NUMERICAL ALGORITHM FOR A TRANSPORT EQUATION WITH PERIODIC SOURCE FUNCTION

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Abstract. An algorithm for determining the solution of a boundary value problem for an integral-differential equation is presented. Using a splitting method for a stationary transport equation with periodic source function and a plan-parallel geometry, we obtain the solution with an algorithm based on the variational form of the integral identity method. An error estimate for the approximate solution of our problem is obtained. Numerical results are presented to prove the accuracy and computational efficiency of the proposed algorithm.

Key Words. integral-differential equation, variational methods, integral identity method.

1. Introduction

In the neutrons transport literature many authors paid attention to the numerical solutions obtained by the methods of Ritz and Galerkin, the method of least squares, [1], [5], [6], [8], [11], the method of finite elements and Nyström method [12].

In this paper we present an algorithm inspired by the variational form of the integral identity method, [5], applied to a diffusion equation. Unlike the Galerkin method, this algorithm allows the using of the discontinuous basis functions for the constructing of the approximate solution. The problem of convergence of these to the exact solution in the case of a positive definite operator can be proved for the more general assumptions regarding to smoothness of the boundary conditions.

In the general case, this method is hard to use, but for any symmetry of the source function, it leads to an algorithm more flexible and computationally more efficient, than the methods mentioned before. The numerical examples prove that the errors, which correspond of the approximate solutions, are minimum.

2. Problem formulation

In the stationary case, we consider a transport equation of the form

\[ \frac{\partial n}{\partial t} + \nabla \cdot (\mathbf{v} n) + q n = f, \]

where \( n \) is the neutron density, \( \mathbf{v} \) is the neutron velocity, \( q \) is the source term, and \( f \) is the right-hand side of the equation.
\[
\mu \frac{\partial \phi(x, \mu)}{\partial x} + \phi(x, \mu) = \frac{1}{4} \int_{-1}^{1} \phi(x, \mu') d\mu' + f(x, \mu)
\]
\[\forall (x, \mu) \in D_1 \times D_2 = [0, H] \times [-1, 1], \quad D_2 = D'_2 \cup D''_2 = [-1, 0] \cup [0, 1].\]

with the boundary conditions are
\[\phi(0, \mu) = 0 \quad \text{if} \quad \mu > 0\]
\[\phi(H, \mu) = 0 \quad \text{if} \quad \mu < 0\]

Here \(\phi\) is the density of neutrons, which migrate in a direction defined by the angle \(\alpha\) against \(Ox\) axis and we denote \(\mu = \cos \alpha\). Let us consider the radioactive source \(f\) as an even function with respect \(\mu\). Using the notations:
\[
\phi^+ = \phi(x, \mu) \quad \text{if} \quad \mu > 0; \quad \phi^- = \phi(x, \mu) \quad \text{if} \quad \mu < 0
\]
and substituting \(\mu'' = -\mu'\), we get
\[
\int_{-1}^{0} \phi(x, \mu') d\mu' = \int_{0}^{1} \phi(x, -\mu') d\mu'' = \int_{0}^{1} \phi^- d\mu''.
\]

Then the conditions (2) become
\[
\phi^+(0, \mu) = 0, \quad \phi^-(H, \mu) = 0
\]
and the equation (1) can be written in the form
\[
\mu \frac{\partial \phi^+}{\partial x} + \phi^+ = \frac{1}{4} \int_{0}^{1} (\phi^+ + \phi^-) d\mu' + f^+
\]
\[-\mu \frac{\partial \phi^-}{\partial x} + \phi^- = \frac{1}{4} \int_{0}^{1} (\phi^+ + \phi^-) d\mu' + f^-
\]

Adding and subtracting the equations (5) and introducing the notations:
\[
u = \frac{1}{2} (\phi^+ + \phi^-), \quad v = \frac{1}{2} (\phi^+ - \phi^-)
\]
\[g = \frac{1}{2} (f^+ + f^-), \quad r = \frac{1}{2} (f^+ - f^-) = 0
\]
we obtain the following system
\[
\mu \frac{\partial u}{\partial x} + u = \frac{1}{2} \int_{0}^{1} u d\mu + g \quad (a)
\]
\[\mu \frac{\partial v}{\partial x} + v = 0 \quad (b)
\]

The boundary conditions become
\[u + v = 0 \quad \text{for} \quad x = 0\]
\[u - v = 0 \quad \text{for} \quad x = H.
\]

Now, we find \(v\) from the second equation (7) and using the first equation, we rewrite the problem (7)-(8) in the following form
\[
-\mu \frac{\partial^2 u}{\partial x^2} + u = \frac{1}{2} \int_{0}^{1} u d\mu + g
\]
Let us now suppose that the functions $u$ and $g$ belong to the Hilbert space $L_2(D)$, where $D = D_1 \times D''_2$ and the norm is defined by the formula

$$
||u|| = \left( \int_0^H \int_0^1 u^2(x, \mu)dxd\mu \right)^{1/2}
$$

The following theorem estimates a bound for the solution $u$ and this inequality is helpful to prove the stability of the numerical algorithm.

