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UNIFORMLY CONVERGENT COMPACT NUMERICAL SCHEME FOR THE NORMALIZED FLUX OF SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

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Abstract. In this paper, we consider singularly perturbed reaction-diffusion two-point boundary value problems. To solve these types of problems we develop a numerical scheme which is a combination of the cubic splines and the classical central difference scheme. The proposed scheme is applied on an appropriate piecewise uniform Shishkin mesh. We prove that, at the mesh points, the method is uniformly convergent of second order. The normalized flux obtained via the cubic spline from the numerical solution is also uniformly convergent. Further, we have constructed the global solution using cubic splines, which is uniformly convergent in the boundary layer regions. One of the merits of the proposed scheme is that it is of compact type. We present some numerical examples illustrating in practice the theoretical error bounds previously proved. These examples reveal the theoretical estimates obtained in this work.

Key Words. Singular perturbation problems, cubic splines, finite difference schemes, reaction–diffusion boundary-value problems, Shishkin mesh, normalized flux, global solution.

Contents

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1. Introduction

In this article, we devise a numerical scheme for the following singularly perturbed reaction–diffusion two-point boundary-value problem (BVP):

$$Lu(x) \equiv -\varepsilon u''(x) + b(x)u(x) = f(x), \quad x \in D = (0,1), \tag{1}$$

$$u(0) = A, \quad u(1) = B, \tag{2}$$

where $\varepsilon > 0$ is a small parameter and b, f are sufficiently smooth functions such that $b(x) > \beta > 0$ on $\overline{D} = [0,1]$. Under these assumptions the BVP (1-2) has a unique solution $u(x) \in \mathcal{C}^2(D) \cap \mathcal{C}(\overline{D})$.

Singular perturbation problems (SPPs) arise in several branches of engineering and applied mathematics which include fluid dynamics, quantum mechanics, elasticity, chemical reactor theory and gas porous electrodes theory, which include linearized Navier-Stokes equation at high Reynolds number and the drift-diffusion equation of semiconductor device modelling. For $\varepsilon \ll 1$ the solution of the BVP (1-2) has two boundary layers of thickness $O(\sqrt{\varepsilon})$ near the boundaries $x = 0$, 1, and it is well known that a classical central difference scheme on a uniform mesh will not give a satisfactory numerical solution in this case. Therefore, a separate treatment is necessary to deal with these types of problems. For details, one may refer to the books of Farrell et al. [2] and Roos et al. [6] and references therein.

From an engineering point of view it is important to obtain accurate numerical solution and normalized flux for the BVP (1-2). For instance, consider a substance in a solution with a flux satisfying Fick's law, and with the distribution given by a diffusion equation. The initial concentration of an admixture at the boundary of the body is known. It is required to find the distribution of the admixture in the material at any given time, as well the quantity of admixture (the diffusive flux) emitted from the boundaries into the exterior environment. Such problems arising in environmental sciences are determining the pollution entering the environment from manufacturing sources. Further, it is interesting to note that the diffusion Fourier number, which is proportional to the diffusion coefficient of the substance, can be sufficiently small to cause large variation of concentration with the depth of sample. For smaller values of Fourier number, a diffusion boundary layer appears in narrow neighbourhood of the boundary. Difficulties arise when it is required to find the solution of these problems. In particular, more difficulties arise when one has to calculate the spatial derivative of $u(x)$, *i.e.*, the normalized fluxes of the solution. The main contribution of this paper is that it proposes a numerical scheme to determine the numerical solution, normalized flux, and global solution of the reaction-diffusion BVP (1-2).

In the present work, we propose a hybrid numerical scheme, which is a combination of the cubic spline scheme and the classical central difference scheme; the method is defined on a piecewise uniform Shishkin mesh (the details of the construction of the mesh are provided in Section 3). In the boundary layer regions $[0, \sigma)$ and $(1-\sigma, 1]$ we discretize the BVP by using the cubic spline difference scheme, whereas in the outer region $[\sigma, 1-\sigma]$ we use the classical central difference scheme. This is mainly to preserve the monotonicity of the difference operator, *i.e.*, to retain the discrete maximum principle of the difference scheme; this property will be crucial in the proof of the ε-uniform convergence of the scheme. The use of the discrete maximum principle gives a different way to the technique developed in the papers [8, 9], in order to prove the uniform convergence of the solution given by the numerical scheme.

We will see that the present method provides an almost second order uniform convergent result throughout the domain of interest. Further, we propose a scheme to determine the normalized fluxes, and we are able to provide ε-uniform convergent results for normalized fluxes. As pointed out above, in several engineering problems it is more interesting to obtain good approximations of the corresponding normalized flux, independently of the value of the diffusion parameter; for instance, in the problem of the diffusion of a substance in a solid, it is required to know the distribution of the substance and also the diffusion fluxes in the material.

The extensions of some of these techniques to singularly perturbed problems is considered in some papers. For instance, in [1, 3], the authors have devised an High-Order Difference Approximation with Identity Expansions (HODIE) schemes for singularly perturbed convection-diffusion and reaction-diffusion problems on Shishkin meshes respectively, In [8], cubic spline was used to solve a nonlinear reaction-diffusion problem, showing their uniform convergence; nevertheless, the proof of the results is incomplete and not clear. In [9] a quadratic spline was used to solve semilinear reaction-diffusion problems, proving the uniform convergence at the mesh points and also the uniform convergence of the global solution given by the spline and the normalized flux obtained from the derivative of the spline.

The paper is organized as follows. Some bounds for the continuous solution and its derivatives are given in Section 2. The cubic spline difference scheme is derived in Section 3; using this scheme we define the hybrid scheme to solve the problem (1-2); also the uniform error estimates for the solution, normalized flux are obtained. In Section 4, we analyze the uniform convergence of the global solution based on both the cubic spline and the piecewise linear Lagrange interpolation. Finally, some numerical examples are presented in Section 5; these numerical experiments show the efficiency of the present method and also permit us to see that the global normalized flux, obtained from the cubic spline, is ε-uniform convergent.

Henceforth, C denotes any positive constant independent of the diffusion parameter ε and the discretization parameter N.

