scaling function, we first approximate \( y(x) \) as

\[
y(x) = C^T \Psi(x),
\]

\[
\end{align*}
\]
where \( C \) and \( \Psi(x) \) are defined similarly to equations (4) and (5). Then from equations (7) and (8) we have

\[
C^T \psi(x) = \alpha - \int_0^x F(t, C^T \psi(t))dt,
\]

we now collocate equation (9) at \((2^k+1)(M+1)\) points \( x_i \) as

\[
C^T \psi(x_i) = \alpha - \int_0^{x_i} F(t, C^T \psi(t))dt,
\]

Suitable collocation points are zeros of Chebyshev polynomials [9]

\[
x_i = \cos((2i-1)\pi/((2^k+1)(M+1))), \quad i = 1, \ldots, (2^k+1)(M+1).
\]

In order to use the Gaussian integration formula for equation (10), we transfer the \( t \)-intervals \([0, x_i]\) into \( \tau \) interval \([-1, 1]\) by means of the transformations

\[
\tau = 2x_i t - 1,
\]

Equation (10) may then be restated as

\[
C^T \psi(x_i) = \alpha - x_i \frac{1}{2} \int_{-1}^{1} F\left(\frac{x_i}{2}(\tau+1), C^T \psi\left(\frac{x_i}{2}(\tau+1)\right)\right) d\tau,
\]

by using the Gaussian integration formula we get

\[
C^T \psi(x_i) \approx \alpha - x_i \sum_{j=1}^{s} w_j F\left(\frac{x_i}{2}(\tau_j+1), C^T \psi\left(\frac{x_i}{2}(\tau_j+1)\right)\right), \quad i = 1, \ldots, (2^k+1)(M+1),
\]

where \( \tau_j \) is zeros of Legendre polynomials \( P_{s+1} \) and \( w_j \) is the corresponding weights given in [9]. The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding \( 2s + 1 \). Equation (11) gives \( 2^{k-1} M \) nonlinear equations which can be solved for the elements of \( C \) in equation (8) using Newton’s iterative method.

### 4. Illustrative Examples

We applied the method presented in this paper and solved three examples given in [6]. This method differs from Adomian approach considered in [6] and thus could be used as a basis for comparison.

**Example 1.** Consider the Emden-fowler equation given in [6] by

\[
y''(x) + \frac{2}{x} y'(x) + x^m y^n(x) = 0, \quad 0 \leq x \leq 1,
\]

\[
y(0) = 1, \quad y'(0) = 0.
\]

The corresponding integral equation is

\[
y(x) = 1 - L_2(x^m y^n(x)).
\]

We applied the method presented in this paper and solved equation (12) with \( k = 0 \) and \( M = 4 \). For this equation we find:

For \( m = n = 0 \)

\[
c_{00} = \frac{17}{18}, \quad c_{01} = -\frac{1}{12\sqrt{3}}, \quad c_{02} = -\frac{1}{36\sqrt{5}}, \quad c_{03} = 0, \quad c_{04} = 0.
\]

Using equation (8) we get

\[
y(x) = c_{00} \Psi^0_{00} + c_{01} \Psi^0_{01} + c_{02} \Psi^0_{02} + c_{03} \Psi^0_{03} + c_{04} \Psi^0_{04} = 1 - \frac{1}{6} x^2
\]
which is the exact solution.
For \( m = 0 \) and \( n = 1 \)

\[
c_{00} = 0.946082, \quad c_{01} = -0.0462223, \quad c_{02} = -0.0113858,
\]

\[
c_{03} = 0.000304602, \quad c_{04} = 0.0000348819.
\]

The exact solution is \( y(x) = \sin(x)/x \).
Table 1 presents values of \( y(x) \) together with the exact values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Legendre scaling function</th>
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Table 1. Estimated and exact values of \( y(x) \).
For \( m = \frac{1}{2} \) and \( n = 0 \)

\[
c_{00} = 0.967324, \quad c_{01} = -0.0313811, \quad c_{02} = -0.0111027,
\]

\[
c_{03} = -0.000970069, \quad c_{04} = 0.000051582.
\]

The exact solution is \( y(x) = 1 - \frac{1}{3} x^2 \).
Table 2. Estimated and exact values of $y(x)$. Table 2 presents values of $y(x)$ together with the exact values.

**Example 2.** Consider the Emden-fowler equation introduced by Richardson [4]

$$y''(x) + \frac{2}{x}y'(x) - e^{-y(x)} = 0, \quad 0 \leq x \leq 1,$$

$$y(0) = 0, \quad y'(0) = 0.$$

This model appear in Richardson’s theory of thermionic current when the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium [1] is to be determined. For a thorough discussion of the formulation of (13) and the physical behavior of the emission of electricity from hot bodies, see [1,4].

It is interesting to point here that

$$y(x) = \ln\left(\frac{x^2}{2}\right)$$

is the only known particular solution for the equation

$$y''(x) + \frac{2}{x}y'(x) - e^{-y(x)} = 0.$$  

The corresponding integral equation is

$$y(x) = L_2(e^{-y(x)}).$$

We applied the method presented in this paper and solved equation (13) with $k = 0$ and $M = 6, 8$.

Let $\hat{y}(x)$ be approximated solution of $y(x)$ and

$$D[y] = y''(x) + \frac{2}{x}y'(x) - e^{-y(x)}.$$

Table 3 presents values of $|D[\hat{y}](x)|$ for $k = 0$ and $M = 6, 8$. 

<table>
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<th>Legendre scaling function</th>
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Example 3. Consider the Emden-fowler equation given in [6] by

\[ y''(x) + \frac{8}{x} y'(x) - 18y(x) - 4y(x) \ln(y(x)) = 0, \quad 0 \leq x \leq 1, \quad (15) \]

\[ y(0) = 1, \quad y'(0) = 0. \]

The corresponding integral equation is

\[ y(x) = 1 - L_8 (18y(x) + 4y(x) \ln(y(x))). \]

We applied the method presented in this paper and solved equation (15) with \( k = 0 \) and \( M = 8, 10 \). Table 4 presents values of \( y(x) \) together with the exact values. The exact solution is \( y(x) = e^{x^2} \).

5. CONCLUSION

The aim of present work is to develope an efficient and accurate method for solv- ing the Emden-fowler equations as singular initial value problems. The properties of the Legendre scaling function together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. Furthermore, since the basis of Legendre scaling function are polynomial, thus the values of integrals for the nonlinear integral equations of the form given in equation (7) are calculated as exact (see example 1).
250  S. A. YOUSEFI

<table>
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Table 4. Estimated and exact values of $y(x)$.

References


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