**Theorem.** If the solution of the problem (9)-(10), $u \in L_2(D)$ and $g \in L_2(D)$, then

$$
||u|| \leq 2||g||
$$

**Proof.** Multiplying (9) by $u$ and integrating with respect to $x$ and $\mu$, we obtain

$$
- \int_D \mu^2 \frac{\partial^2 u}{\partial x^2} u dxd\mu + \int_D u^2 dxd\mu =
$$

$$
= \frac{1}{2} \int_0^H dx \left( \int_0^1 u(x, \mu)d\mu \right) \left( \int_0^1 u(x, \mu')d\mu' \right) + \int_D gudxd\mu
$$

We now get two inequalities for the boundary conditions (10). Using integration by parts we have

$$
- \int_D \mu^2 \frac{\partial^2 u}{\partial x^2} u dxd\mu = - \int_0^1 \mu(-u^2(H, \mu) - u^2(0, \mu))d\mu
$$

$$
+ \int_D \mu^2 \left( \frac{\partial u}{\partial x} \right)^2 dxd\mu \geq 0
$$

On the other hand, we get from the Cauchy-Schwarz inequality

$$
\left( \int_0^1 u(x, \mu)d\mu \right)^2 \leq \int_0^1 u^2(x, \mu)d\mu
$$

With the help of these inequalities, (13) can be written in the form

$$
\int_0^H \int_0^1 u^2 dxd\mu \leq \frac{1}{2} \int_0^H \int_0^1 u^2 dxd\mu + \int_0^H \int_0^1 gudxd\mu.
$$

Finally, we arrive at inequality (12).

In order to get a solution of the problem (9)-(10), we consider on $x$ axis two points systems:

- a principal system: $\{x_k\} = \Delta'_1$, $k \in \{0, 1, \ldots, N\}$, with $x_0 = 0$, $x_N = H$ and $h = x_{k+1} - x_k$;

- a secondary system, $\{x_{k+1/2}\} = \Delta''_1$, $k \in \{0, 1, 2, \ldots, N - 1\}$, which verifies the inequalities: $x_{k-1/2} < x_k < x_{k+1/2}$, where $x_{k+1/2} = (x_k + x_{k+1})/2$ and $0 = x_0 < x_{1/2} < \ldots < x_{N-1/2} < x_N = H$.

Besides, let $\Delta_2 = \{\mu_l\}$, $l = \{0, 1, \ldots, L\}$ be a partition of the interval $D''_2 = [0, 1]$ and $\tau = \mu_{l+1} - \mu_l$, $l \in \{0, 1, \ldots, L - 1\}$. 

Further on, we consider $H = 1$. For every value $\mu_1 \in \Delta_2$, the problem (9)-(10) becomes:

$$-\mu_1^2 \frac{d^2 u(x, \mu_1)}{dx^2} + u(x, \mu_1) = f_1(x, \mu_1)$$

where

$$f_1(x, \mu_1) = S(x) + g(x, \mu_1), S(x) = \frac{1}{2} \int_0^1 u(x, \mu) d\mu$$

and

$$\left. \left( u(x, \mu_1) - \mu_1 \frac{du(x, \mu_1)}{dx} \right) \right|_{x=0} = 0$$

$$\left. \left( u(x, \mu_1) + \mu_1 \frac{du(x, \mu_1)}{dx} \right) \right|_{x=1} = 0$$

Now (14)-(15) are a boundary problem for a one-dimensional diffusion equation (14). Integrating (14) with respect to $x$ on the intervals: $(x_{k-1/2}, x_{k+1/2})$, we obtain

$$-J_{k+1/2} + J_{k-1/2} + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) dx = 0$$

where

$$J_{k\pm 1/2} = J(x_{k \pm 1/2}, \mu_1), \quad J(x, \mu_1) = \mu_1^2 \frac{du(x, \mu_1)}{dx}.$$ We find $J_{k-1/2}$ integrating (14) on the interval $(x_{k-1/2}, x)$, where $x \in (x_{k-1}, x_k)$. We get

$$\mu_1^2 \frac{du(x, \mu_1)}{dx} = J_{k-1/2} + \int_{x_{k-1/2}}^{x_k} (u(\xi, \mu_1) - f_1(\xi, \mu_1)) d\xi$$

Then, dividing (17) by $\mu_1^2$ and integrating on $(x_{k-1}, x_k)$ we have

$$u_k - u_{k-1} = J_{k-1/2} \int_{x_{k-1}}^{x_k} \frac{dx}{\mu_1^2} + \int_{x_{k-1}}^{x} \frac{dx}{\mu_1^2} \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) d\xi$$