2. Bounds for the continuous solution

Here, we recall some standard results on the bounds for the solution of the continuous problem and its regular and singular components. One can find the details of the proofs in [4]. Let $y(x)$ be the solution of the boundary value problem

$$Ly(x) \equiv -\varepsilon y''(x) + b(x)y(x) = g(x,\varepsilon), \quad x \in D, \tag{3}$$

$$y(0) = A, \quad y(1) = B, \tag{4}$$

where $b, g \in \mathcal{C}^3(\overline{D})$, with $b(x) > \beta > 0$, on $\overline{D}$, and the right hand side function $g(x,\varepsilon)$ satisfies the following bounds:

$$|g^{(k)}(x,\varepsilon)| \le C\left(1 + \varepsilon^{-k/2} e(x,x,\beta,\varepsilon)\right), \quad 0 \le k \le 3,$$

where

$$e(\xi_1,\xi_2,\beta,\varepsilon) = \exp(-\sqrt{\beta}\xi_1/\sqrt{\varepsilon}) + \exp(-\sqrt{\beta}(1-\xi_2)/\sqrt{\varepsilon}).$$

Then, the following estimates hold

$$|y^{(k)}(x)| \le C\left(1 + \varepsilon^{-k/2} e(x,x,\beta,\varepsilon)\right), \quad 0 \le k \le 4. \tag{5}$$

Further, we can decompose the solution $u(x)$ of (1-2) as $u(x) = v(x) + w(x)$, where v and w are the regular and singular components of u respectively and they are the solutions of suitable problems. Using the technique applied in [4], one can prove the following bounds for these components:

$$|v^{(k)}(x)| \le C, \quad |w^{(k)}(x)| \le C\varepsilon^{-k/2} e(x, x, \beta, \varepsilon), \quad 0 \le k \le 4. \tag{6}$$

3. The numerical method

In this section, first we derive the cubic spline scheme on a general nonuniform mesh, and then we propose the hybrid scheme. Let

$$\overline{D}^N \equiv \left\{ x_i : x_0 = 0,\, x_N = 1,\, x_i \sum_{k=0}^{i-1} h_k,\, h_k = x_{k+1} - x_k,\, i = 1, 2, \dots, N-1 \right\}$$

be the discretization of the domain $\overline{D}$. On this mesh we construct the numerical scheme as follows. We seek a $\mathcal{C}^2$-spline $s(x)$ with nodal values $s(x_i) = U_i$ that satisfies the collocation conditions $Ls(x_i) = f(x_i)$ and the boundary conditions $U_0^N = A$, $U_N^N = B$. Using the 'continuity conditions' of first derivatives of cubic spline at nodal points, one can obtain the following difference scheme on variable mesh for the BVP (1):

$$\begin{cases} \left[\dfrac{-3\varepsilon}{h_{i-1}(h_i + h_{i-1})} + \dfrac{h_{i-1}}{2(h_i + h_{i-1})} b_{i-1} \right] U_{i-1}^N + \left[\dfrac{3\varepsilon}{h_i h_{i-1}} + b_i \right] U_i^N + \\ + \left[\dfrac{-3\varepsilon}{h_i(h_i + h_{i-1})} + \dfrac{h_i}{2(h_i + h_{i-1})} b_{i+1} \right] U_{i+1}^N \left[\dfrac{h_{i-1}}{2(h_i + h_{i-1})} \right] f_{i-1} + f_i + \\ + \left[\dfrac{h_i}{2(h_i + h_{i-1})} \right] f_{i+1}, \end{cases} \tag{7}$$

where $f_i = f(x_i)$, $b_i = b(x_i), i = 1, \cdots, N-1$, and $U_0^N = A$, $U_N^N = B$. One can refer [5] for the detailed derivation of this scheme.

The cubic spline scheme given in (7) is analyzed for stability, and it has been observed that for the corresponding stiffness matrix to be an M-matrix, a very restrictive condition is needed on the mesh size, specially in the outer region. In other words, the discrete operator is not monotone in the outer region $[\sigma, 1-\sigma]$, where the mesh is coarse. To preserve the monotonicity of the discrete operator one has to use a smaller mesh size in the outer region; nevertheless, this is not possible because we are using the piecewise uniform Shishkin mesh whose outer mesh size is fixed for each value of ε and N. To overcome this difficulty, the following hybrid scheme is proposed, in which the well known classical central difference scheme is taken in the outer region and the above cubic spline scheme is considered in the boundary layer regions:

$$L^N U_i^N \equiv r_i^- U_{i-1}^N + r_i^c U_i^N + r_i^+ U_{i+1}^N q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1}, \tag{8}$$

along with the boundary conditions $U_0^N = A$ and $U_N^N = B$, where, for indices $i = 1, \cdots, N/4 - 1$ and also $3N/4 + 1, \cdots, N-1$, the coefficients are defined by

$$\begin{cases} r_i^- = \dfrac{-3\varepsilon}{h_{i-1}(h_i + h_{i-1})} + \dfrac{h_{i-1}}{2(h_i + h_{i-1})} b_{i-1}, \quad r_i^c = \dfrac{3\varepsilon}{h_i h_{i-1}} + b_i, \\ r_i^+ = \dfrac{-3\varepsilon}{h_i(h_i + h_{i-1})} + \dfrac{h_i}{2(h_i + h_{i-1})} b_{i+1}, \\ q_i^- = \dfrac{h_{i-1}}{2(h_i + h_{i-1})}, \quad q_i^c = 1, \quad q_i^+ = \dfrac{h_i}{2(h_i + h_{i-1})}, \end{cases} \tag{9}$$

and for $i = N/4, \cdots, 3N/4$, the coefficients are given by

$$\left\{ \begin{array}{l} r_i^- = \dfrac{-2\varepsilon}{h_{i-1}(h_i + h_{i-1})}, \quad r_i^c = \dfrac{2\varepsilon}{h_i h_{i-1}} + b_i, \quad r_i^+ = \dfrac{-2\varepsilon}{h_i(h_i + h_{i-1})}, \\ q_i^- = 0, \quad q_i^c = 1, \quad q_i^+ = 0. \end{array} \right. \tag{10}$$

3.1. The piecewise uniform Shishkin Mesh. Before proving the uniform convergence of the hybrid scheme, let us define an appropriate Shishkin mesh for the boundary value problem (1-2). On $\overline{D}$ a piecewise uniform mesh of N mesh intervals is constructed as follows: the domain $\overline{D}$ is divided into three subintervals as $\overline{D} = [0, \sigma) \cup [\sigma, 1 - \sigma] \cup (1 - \sigma, 1]$, for some σ such that $0 < \sigma \leq 1/4$. On the subintervals $[0, \sigma]$ and $[1-\sigma, 1]$ a uniform mesh with $N/4$ mesh intervals are placed, while $[\sigma, 1-\sigma]$ has a uniform mesh with $N/2$ mesh intervals. It is obvious that the mesh is uniform when $\sigma = 1/4$ and it is fitted to the problem by choosing σ as the following function of N, ε and σ_0

$$\sigma = \min \left\{ \frac{1}{4}, \sigma_0 \sqrt{\varepsilon} \ln N \right\}, \tag{11}$$

where σ_0 is a constant to be fixed later. Here we are interested in the case that $\sigma = \sigma_0 \sqrt{\varepsilon} \ln N$; in the other case, a classical analysis can be done to prove the convergence of our numerical method. Further, we denote the mesh size in the region $[\sigma, 1 - \sigma]$ as $H2(1 - 2\sigma)/N$, and in the regions $[0, \sigma], [1 - \sigma, 1]$ by $h = 4\sigma/N$.

In [5], the following theorem was proved which gives the uniform convergence of the numerical solution at the mesh points of the Shishkin mesh.

Theorem 3.1. *Let $u(x)$ be the solution of (1-2) and U^N be the numerical solution of the hybrid finite difference scheme (8). Then, the error satisfies*

$$|u(x_i) - U_i^N| \leq C \left(N^{-2} \ln^2 N + N^{-\sqrt{\beta}\sigma_0} \right), \quad i = 0, 1, \cdots, N, \tag{12}$$

and therefore if $\sigma_0 \geq 2/\sqrt{\beta}$ the method (8) is uniformly convergent of order almost two.

Remark 3.2. *In the sequel we always assume that the constant σ_0 satisfies the condition $\sigma_0 \geq 2/\sqrt{\beta}$.*

3.2. The Normalized flux. Next, we are interested in proving the uniform convergence for the normalized flux, which is defined by $\sqrt{\varepsilon} u'(x)$. Using the numerical solution U_i^N, $i = 0, \cdots, N$, we can define a global solution via the cubic spline by interpolating these values. Then, with the values of $\overline{M}_i = (b_i U_i^N - f_i)/\varepsilon$, $i = 0, \cdots, N$, the cubic spline is given by

$$\begin{aligned} S(x) \quad = \quad & \frac{(x_{i+1} - x)^3}{6h_i} \overline{M}_i + \frac{(x - x_i)^3}{6h_i} \overline{M}_{i+1} + \left(U_i^N - \frac{h_i^2}{6} \overline{M}_i \right) \left(\frac{x_{i+1} - x}{h_i} \right) + \\ & + \left(U_{i+1}^N - \frac{h_i^2}{6} \overline{M}_{i+1} \right) \left(\frac{x - x_i}{h_i} \right), \quad x_i \leq x \leq x_{i+1}, \ i = 0, \cdots, N - 1. \end{aligned} \tag{13}$$

Then, for $i = N/4, 3N/4 - 1$ and N, we consider the interval $[x_{i-1}, x_i]$ for first derivative at x_i and we define the normalized flux at these points as

$$\sqrt{\varepsilon} S'(x_i) = \frac{\sqrt{\varepsilon} h_{i-1}}{6} \overline{M}_{i-1} + \frac{\sqrt{\varepsilon} h_{i-1}}{3} \overline{M}_i + \sqrt{\varepsilon} \left(\frac{U_i^N - U_{i-1}^N}{h_{i-1}} \right). \tag{14}$$

In the other case, *i.e.*, for the rest of the points x_i, we take the interval $[x_i, x_{i+1}]$ for first derivative at x_i and we define the normalized flux at these points as

$$\sqrt{\varepsilon}S'(x_i) = -\frac{\sqrt{\varepsilon}h_i}{3}\overline{M}_i - \frac{\sqrt{\varepsilon}h_i}{6}\overline{M}_{i+1} + \sqrt{\varepsilon}\left(\frac{U_{i+1}^N - U_i^N}{h_i}\right). \tag{15}$$

The different approximation of the normalized flux at the mesh point x_N is obvious and in the other two points is related with that these points are the transition points of the Shishkin mesh, where the mesh sizes are different.

Theorem 3.3. *Let $u(x)$ be the solution of (1-2) and U^N be the numerical solution of the hybrid finite difference scheme (8). Then, the error of the normalized flux satisfies*

$$\sqrt{\varepsilon}|u'(x_i) - S'(x_i)| \le C\left(N^{-1}\ln N + N^{-\sqrt{\beta}\sigma_0}\right), \quad i = 0, 1, \cdots, N. \tag{16}$$

Proof. We assume that $x_i \le 1/2$ (the other case is symmetric). From Theorem 3.1, we have

$$|u(x_i) - S(x_i)| \le C(N^{-2}\ln^2 N). \tag{17}$$

Also, invoking the differential equation one can compute approximations of the second order derivative: $u''(x_i) \approx \overline{M}_i = (b(x_i)U_i^N - f(x_i))/\varepsilon$, we get

$$|u''(x_i) - S''(x_i)| \le C\varepsilon^{-1}(N^{-2}\ln^2 N). \tag{18}$$

For $x_i \in [0, \sigma]$, the above two equations give

$$|u'(x_i) - S'(x_i)| \le C\varepsilon^{-1/2}(N^{-2}\ln^2 N), \tag{19}$$

since

$$\|g'\|_{[a,b]} \le 2\|g\|_{[a,b]}/(b-a) + (b-a)\|g''\|_{[a,b]}.$$

Therefore, we have

$$\sqrt{\varepsilon}|u'(x_i) - S'(x_i)| \le C(N^{-2}\ln^2 N). \tag{20}$$

For the second case, *i.e.*, when $x_i \in (\sigma, 1/2]$, from Taylor expansion it follows that

$$\begin{aligned}\sqrt{\varepsilon}|u'(x_i) - S'(x_i)| &\le C\left(\frac{\sqrt{\varepsilon}}{h_i} + \frac{h_i}{\sqrt{\varepsilon}}\right)(|u(x_i) - U_i^N| + |u(x_{i+1}) - U_{i+1}^N|) + \\ &\quad + C\sqrt{\varepsilon}h_i^3|u^{iv}(\xi)|, \quad \xi \in [x_i, x_{i+1}].\end{aligned} \tag{21}$$

Now, we distinguish the following two cases: first, if $H \le \sqrt{\varepsilon}$, then from (21) it follows that

$$\begin{aligned}\sqrt{\varepsilon}|u'(x_i) - S'(x_i)| &\le C(N^{-1}\sqrt{\varepsilon}\ln^2 N + N^{-2}\ln^2 N + N^{-\sqrt{\beta}\sigma_0}) \\ &\le C(N^{-1}\ln N + N^{-2}\ln^2 N + N^{-\sqrt{\beta}\sigma_0}).\end{aligned} \tag{22}$$