Finally, we get

$$J_{k-1/2} = \frac{\mu_1^2}{h} \left[ u_k - u_{k-1} - \int_{x_{k-1}}^{x_k} \frac{dx}{\mu_1^2} \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) d\xi \right]$$

In a similar manner, we obtain $J_{k+1/2}$ replacing $k$ by $k + 1$. Consequently, the equality (16) becomes:

$$\mu_1^2 \left( \frac{u_k - u_{k+1}}{h} + \frac{u_{k-1} - u_k}{h} \right) + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) d\xi =$$

$$= -\frac{1}{h} \int_{x_k}^{x_{k+1}} dx \int_{x_{k+1/2}}^{x} (u - f_1) d\xi + \frac{1}{h} \int_{x_{k-1}}^{x_k} dx \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1) d\xi$$

where $k \in \{1, 2, \ldots, N - 1\}$.

Now we shall denote:

$$\psi(x) = u(x, \mu_1) - f_1(x, \mu_1)$$

$$\rho_k(x) = (x - x_k)/h.$$
Applying the method of integration by parts we obtain

$$-\frac{1}{h} \int_{x_k}^{x_{k+1}} d(\int_{x_k}^{x} \psi(x) d\xi) =$$

$$=-\frac{1}{h} \left[ h \int_{x_{k+1/2}}^{x_{k+1}} \psi(x) d\xi - \int_{x_k}^{x_{k+1}} (x-x_k) \psi(x) dx \right] =$$

$$= - \int_{x_{k+1/2}}^{x_{k+1}} \psi(x) d\xi + \int_{x_k}^{x_{k+1}} \rho_k(x) \psi(x) dx$$

(22)

Analogously, if we denote

$$\tilde{\rho}_k(x) = \frac{x_k - x}{h}$$

the equation (20) is now of the form

$$\mu_l u(x_0, \mu_l) + \mu_l^2 \left( \frac{u_k - u_{k+1}}{h} + \frac{u_k - u_{k-1}}{h} \right) + \int_{x_{k-1}}^{x_k} (1 - \tilde{\rho}_k(x)) \psi(x) dx +$$

$$+ \int_{x_k}^{x_{k+1}} (1 - \rho_k(x)) \psi(x) dx = 0, \quad k \in \{1, 2, \ldots, N-1\}.$$  

(24)

It should be observed that the integral from the left-hand side of (20) was decomposed in the intervals: $$(x_{k-1/2}, x_k) \cup (x_k, x_{k+1/2}).$$ Let us introduce the functions:

$$Q_k(x) = \begin{cases} 
1 - \frac{\tilde{\rho}_k}{\sqrt{h}} = \frac{x - x_{k-1}}{h \sqrt{h}}, & x \in [x_{k-1}, x_k] \\
1 - \frac{\rho_k}{\sqrt{h}} = \frac{x_{k+1} - x}{h \sqrt{h}}, & x \in [x_k, x_{k+1}] \\
0, & x \notin [x_{k-1}, x_{k+1}] 
\end{cases}$$

(25)

where

$$Q_k(x_k) = \frac{1}{\sqrt{h}}.$$  

Now we suppose that the function $u(x, \mu_l) \in L_2([0, 1])$, the Hilbert space with the scalar product defined by formula

$$\langle w, v \rangle = \int_0^1 w(x) v(x) dx$$

(26)

Then, using the scalar product and (15), the system for our boundary problem (14)-(15) becomes

$$\mu_l u(x_0, \mu_l) + \mu_l^2 \frac{u(x_0, \mu_l) - u(x_1, \mu_l)}{h} + \langle u, \sqrt{h} Q_0 \rangle = (f_1, \sqrt{h} Q_0),$$

$$\mu_l^2 \left( \frac{u(x_k, \mu_l) - u(x_{k+1}, \mu_l)}{h} + \frac{u(x_k, \mu_l) - u(x_{k-1}, \mu_l)}{h} \right) +$$

$$+ \langle u, \sqrt{h} Q_k \rangle = (f_1, \sqrt{h} Q_k) \quad k \in \{1, \ldots, N-1\}$$

$$\mu_l u(z_N, \mu_l) + \mu_l^2 \frac{u(z_N, \mu_l) - u(z_{N-1}, \mu_l)}{h} + \langle u, \sqrt{h} Q_N \rangle = (f_1, \sqrt{h} Q_N).$$