On the other hand, if $H > \sqrt{\varepsilon}$, then we have

$$\sqrt{\varepsilon}|u'(x_i) - S'(x_i)| \le CN^{-2}\ln^2 N + C\frac{h_i}{\sqrt{\varepsilon}}|\tau_{i,u}| + C\sqrt{\varepsilon}h_i^3|u^{iv}(\xi)|, \tag{23}$$

with $\xi \in (x_i, x_{i+1})$. Now the local truncation error can be written as

$$\tau_{i,u} = \frac{2\varepsilon}{h_i + h_{i-1}}\left[\frac{R_3(x_i, x_{i+1}, u)}{h_i} + \frac{R_3(x_i, x_{i-1}, u)}{h_{i-1}}\right],$$

where

$$R_k(a, b, g) = \frac{1}{k!}\int_a^b (b-s)^k g^{(k+1)} ds,$$

is the remainder of the Taylor expansion. Using the bounds for the derivatives from (5), we have

$$\frac{h_i}{\sqrt{\varepsilon}}\frac{2\varepsilon}{h_i+h_{i-1}}\left|\frac{R_3(x_i,x_{i-1},u)}{h_{i-1}}\right| \le \frac{C\sqrt{\varepsilon}}{h_i+h_{i-1}}\int_{x_{i-1}}^{x_i}(s-x_{i-1})^3\left(1+\varepsilon^{-2}e(s,s,\beta,\varepsilon)\right)ds,$$

and integrating by parts it follows that

$$\begin{aligned}\frac{h_i}{\sqrt{\varepsilon}}\frac{2\varepsilon}{h_i+h_{i-1}}\left|\frac{R_3(x_i,x_{i-1},u)}{h_{i-1}}\right| \quad &\le \quad C\left(h_{i-1}^3\sqrt{\varepsilon}+h_{i-1}^2\varepsilon^{-1}e(x_i,x_i,\beta,\varepsilon)+\right.\\ &\quad +h_{i-1}\varepsilon^{-1/2}e(x_i,x_i,\beta,\varepsilon)+e(x_i,x_i,\beta,\varepsilon)+\\ &\quad \left.+e(x_{i-1},x_{i-1},\beta,\varepsilon)\right)\le C\left(N^{-3}\sqrt{\varepsilon}+N^{-\sqrt{\beta}\sigma_0}\right). \qquad (24)\end{aligned}$$

In a similar way, we can prove that

$$\frac{h_i}{\sqrt{\varepsilon}}\frac{2\varepsilon}{h_i+h_{i-1}}\left|\frac{R_3(x_i,x_{i+1},u)}{h_i}\right| \le C\left(N^{-3}\sqrt{\varepsilon}+N^{-\sqrt{\beta}\sigma_0}\right). \qquad (25)$$

Using again the bounds (5) we obtain

$$\sqrt{\varepsilon}h_i^3|u^{iv}(\xi)| \le C\left(\frac{h_i}{\sqrt{\varepsilon}}\right)^3 e(x_i,x_{i+1},\beta,\varepsilon)\le CN^{-\sqrt{\beta}\sigma_0}. \qquad (26)$$

Therefore, from (23-26) it follows that

$$\sqrt{\varepsilon}|u'(x_i)-S'(x_i)| \le C(N^{-2}\ln^2 N+N^{-3}\sqrt{\varepsilon}+N^{-\sqrt{\beta}\sigma_0}). \qquad (27)$$

Combining the estimates in (20), (22), and (27), we obtain the required result. ■

Remark 3.4. *In above theorem, it is observed that the we are getting almost second order convergence except in the outer region for the case when $H \le \sqrt{\varepsilon}$, which is not the practical case, so we conclude that the approximation of the normalized flux based on the cubic spline gives almost second order of uniform convergence. Our numerical results also confirms this.*

4. The Global Solution

In this section, we will analyze the uniform convergence of the global solution at any arbitrary point of the domain by using the cubic spline $S(x)$ given in (13).

Theorem 4.1. *(***Global error in the boundary layers***) Let $u(x)$ be the solution of (1-2) and U^N be the numerical solution of the hybrid finite difference scheme (8). Then, the error associated with the global solution satisfies*

$$|u(x)-S(x)| \le CN^{-2}\ln^2 N, \quad x\in[0,\sigma]\bigcup[1-\sigma,1]. \qquad (28)$$

Proof. Here, we supply a very simple proof by using Taylor's expansions. Let $x\in(x_i,x_{i+1})$; then, $x=x_i+\theta h_i$, with $0<\theta<1$. From the Taylor expansion, we obtain the following

$$\begin{aligned}|u(x)-S(x)| \quad &\le \quad |u(x_i)-S(x_i)|+\theta h_i|u'(x_i)-S'(x_i)|+\frac{\theta^2h_i^2}{2!}|u''(x_i)-S''(x_i)|+\\ &\quad +\frac{\theta^3h_i^3}{3!}|u'''(x_i)-S'''(x_i)|+\frac{\theta^4h_i^4}{4!}|u^{iv}(\xi)|, \qquad (29)\end{aligned}$$

with $\xi\in(x_i,x_{i+1})$. From Theorem 3.1, we know that

$$|u(x_i)-S(x_i)| \le CN^{-2}\ln^2 N. \qquad (30)$$

Also, from (19), and using that the step size is $h_i = h = 4N^{-1}\sigma_0\sqrt{\varepsilon}\ln N$, it is easy to see

$$h_i|u'(x_i) - S'(x_i)| \le CN^{-2}\ln^2 N. \tag{31}$$

In the third place, using (18), we have

$$h_i^2|u''(x_i) - S''(x_i)| \le C\frac{h_i^2}{\varepsilon}|u(x_i) - U_i^N| \le CN^{-4}\ln^4 N. \tag{32}$$

In fourth place, using that

$$u'''(x_i) - S'''(x_i) = \frac{u''(x_{i+1}) - u''(x_i)}{h_i} - \frac{\overline{M}_{i+1} - \overline{M}_i}{h_i} - \frac{h_i}{2}u^{iv}(\eta),$$

with $\eta \in (x_i, x_{i+1})$, from the bounds (5) it follows that

$$h_i^3|u'''(x_i) - S'''(x_i)| \le CN^{-4}\ln^4 N. \tag{33}$$

Finally, using again (6) to bound the fourth derivative, it is straightforward to prove that

$$\frac{\theta^4 h_i^4}{4!}|u^{iv}(\xi)| \le CN^{-4}\ln^4 N. \tag{34}$$

The results follows from (29-34). ■

Theorem 4.2. *(***Global error outside the boundary layers***) Let $u(x)$ be the solution of (1-2) and U^N be the numerical solution of the hybrid finite difference scheme (8). Then, the error associated to the global solution satisfies*

$$|u(x) - S(x)| \;\le\; C(N^{-2}\ln^2 N + N^{-\sqrt{\beta}\sigma_0}),\;\; x \in [\sigma + H, 1 - \sigma - H]. \tag{35}$$

Proof. The proof is analogous to the previous one. From Taylor expansion it follows that

$$\begin{aligned} |u(x) - S(x)| \;\le\; & |u(x_i) - S(x_i)| + \theta h_i|u'(x_i) - S'(x_i)| + \frac{\theta^2 h_i^2}{2!}|u''(x_i) - S''(x_i)| + \\ & + \frac{\theta^3 h_i^3}{3!}|u'''(x_i) - S'''(x_i)| + \frac{\theta^4 h_i^4}{4!}|u^{iv}(\xi)|, \end{aligned} \tag{36}$$

with $\xi \in (x_i, x_{i+1})$. For the first term, again from Theorem 3.1 it follows

$$|u(x_i) - S(x_i)| \le CN^{-2}\ln^2 N. \tag{37}$$

For the second term, if $H \le \sqrt{\varepsilon}$, it is trivial that

$$H|u'(x_i) - S'(x_i)| \le CN^{-2}\ln^2 N. \tag{38}$$

In the other case, when $H > \sqrt{\varepsilon}$, we have

$$H|u'(x_i) - S'(x_i)| \le C\left(\frac{H^2}{\varepsilon}|\tau_{i,u}| + H^4|u^{(iv)}(\xi)|\right),$$

with $\xi \in (x_i, x_{i+1})$. Then, using that

$$\begin{aligned} \frac{H^2}{\varepsilon}|\tau_{i,u}| \le \;\; & C\left(H^4 + \int_{x_i}^{x_{i+1}}(x_{i+1} - \xi)^3\varepsilon^{-2}e(\xi,\xi,\beta,\varepsilon)d\xi + \right. \\ & \left. + \int_{x_{i-1}}^{x_i}(\xi - x_{i-1})^3\varepsilon^{-2}e(\xi,\xi,\beta,\varepsilon)d\xi\right), \end{aligned}$$

and integrating by parts, we can obtain

$$\begin{aligned} H|u'(x_i) - S'(x_i)| &\leq CN^{-4} + Ce(x_i, x_i, \beta, \varepsilon)(H^4\varepsilon^{-2} + H^3\varepsilon^{-3/2} + H^2\varepsilon^{-1} + \\ &\quad +H\varepsilon^{-1/2}) + Ce(x_i, x_{i+1}, \beta, \varepsilon) + Ce(x_{i-1}, x_i, \beta, \varepsilon) \\ &\leq C(N^{-4} + N^{-\sqrt{\beta}\sigma_0}). \end{aligned} \tag{39}$$

In a similar way, we can prove that

$$H^2|u''(x_i) - S''(x_i)| \leq C(N^{-4} + N^{-\sqrt{\beta}\sigma_0}), \tag{40}$$

and also that

$$H^3|u'''(x_i) - S'''(x_i)| \leq C(N^{-4} + N^{-\sqrt{\beta}\sigma_0}). \tag{41}$$

Finally, using (5), for the remainder term we have

$$H^4|u^{(iv)}(\xi)| \leq C\left(\frac{H}{\sqrt{\varepsilon}}\right)^4 (e(x_i, x_i, \beta, \varepsilon) + e(x_{i-1}, x_{i-1}, \beta, \varepsilon)) \leq CN^{-\sqrt{\beta}\sigma_0}. \tag{42}$$

We can obtain the required result from (37-42). ∎

Remark 4.3. *Note that from the two previous theorems we cannot obtain uniform convergence of the global solution at the subintervals $(\sigma, \sigma+H)\bigcup(1-\sigma-H, 1-\sigma)$. Later on, we will corroborate this property from the numerical results.*

Another way to approximate the global solution uses the piecewise linear Lagrange interpolation. Then, we can define a global linear approximation to the solution u of (1-2) as

$$\overline{U}^N(x) = \sum_{i=0}^{N} U_i^N \phi_i(x), \tag{43}$$

where $\phi_i(x)$ is the standard piecewise linear basis function associated with the interval $[x_{i-1}, x_{i+1}]$.

Theorem 4.4. *Let $u(x)$ be the solution of (1-2) and $\overline{U}^N(x)$ be the linear piecewise Lagrange interpolated solution defined in (43). Then, the global error satisfies*

$$|u(x) - \overline{U}^N(x)| \leq C(N^{-2}\ln^2 N + N^{-\sqrt{\beta}\sigma_0}), \quad x \in [0,1]. \tag{44}$$

Proof. Note that from Theorem 3.1, using that functions $\phi_i(x) \geq 0$ and also that it holds $\|\sum_{i=0}^{N}\phi_i(x)\| \leq 1$, it immediately follows that

$$|u(x) - \overline{U}^N(x)| \leq |u(x) - u_I(x)| + CN^{-2}\ln^2 N. \tag{45}$$

Using the decomposition of the exact solution into its regular and singular component, the bounds (6) for the derivatives of the regular and the singular components and the same argument to this one used in [2] for the upwind scheme, it is straightforward to prove that

$$|u(x) - u_I(x)| \leq C(N^{-2}\ln^2 N + N^{-\sqrt{\beta}\sigma_0}). \tag{46}$$

From (45) and (46) the result follows. ∎

TABLE 1. *Maximum point-wise errors and rates of convergence for Example 5.1*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	1.4341E-2	3.6326E-3	9.1085E-4	2.2784E-4	5.6964E-5	1.4241E-5	3.5600E-6
	1.9810	1.9957	1.9992	1.9999	2.0000	2.0000	
2^{-8}	1.6316E-2	3.7861E-3	9.3019E-4	2.3158E-4	5.7840E-5	1.4457E-5	3.6146E-6
	2.1075	2.0251	2.0060	2.0014	2.0003	1.9999	
2^{-16}	3.3124E-2	1.2267E-2	4.2436E-3	1.4157E-3	4.6119E-4	1.4572E-4	4.4947E-5
	1.4331	1.5314	1.5838	1.6181	1.6622	1.6969	
2^{-24}	3.3127E-2	1.2267E-2	4.2437E-3	1.4157E-3	4.6119E-4	1.4572E-4	4.4947E-5
	1.4332	1.5314	1.5838	1.6181	1.6622	1.6969	
2^{-32}	3.3128E-2	1.2267E-2	4.2437E-3	1.4157E-3	4.6119E-4	1.4572E-4	4.4947E-5
	1.4333	1.5314	1.5838	1.6181	1.6622	1.6969	
E^N	3.3128E-2	1.2267E-2	4.2437E-3	1.4157E-3	4.6119E-4	1.4572E-4	4.4947E-5
p^N	1.4333	1.5314	1.5838	1.6181	1.6622	1.6969	