(27)
On the other hand, we observe that

\[
\left( \mu_l^2 \frac{du(x, \mu_l)}{dx}, \frac{dQ_k}{dx} \right) = \\
\mu_l^2 \int_{x_{k-1}}^{x_k} \frac{du(x, \mu_l)}{dx} \cdot \frac{1}{h \sqrt{h}} dx - \mu_l^2 \int_{x_k}^{x_{k+1}} \frac{du(x, \mu_l)}{dx} \cdot \frac{1}{h \sqrt{h}} dx = \\
= \mu_l^2 \left( \frac{u_k - u_{k-1}}{h \sqrt{h}} - \frac{u_{k+1} - u_k}{h \sqrt{h}} \right), \quad k \in \{1, 2, \ldots, N - 1\}.
\]

and (27) can be written in the following form:

\[
(29) \quad \left( \mu_l^2 \frac{du}{dx}, \frac{dQ_k}{dx} \right) + \mu_l (uQ_k |_{z=0} + uQ_k |_{z=H}) + (u, Q_k) = (f_1, Q_k), \quad k \in \{0, 1, \ldots, N\}.
\]

This system allows us to consider the integral method as a variational method and the equations (29) may be used for determining the approximate solutions using a sequence of coordinate functions. It should be noted that the equations (29) coincide with the relations obtained by Galerkin method, where \(Q_k(x)\) are the coordinate functions. Then, the solution of the system (27) can be defined in the following way

\[
(30) \quad \tilde{u}(x, \mu_l) = \sum_{k=0}^{N} a_k(\mu_l)Q_k(x)
\]

where

\[
a_k(\mu_l) = 2\beta_k \mu_l^2.
\]

We shall now determine the coefficients \(\beta_k\) form the condition that (30) is a solution of the system (27). Also the boundary conditions must be satisfied. Since the functions \(\tilde{u}\) are linear with respect to \(x\), we get from (27) for \(k\) segment \((x_{k-1}, x_{k+1})\), \(k \in \{1, 2, \ldots, N_1\}\) and \(x_k = kh\):

\[
(31) \quad \mu_l^2 \left( \frac{\tilde{u}_k - \tilde{u}_{k+1}}{h \sqrt{h}} + \frac{\tilde{u}_k - \tilde{u}_{k-1}}{h \sqrt{h}} \right) = \frac{\mu_l^2}{h^2} (2\tilde{u}_k - \tilde{u}_{k-1} - \tilde{u}_{k+1}) = \\
\quad = -\frac{2 \mu_l^4}{h^2} (\beta_{k-1} - 2\beta_k + \beta_{k+1})
\]
\[(\bar{u}, Q_k) = \int_{x_{k-1}}^{x_k} \left( \sum_{j=0}^{N} 2\beta_j \mu^2 Q_j(x) \right)Q_k(x)dx + \int_{x_k}^{x_{k+1}} \left( \sum_{j=0}^{N} 2\beta_j \mu^2 Q_j(x) \right)Q_k(x)dx =
\]

\[= \frac{2\mu^2}{3} \int_{x_{k-1}}^{x_k} \left[ \beta_{k-1} \frac{x_k - x}{h\sqrt{h}} - \frac{x - x_{k-1}}{h\sqrt{h}} + \beta_k \left( \frac{x - x_{k-1}}{h\sqrt{h}} \right)^2 \right]dx + \frac{2\mu^2}{3} \int_{x_k}^{x_{k+1}} \left[ \beta_{k+1} \frac{x - x_k}{h\sqrt{h}} - \frac{x_{k+1} - x}{h\sqrt{h}} + \beta_k \left( \frac{x_{k+1} - x}{h\sqrt{h}} \right)^2 \right]dx = \]

\[= \frac{\mu^2}{3}(\beta_{k-1} + 4\beta_k + \beta_{k+1})
\]

\[S(x) = \frac{1}{2} \int_0^1 \bar{u}(x, \mu)d\mu = \frac{1}{2} \sum_{j=0}^{N} \int_0^1 2\beta_j \mu^2 Q_j(x)d\mu = \frac{1}{3} \sum_{j=0}^{N} \beta_j Q_j(x)
\]

and

\[(S(x), Q_k(x)) = \frac{1}{3} \int_{x_{k-1}}^{x_k} \left( \sum_{j=0}^{N} \beta_j Q_j \right)Q_k(x)dx + \frac{1}{3} \int_{x_k}^{x_{k+1}} \left( \sum_{j=0}^{N} \beta_j Q_j \right)Q_k(x)dx = \frac{1}{18}(\beta_{k-1} + 4\beta_k + \beta_{k+1}).
\]

Thus, we arrive at the following system:

\[
\frac{2\mu^3 \beta_0}{h} + 2\mu^4 \frac{\beta_0 - \beta_1}{h^2} + \frac{6\mu^2 - 1}{18}(2\beta_0 + \beta_1) = \gamma_0,
\]

\[\frac{2\mu^4}{h^2}(\beta_{k-1} - 2\beta_k + \beta_{k+1}) + \frac{3\mu^2 - 2}{9}(\beta_{k-1} + 4\beta_k + \beta_{k+1}) = \gamma_k
\]

\[k \in \{1, \ldots, N - 1\},
\]

\[
\frac{2\mu^3 \beta_N}{h} + 2\mu^4 \frac{\beta_N - \beta_{N-1}}{h^2} + \frac{6\mu^2 - 1}{18}(\beta_{N-1} + 2\beta_N) = \gamma_N
\]

where

\[
\gamma_0 = \int_0^{x_1} g(x)Q_0(x)dx, \quad \gamma_k = (g, Q_k) = \int_{x_k}^{x_{k+1}} g(x)Q_k(x)dx, \quad k \in \{1, \ldots, N - 1\},
\]

\[
\gamma_N = \int_{x_{N-1}}^{x_n} g(x)Q_N(x)dx.
\]
Now we denote:

\[ a_{00} = \frac{2\mu_l^3}{h} + \frac{2\mu_l^4}{h^2} + \frac{6\mu_l^2}{9} - \frac{1}{18}; \quad a_{01} = -\frac{2\mu_l^4}{h^2} + \frac{6\mu_l^2}{9} \]

\[ a_{k,k-1} = -\frac{2\mu_l^4}{h^2} + \frac{3\mu_l^2}{9}, \quad k \in \{1, \ldots, N\} \]

\[ a_{k,k} = \frac{4\mu_l^4}{h^2} + \frac{4(3\mu_l^2 - 2)}{9}, \quad k \in \{1, \ldots, N - 1\} \]

\[ a_{k,k+1} = a_{k,k-1}, \quad k \in \{1, \ldots, N - 1\} \]

\[ a_{N,N-1} = a_{01}; \quad a_{NN} = a_{00} \]

The formula (34) can be written as a matrix equation

\[ \mathbf{A} \cdot \mathbf{B} = \Gamma \]

where

- \( \mathbf{A} \) is a matrix of the form

\[
\begin{bmatrix}
  a_{00} & a_{01} & 0 & \ldots & 0 & 0 & 0 \\
  a_{10} & a_{11} & a_{12} & \ldots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & a_{N-1,N-2} & a_{N-1,N-1} & a_{N-1,N} \\
  0 & 0 & 0 & \ldots & 0 & a_{N,N-1} & a_{N,N}
\end{bmatrix}
\]

- \( \mathbf{B} \) is a column matrix: \( [\beta_0 \beta_1 \ldots \beta_N]^T \)

- \( \Gamma \) is a column matrix: \( [\gamma_0 \gamma_1 \ldots \beta_N]^T \).

Here one property of approximation (30) should be noted down. The functions \( Q_{k-1} \) and \( Q_{k+1} \) are orthogonal, since at such places where one of these is nonzero, the other is equal to zero. Thus, the basis introduced is "almost orthogonal". This is the reason behind the appearance of band in the matrix \( \mathbf{A} \).

Solving the system of equations (36) we can find the values of coefficients \( \beta_k \), i.e. we can construct by (30) the solution \( u \) of (14)-(15). Let us now consider

\[ x_k = y_0 < y_1 < \ldots < y_m = x_{k+1} \]

for every interval \((x_k, x_{k+1})\), \( k = 0, 1, \ldots, N - 1 \). Using the \( Q_k(x) \), we can find the values of \( \tilde{u}_{k,j} = \tilde{u}(y_j, \mu_l), \quad k \in \{0, 1, 2, \ldots, N-1\}, \quad j \in \{1, 2, \ldots, m\} \). In order to get \( \tilde{v}_{k,j} \), we use \( \tilde{u}_{k,j} \) and the numerical derivative for equations (7b):

\[ \tilde{v}_{k,j} = -\frac{\tilde{u}_{k,j+1} - \tilde{u}_{k,j-1}}{2(h/m)} \cdot \mu_l, \quad k \in \{0, 1, \ldots, N\}, \quad j \in \{1, 2, \ldots, m-1\} \]

\[ \tilde{v}_{k,4} = -\frac{\tilde{u}_{k+1,1} - \tilde{u}_{k,3}}{2(h/m)} \cdot \mu_l, \quad k \in \{0, 1, \ldots, N - 2\} \]

\[ \tilde{v}_{0,1} = -\frac{\tilde{u}_{0,1} \cdot \mu_l}{h/m}, \quad \tilde{v}_{N-1,m} = \frac{\tilde{u}_{N-1,m} - 1}{h/m} \cdot \mu_l \]
\[ \tilde{u}_{k,j} = 2\mu_i^2 \left( \beta_k \frac{x_{k+1} - y_j}{h} + \beta_{k+1} \frac{y_j - x_k}{h} \right), \]

where

\( k \in \{0, 1, 2, \ldots, N - 1\}, \ j \in \{0, 1, \ldots, m\}. \)