5. Numerical Experiments

In this section, we show the results obtained via our hybrid scheme for two test problems. In both cases we present the results in the form of tables containing, for the numerical solution or for the numerical normalized flux, the maximum errors and the rates of convergence for some values of ε and N, or the maximum uniform errors and the numerical order of uniform convergence for the range of values $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}, \cdots, 2^{-32}$. In all cases we take the constant $\sigma_0 = 2$.

Example 5.1. *The first test example is the self-adjoint problem*

$$\left\{ \begin{array}{l} -\varepsilon u''(x) + u(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x), \quad x \in (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{array} \right. \tag{47}$$

for which the exact solution is given by

$$u(x) = \frac{\exp(-x/\sqrt{\varepsilon}) + \exp(-(1-x)/\sqrt{\varepsilon})}{1 + \exp(-1/\sqrt{\varepsilon})} - \cos^2(\pi x).$$

Then, we can calculate exactly the maximum errors and the ε-uniform errors by

$$E_\varepsilon^N = \max_{x_i \in \overline{D}^N} |u(x_i) - U^N(x_i)|, \quad E^N = \max_\varepsilon E_\varepsilon^N,$$

respectively, where u denotes the exact solution and U^N is the numerical solution obtained by using N mesh intervals in the domain $\overline{D}^N$. In addition, in a standard way, the rates of convergence and the ε-uniform order of convergence are calculated by

$$p_\varepsilon^N \log_2\left(E_\varepsilon^N / E_\varepsilon^{2N}\right), \quad p^N = \log_2\left(E^N / E^{2N}\right).$$

The errors associated to the normalized flux are obtained by

$$F_\varepsilon^N = \max_{x_i \in \overline{D}^N} \sqrt{\varepsilon}|u'(x_i) - S'(x_i)|, \quad F^N = \max_\varepsilon F_\varepsilon^N,$$

where u is the exact solution and S is the cubic spline given by (13). From these values we obtain the rates of convergence and the ε-uniform order of convergence for the flux, by using

$$q_\varepsilon^N \log_2\left(F_\varepsilon^N / F_\varepsilon^{2N}\right), \quad q^N = \log_2\left(F^N / F^{2N}\right).$$

From Tables 1 and 2 we clearly deduce the second order of convergence (except by a logarithmic factor) of both the numerical solution and the approximation to the

TABLE 2. *Maximum point-wise errors for the normalized flux and rates of convergence for Example 5.1*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	3.0369E-2	7.7265E-3	1.9394E-3	4.8525E-4	1.2133E-4	3.0332E-5	7.5827E-6
	1.9747	1.9942	1.9988	1.9998	2.0000	2.0000	
2^{-8}	3.9878E-2	1.0133E-2	2.5440E-3	6.3672E-4	1.5923E-4	3.9811E-5	9.9529E-6
	1.9766	1.9938	1.9984	1.9996	1.9999	2.0000	
2^{-16}	7.7112E-2	3.0806E-2	1.1198E-2	3.8244E-3	1.2504E-3	3.9580E-4	1.2218E-4
	1.3237	1.4600	1.5499	1.6129	1.6595	1.6958	
2^{-24}	7.7113E-2	3.0806E-2	1.1198E-2	3.8244E-3	1.2504E-3	3.9580E-4	1.2218E-4
	1.3238	1.4600	1.5499	1.6129	1.6595	1.6958	
2^{-32}	7.7114E-2	3.0806E-2	1.1198E-2	3.8244E-3	1.2504E-3	3.9580E-4	1.2218E-4
	1.3238	1.4600	1.5499	1.6129	1.6595	1.6958	
F^N	7.7114E-2	3.0806E-2	1.1198E-2	3.8244E-3	1.2504E-3	3.9580E-4	1.2218E-4
q^N	1.3238	1.4600	1.5499	1.6129	1.6595	1.6958	

normalized flux; these numerical errors coincide clearly with the theoretical results given in Theorems 3.1 and 3.3.

Now we would like to see which are the errors associated to the global solution. To do that we calculate the maximum errors at the midpoints $x = (x_i + x_{i+1})/2$, of the corresponding Shishkin mesh, by $E_\varepsilon^N(x) = \max_x |u(x) - S(x)|$, where u is the exact solution and S is the cubic spline given by (13). Tables 3 and 4 display the errors at different regions of the domain. From these tables we see that in the boundary layer regions the global solution gives uniformly convergent results; nevertheless, when $x \in (\sigma, \sigma + H) \bigcup (1 - \sigma - H, 1 - \sigma)$, then the global solution is a bad approximation, because the maximum errors are increasing for ε sufficiently small and therefore we do not have the necessary stabilization of the errors to deduce the uniform convergence of the numerical scheme. This behavior agrees with Theorems 4.1 and 4.2 and Remark 4.3.

TABLE 3. *Maximum errors at the midpoints excluding $(\sigma, \sigma + H) \bigcup (1 - \sigma - H, 1 - \sigma)$ and rates of convergence for Example 5.1*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	1.3966E-2	3.6090E-3	9.0937E-4	2.2775E-4	5.6958E-5	1.4240E-5	3.5600E-6
	1.9522	1.9887	1.9974	1.9995	1.9999	2.0000	
2^{-8}	1.5070E-2	3.7389E-3	9.2848E-4	2.3160E-4	5.7856E-5	1.4460E-5	3.6146E-6
	2.0110	2.0097	2.0032	2.0011	2.0004	2.0002	
2^{-16}	3.7725E-2	1.2200E-2	4.2181E-3	1.4264E-3	4.6194E-4	1.4581E-4	4.4970E-5
	1.6287	1.5322	1.5642	1.6266	1.6637	1.6970	
2^{-24}	3.7726E-2	1.2200E-2	4.2181E-3	1.4264E-3	4.6195E-4	1.4581E-4	4.4970E-5
	1.6287	1.5323	1.5642	1.6266	1.6637	1.6970	
2^{-32}	3.7726E-2	1.2200E-2	4.2181E-3	1.4264E-3	4.6195E-4	1.4581E-4	4.4970E-5
	1.6287	1.5323	1.5642	1.6266	1.6637	1.6970	
E^N	3.7726E-2	1.2200E-2	4.2181E-3	1.4264E-3	4.6195E-4	1.4581E-4	4.4970E-5
p^N	1.6287	1.5323	1.5642	1.6266	1.6637	1.6970	