According to the continuity of function \( u \) we get

\[ \tilde{u}_{k-1,m} = \tilde{u}_{k,0}, \ k \in \{1, 2, \ldots, N - 1\}. \]

Finally, the values of \( \varphi \) obtained by this algorithm will be

\[ \begin{align*}
\tilde{\varphi}^{+}_{k,j} &= \tilde{u}_{k,j} + \tilde{v}_{k,j} \quad \text{for} \quad \mu_i > 0 \\
\tilde{\varphi}^{-}_{k,j} &= \tilde{u}_{k,j} - \tilde{v}_{k,j} \quad \text{for} \quad \mu_i < 0
\end{align*} \]

\( k \in \{0, 1, \ldots, N\}, \ j \in \{0, 1, \ldots, m\}. \)

3. Approximation study

Let be consider an interpolation polynomial of solution \( u(x, \mu) \):

\[ U(x, \mu_l) = \sum_{k=0}^{N} u(x_k, \mu_l)Q_k(x) \sqrt{h} \]

hence

\[ U(x_k, \mu) = u(x_k, \mu), \quad k \in \{0, 1, \ldots, N\}, \quad \forall \mu \in (0, 1). \]

Let us now the approximate solution of (29) of the form (30)

\[ \tilde{u}(x, \mu) = \sum_{k=0}^{N} a_k(\mu)Q_k(x) \]

where \( a_k(\mu) = 2\beta_k\mu_i^2 \). We get

\[ \begin{align*}
\left( \mu^2 \frac{d\tilde{u}}{dx}, dQ_k \right) + \mu(\tilde{u}Q_k|_{z=0} + \tilde{u}Q_k|_{z=H}) + (\tilde{u}, Q_k) = (f_1, Q_k),
\end{align*} \]

\( k \in \{0, 1, \ldots, N\}. \)

To find the aprioristic estimation of the numerical solution, we multiply (43) by \( a_k(\mu) \), sum the product \( a_k(\mu)Q_k(x) \) with respect to \( k \) and then integrate between \( \mu = 0 \) and \( \mu = 1 \). The system (43) becomes:

\[ \int_0^1 \left[ \left( \mu^2 \frac{d\tilde{u}}{dx}, \frac{d\tilde{u}}{dx} \right) + \mu(\tilde{u}Q_k|_{z=0} + \tilde{u}Q_k|_{z=H}) + (\tilde{u}, \tilde{u}) - (S\tilde{u}, \tilde{u}) \right] d\mu = \]

\[ \int_0^1 (g, \tilde{u})d\mu, \quad k \in \{0, 1, \ldots, N\} \]

where

\[ (S\tilde{u}, \tilde{u}) = \frac{1}{2} \int_0^H \int_0^1 \tilde{u}(x, \mu')dx d\mu' \]