The alternate way to obtain a global solution uses the piecewise linear Lagrange interpolation. Again we calculate the maximum errors at the midpoints $(x_i + x_{i+1})/2$ of the corresponding Shishkin mesh. The results obtained are displayed in Table 5; from it we see that the errors are uniformly convergent in all domain $[0, 1]$ in agreement with Theorem 4.4. Before we conclude this example, we wish to analyze numerically, whether the global normalized flux based on the cubic spline $S(x)$ is uniformly convergent. For that, we approximate the errors of the normalized flux at the midpoints $x = (x_i + x_{i+1})/2$. These errors are calculated by

Table 4. *Maximum errors at the midpoints only in $(\sigma, \sigma + H) \bigcup (1 - \sigma - H, 1 - \sigma)$ and rates of convergence for Example 5.1*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	1.0294E-2 2.0236	2.5318E-3 2.0234	6.2277E-4 2.0145	1.5413E-4 2.0079	3.8322E-5 2.0041	9.5530E-6 2.0021	2.3847E-6
2^{-8}	7.1554E-4 1.3680	2.7723E-4 1.5943	9.1811E-5 1.7897	2.6554E-5 1.8944	7.1428E-6 1.9472	1.8522E-6 1.9736	4.7160E-7
2^{-16}	2.1002E-2 3.0096	2.6077E-3 3.2357	2.7684E-4 3.5844	2.3079E-5 3.6712	1.8117E-6 3.0485	2.1897E-7 2.3489	4.2985E-8
2^{-20}	1.0494E-1 2.9549	1.3534E-2 2.9954	1.6972E-3 3.0895	1.9939E-4 3.2824	2.0492E-5 3.6133	1.6745E-6 3.8710	1.1444E-7
2^{-24}	4.4157E-1 2.9503	5.7130E-2 2.9609	7.3375E-3 2.9932	9.2154E-4 3.0374	1.1224E-4 3.1146	1.2959E-5 3.2896	1.3252E-6
2^{-28}	1.7882E+0 2.9497	2.3146E-1 2.9539	2.9872E-2 2.9751	3.7990E-3 2.9964	4.7607E-4 3.0178	5.8781E-5 3.0500	7.0971E-6
2^{-32}	7.1749E+0 2.9496	9.2877E-1 2.9522	1.2000E-1 2.9710	1.5305E-2 2.9875	1.9299E-3 2.9990	2.4140E-4 3.0091	2.9986E-5
2^{-34}	1.4357E+1 2.9496	1.8585E+0 2.9520	2.4018E-1 2.9703	3.0647E-2 2.9860	3.8681E-3 2.9961	4.8482E-4 3.0031	6.0475E-5

Table 5. *Maximum errors at the midpoints and rates of convergence for Example 5.1 by using Lagrange interpolation*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	1.4275E-2 1.9797	3.6195E-3 1.9951	9.0798E-4 1.9988	2.2718E-4 1.9998	5.6803E-5 2.0000	1.4201E-5 2.0000	3.5502E-6
2^{-8}	7.8674E-2 1.6377	2.5284E-2 1.8010	7.2560E-3 1.8951	1.9508E-3 1.9461	5.0627E-4 1.9726	1.2899E-4 1.9862	3.2557E-5
2^{-16}	1.0851E-1 0.9621	5.5699E-2 1.1898	2.4417E-2 1.3669	9.4671E-3 1.4971	3.3538E-3 1.5898	1.1142E-3 1.6552	3.5376E-4
2^{-24}	1.0844E-1 0.9619	5.5671E-2 1.1896	2.4407E-2 1.3668	9.4637E-3 1.4971	3.3527E-3 1.5898	1.1139E-3 1.6552	3.5365E-4
2^{-32}	1.0844E-1 0.9619	5.5671E-2 1.1896	2.4407E-2 1.3668	9.4636E-3 1.4971	3.3527E-3 1.5898	1.1139E-3 1.6552	3.5365E-4
E^N p^N	1.0960E-1 0.9655	5.6124E-2 1.1917	2.4570E-2 1.3680	9.5191E-3 1.4977	3.3708E-3 1.5901	1.1196E-3 1.6554	3.5542E-4

$F_\varepsilon^N(x) = \max_x \sqrt{\varepsilon}|u'(x) - S'(x)|$, where u is the exact solution and S is the cubic spline given by (13). The results are displayed in Table 6; from it we clearly deduce the uniform convergence of the global normalized flux, but at the moment we do not dispose the theoretical proof of this subject.

Table 6. *Maximum errors at the midpoints for the normalized flux and rates of convergence for Example 5.1*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	2.6608E-2 1.8764	7.2468E-3 1.9474	1.8790E-3 1.9758	4.7769E-4 1.9885	1.2038E-4 1.9944	3.0213E-5 1.9972	7.5679E-6
2^{-8}	1.4889E-02 1.2679	6.1826E-03 1.6406	1.9829E-03 1.8197	5.6171E-04 1.9095	1.4952E-04 1.9546	3.8576E-05 1.9773	9.7973E-06
2^{-16}	1.9670E-02 0.5683	1.3266E-02 0.9772	6.7386E-03 1.2478	2.8375E-03 1.4298	1.0532E-03 1.5518	3.5924E-04 1.6338	1.1576E-04
2^{-24}	1.9672E-2 0.5684	1.3266E-2 0.9772	6.7387E-3 1.2478	2.8375E-3 1.4298	1.0532E-3 1.5518	3.5924E-4 1.6338	1.1576E-4
2^{-32}	1.9672E-2 0.5684	1.3266E-2 0.9772	6.7387E-3 1.2478	2.8375E-3 1.4298	1.0532E-3 1.5518	3.5924E-4 1.6338	1.1576E-4
F^N q^N	1.9672E-2 0.5684	1.3266E-2 0.9772	6.7387E-3 1.2478	2.8375E-3 1.4298	1.0532E-3 1.5518	3.5924E-4 1.6338	1.1576E-4