On the basis of Cauchy-Schwarz’s inequality

\[ \left( \int_0^1 fg \right)^2 \leq \int_0^1 f^2 \int_0^1 g^2 \quad \text{and} \quad g = 1 \]
we get
\[ \int_0^1 [(\tilde{u}, \tilde{u}) - (Su, \tilde{u})]d\mu = \]
\[ = \int_0^1 \left\{ \int_0^H \tilde{u}^2(x, \mu)d\mu - \int_0^H \left[ \frac{1}{2} \tilde{u}(x, \mu) \int_0^1 \tilde{u}(x, \mu')d\mu' \right] dx \right\} d\mu = \]
\[ = \int_0^H \left[ \int_0^1 \tilde{u}^2(x, \mu)d\mu - \frac{1}{2} \left( \int_0^1 \tilde{u}(x, \mu)d\mu \right)^2 \right] dx \geq \]
\[ \geq \left( 1 - \frac{1}{2} \right) \int_0^H dx \int_0^1 \tilde{u}^2(x, \mu)d\mu = \frac{1}{2} \int_0^1 (\tilde{u}, \tilde{u})d\mu \]
when \( H = 1 \). Then, using formula (44), Cauchy-Schwarz’s inequality and (46), we arrive at the inequality
\[ \int_0^1 \left[ \left( \mu^2 \frac{d\tilde{u}}{dx} \cdot \frac{d\tilde{u}}{dx} \right) + \mu(\tilde{u}^2)_{|z=0} + \tilde{u}^2_{|z=H} \right] d\mu \leq \int_0^1 (g, \tilde{u})d\mu \leq \]
\[ \leq \left( \int_0^1 (g, g)d\mu \right)^{1/2} \left( \int_0^1 (\tilde{u}, \tilde{u})d\mu \right)^{1/2} \leq \alpha \int_0^1 (g, g)d\mu + \frac{1}{4\alpha} \int_0^1 (\tilde{u}, \tilde{u})d\mu \]
in accordance with inequality
\[ |ab| \leq \alpha a^2 + \frac{b^2}{4\alpha}, \quad \alpha > 0. \]
Let us consider \( \alpha = 1 \) and we get
\[ \int_0^1 \left[ \left( \mu^2 \frac{d\tilde{u}}{dx} \cdot \frac{d\tilde{u}}{dx} \right) + \mu(\tilde{u}^2)_{|z=0} + \tilde{u}^2_{|z=H} \right] d\mu \leq \int_0^1 (g, g)d\mu. \]
It follows from inequality (49) that exists only one solution \( \tilde{u} \). Indeed, if assume that \( \tilde{u}_1 \neq \tilde{u} \) verifies (43) and hence (49), we obtain
\[ \int_0^1 \left[ \left( \mu^2 \frac{d\tilde{u}_1}{dx} \cdot \frac{d\tilde{u}_1}{dx} \right) + \mu(\tilde{u}_1^2)_{|z=0} + \tilde{u}_1^2_{|z=H} \right] d\mu \leq \int_0^1 (g, g)d\mu. \]
We denote \( \tilde{u}_2 = \tilde{u} - \tilde{u}_1 \) and subtracting (50) from (48) we get
\[ \int_0^1 \left[ \left( \mu^2 \frac{d\tilde{u}_2}{dx} \cdot \frac{d\tilde{u}_2}{dx} \right) + \mu(\tilde{u}_2^2)_{|z=0} + \tilde{u}_2^2_{|z=H} \right] d\mu \leq 0. \]
Hence, \( \tilde{u}_2 = 0 \Rightarrow \tilde{u}_1 = \tilde{u} \).
Now, to find the error estimate we observe that
\[ \mu^2 \left( \frac{d\tilde{u}(x, \mu)}{dx}, \frac{dQ_k}{dx} \right) = \mu^2 \left( \frac{u_k - u_{k-1}}{h\sqrt{h}} - \frac{u_{k+1} - u_k}{h\sqrt{h}} \right) = \]
\[ = \mu^2 \left( \frac{U_k - U_{k-1}}{h\sqrt{h}} - \frac{U_{k+1} - U_k}{h\sqrt{h}} \right) = \mu^2 \left( \frac{dU}{dx}, \frac{dQ_k}{dx} \right) \]
Then, it follows from equations (28)-(29) that for \( U \) and \( \tilde{u} \) we have
\[ \left( \mu^2 \frac{dU}{dx}, \frac{dQ_k}{dx} \right) + (u, Q_k) + \mu(UQ_k)_{|z=0} + UQ_k_{|z=H} - (Su, Q_k) = (g, Q_k) \]
\[ \left( \mu^2 \frac{d\tilde{u}}{dx}, \frac{dQ_k}{dx} \right) + (\tilde{u}, Q_k) + \mu(\tilde{u}Q_k)_{|z=0} + \tilde{u}Q_k_{|z=H} - (S\tilde{u}, Q_k) = (g, Q_k) \]
Finally, from (57) and (58), we obtain

\[
\int_0^1 \left[ \mu^2 \left( \frac{d(U - \tilde{u})}{dx}, \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right]_{x=0} + \mu(U - \tilde{u})^2 \right]_{x=H} \right] d\mu + \\
+ \int_0^1 \left[ (u - \tilde{u}, U - \tilde{u}) - S(u - \tilde{u}, U - \tilde{u}) \right] d\mu = 0
\]

Adding and subtracting \(u\) in the second integral we rewrite (54) in the form

\[
\int_0^1 \left[ \mu^2 \left( \frac{d(U - \tilde{u})}{dx}, \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right]_{x=0} + \mu(U - \tilde{u})^2 \right]_{x=H} \right] d\mu + \\
+ \int_0^1 \left[ (u - \tilde{u}, U - \tilde{u}) - (S(u - \tilde{u}), u - \tilde{u}) \right] d\mu = \\
= \int_0^1 \left[ (u - \tilde{u}, u - U) - (S(u - \tilde{u}), u - U) \right] d\mu.
\]

Accordingly with (46) we have

\[
\int_0^1 [(u - \tilde{u}, u - U) - (S(u - \tilde{u}), u - \tilde{u})] d\mu \geq \frac{1}{2} \int_0^1 (u - \tilde{u}, \tilde{u} - \tilde{u}) d\mu
\]

and using Cauchy-Schwarz’s inequality, we obtain for the right hand of relation (55) the inequalities:

\[
\left| \int_0^1 (u - \tilde{u}, u - U) d\mu \right| \\
\leq \int_0^H \left( \int_0^1 (u - \tilde{u})^2 d\mu \right)^{1/2} \left( \int_0^1 (u - U)^2 d\mu \right)^{1/2} dx \\
\leq \left( \int_0^H dx \int_0^1 (u - \tilde{u})^2 d\mu \right)^{1/2} \left( \int_0^H dx \int_0^1 (u - U)^2 d\mu \right)^{1/2}
\]

\[
\left| \int_0^1 (S(u - \tilde{u}), u - U) d\mu \right| \leq \frac{1}{2} \int_0^H \left( \int_0^1 (u - \tilde{u}) d\mu \right) \left( \int_0^1 (u - U) d\mu \right) dx \\
\leq \frac{1}{2} \left( \int_0^H dx \int_0^1 (u - \tilde{u}) d\mu \right)^{1/2} \left( \int_0^H dx \int_0^1 (u - U) d\mu \right)^{1/2}
\]