Example 5.2. *The second test problem that we consider is given by*

$$\begin{cases} -\varepsilon u''(x) + (1 + x^2 + \cos x)u(x) = x^{4.5} + \sin x,\ x \in (0,1), \\ u(0) = 1, \quad u(1) = 1. \end{cases} \tag{48}$$

The exact solution of this example is not available. Therefore, to obtain the maximum point-wise errors and the rates of convergence, we use a variant of the double mesh principle (see [7, 8]). We calculate the numerical solution U^N on D^N and the numerical solution $\widetilde{U}^N$ on the mesh $\widetilde{D}^N$ where the transition parameter is now given by

$$\widetilde{\sigma} = \min\left\{\frac{1}{4}, \sigma_0\sqrt{\varepsilon}\ln(N/2)\right\}.$$

Then the approximated errors and the ε-uniform errors are calculated respectively by

$$G_\varepsilon^N = \max_{x_j \in \overline{D}^N} |U^N(x_j) - \widetilde{U}^{2N}(x_j)|, \quad \text{and} \quad G^N \max_\varepsilon G_\varepsilon^N.$$

From these values the parameter-robust orders of convergence are given by

$$p_\varepsilon^N \log_2\left(G_\varepsilon^N / G_\varepsilon^{2N}\right), \quad p^N = \log_2\left(G^N / G^{2N}\right).$$

In this case the errors associated to the normalized flux are obtained by

$$H_\varepsilon^N = \max_{x_i \in \overline{D}^N} \sqrt{\varepsilon}|S_N'(x_i) - \widetilde{S}_{2N}'(x_i)|, \quad H^N = \max_\varepsilon H_\varepsilon^N,$$

where S_N and $\widetilde{S}_{2N}$ are the splines defined by (13) on the Shishkin meshes D^N and $\widetilde{D}^{2N}$ respectively. From these values we obtain the rates of convergence and the ε-uniform order of convergence for the flux, by using

$$q_\varepsilon^N \log_2\left(H_\varepsilon^N / H_\varepsilon^{2N}\right), \quad q^N = \log_2\left(H^N / H^{2N}\right).$$

TABLE 7. *Maximum point-wise errors and rates of convergence for Example 5.2*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	2.0489E-04	4.5562E-05	1.0698E-05	2.5885E-06	6.3637E-07	1.5775E-07	3.9269E-08
	2.1689	2.0905	2.0472	2.0242	2.0122	2.0062	
2^{-8}	2.6907E-02	5.7967E-03	1.4539E-03	3.5962E-04	8.9800E-05	2.2440E-05	5.6098E-06
	2.2147	1.9953	2.0154	2.0017	2.0007	2.0000	
2^{-16}	4.9652E-02	2.0091E-02	6.3675E-03	2.1372E-03	6.9509E-04	2.1880E-04	6.7469E-05
	1.3053	1.6577	1.5750	1.6205	1.6676	1.6973	
2^{-24}	4.9652E-02	2.0091E-02	6.3675E-03	2.1372E-03	6.9509E-04	2.1880E-04	6.7469E-05
	1.3053	1.6577	1.5750	1.6205	1.6676	1.6973	
2^{-32}	4.9652E-02	2.0091E-02	6.3675E-03	2.1372E-03	6.9509E-04	2.1880E-04	6.7469E-05
	1.3053	1.6577	1.5750	1.6205	1.6676	1.6973	
G^N	4.9652E-02	2.0091E-02	6.3675E-03	2.1372E-03	6.9509E-04	2.1880E-04	6.7469E-05
p^N	1.3053	1.6577	1.5750	1.6205	1.6676	1.6973	

From Tables 7 and 8 again we deduce the second order of convergence (except by the logarithmic factor) in agreement with Theorems 3.1 and 3.3.

Further, we have carried out the numerical experiments for the global solution, and the global normalized flux for Example 5.2. And found exactly a similar behavior as like in the previous example.

TABLE 8. *Maximum point-wise errors for the normalized flux and rates of convergence for Example 5.2*

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
2^{-2}	6.1071E-04	1.7365E-04	4.5943E-05	1.1796E-05	2.9876E-06	7.5168E-07	1.8852E-07
	1.8143	1.9183	1.9615	1.9813	1.9908	1.9954	
2^{-8}	8.4129E-02	2.1816E-02	5.5069E-03	1.3801E-03	3.4524E-04	8.6323E-05	2.1582E-05
	1.9472	1.9861	1.9965	1.9991	1.9998	1.9999	
2^{-16}	1.5525E-01	6.3917E-02	2.3559E-02	8.0897E-03	2.6500E-03	8.3938E-04	2.5916E-04
	1.2803	1.4399	1.5421	1.6101	1.6586	1.6955	
2^{-24}	1.5525E-01	6.3917E-02	2.3559E-02	8.0897E-03	2.6500E-03	8.3938E-04	2.5916E-04
	1.2803	1.4399	1.5421	1.6101	1.6586	1.6955	
2^{-32}	1.5525E-01	6.3917E-02	2.3559E-02	8.0897E-03	2.6500E-03	8.3938E-04	2.5916E-04
	1.2803	1.4399	1.5421	1.6101	1.6586	1.6955	
H^N	1.5525E-01	6.3917E-02	2.3559E-02	8.0897E-03	2.6500E-03	8.3938E-04	2.5916E-04
q^N	1.2803	1.4399	1.5421	1.6101	1.6586	1.6955	

Remark 5.3. *We would like to remark that in our numerical experiments, using the finite difference scheme (7) based only on cubic spline, which does not satisfy the discrete maximum principle, we have obtained very similar results to these ones given by the hybrid scheme; therefore, we can conclude that the maximum errors occur in the boundary layer regions, that the numerical solution at the mesh points located out of the boundary layers have not influence in the maximum errors and also that the solutions of linear systems associated to the cubic spline finite difference scheme are good, even in this case when the associated matrix is not an M-matrix.*

6. Conclusions

An ε-uniform compact numerical scheme is derived in this paper for the solution of the reaction-diffusion BVP (1-2). Further, an efficient method is proposed for the normalized flux and global solution of this BVP. All these methods converge independently of the singular perturbation parameter ε. As mentioned in the introduction section, it is too difficult to determine the normalized fluxes for SPPs, and this paper provides an easy way to determine it. Also we have constructed the global solution using cubic spline. One of the main merits of the paper is that the proposed scheme is of compact type. The present method can be extended to nonlinear reaction-diffusion problems after linearizing the problems.

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