Finally, from (57) and (58), we have

\[
\left| \int_0^1 [(u - \tilde{u}, u - U) - S(u - \tilde{u}), u - U) d\mu \right| \\
\leq \frac{3}{2} \left( \int_0^H dx \int_0^1 (u - \tilde{u}) d\mu \right)^{1/2} \left( \int_0^H dx \int_0^1 (u - U)^2 d\mu \right)^{1/2} = \\
= \frac{3}{2} \left( \int_0^1 (u - \tilde{u}, u - \tilde{u}) d\mu \right)^{1/2} \left( \int_0^1 (u - U, u - U) d\mu \right)^{1/2}
\]
Using in (55) the inequalities (56), (59) and (48), we obtain the estimation of the error for the approximate solution $\tilde{u}$

$$
\int_0^1 \left[ \mu^2 \left( \frac{d(U - \tilde{u})}{dx}, \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right] d\mu + \\
+ \frac{1}{2} \int_0^1 (u - \tilde{u}, u - \tilde{u}) d\mu \leq \left| \int_0^1 [(u - \tilde{u}, u - U) - (S(u - \tilde{u}), u - U)] d\mu \right| \leq \\
\frac{3}{2} \left( \int_0^1 (u - \tilde{u}, u - \tilde{u}) d\mu \right)^{1/2} \left( \int_0^1 (u - U, u - U) d\mu \right)^{1/2} \leq \\
\frac{3\varepsilon}{2} \int_0^1 (u - \tilde{u}, u - \tilde{u}) d\mu + \frac{1}{4\varepsilon} \int_0^1 (u - U, u - U) d\mu \\
\Rightarrow \int_0^1 \left[ \mu^2 \left( \frac{d(U - \tilde{u})}{dx}, \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right] d\mu + \\
+ \left( \frac{1 - 3\varepsilon}{2} \right) \int_0^1 (u - \tilde{u}, u - \tilde{u}) d\mu \leq \frac{1}{4\varepsilon} ||u - U||^2.
$$

In the approximation theory, the next Agoskov inequality is used, [5],

$$
||u - U|| \leq C \frac{h^{1/2}}{\varepsilon} ||g||_{\infty}, \ g \in L_{\infty}(0, H) \times (0, 1)) \Rightarrow
$$

$$
\int_0^1 \left[ \mu^2 \left( \frac{d(U - \tilde{u})}{dx}, \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right] d\mu + \\
+ \left( \frac{1 - 3\varepsilon}{2} \right) \int_0^1 (u - \tilde{u}, u - \tilde{u}) d\mu \leq C_1 h ||g||_{\infty}^2
$$

In accordance with (46), (52) and (60), the error estimate for $\varepsilon = 1/3$ is given by

$$
\int_0^1 \left[ \mu \left( \frac{d(U - \tilde{u})}{dx}, \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right] d\mu + \\
+ \frac{1}{2} \left( u - \tilde{u}, u - \tilde{u} \right) d\mu \leq C_1 h ||g||_{\infty}^2
$$

Thus, we have obtained that the approximation of $u$ is of the order $h$.

4. Numerical examples

Let us consider the stationary transport equation

$$
\mu \frac{\partial \varphi(x, \mu)}{\partial x} + \varphi(x, \mu) = \int_{-1}^1 \varphi(x, \mu') d\mu' + f(x, \mu) \\
\forall (x, \mu) \in D_1 \times D_2, \ D_1 = [0, 1], \ D_2 = [-1, 1]
$$

with the boundary conditions:

$$
u(0, \mu) = 0, \ \mu > 0; \ u(1, \mu) = 0, \ \mu < 0.
$$
Now, we choose an even function $f$ with respect to $\mu$. Also, it is a periodical function with respect to $x$ of the form

$$f(x, \mu) = -2\pi^2 \mu^4 \cos(2\pi x) + \left(\mu^2 - \frac{2}{3}\right) \sin^2(\pi x)$$

Hence, $r = (f^+ - f^-)/2 = 0$ and we obtain $g = f$. We break up the closed interval $D_1$ into $N = 8$ segments of length $h = 1/8$ and for $D_2 = [0, 1]$ we have $L = 4$. Some computational results will illustrate the application of above algorithm. Let us now consider that $\mu_l = 1/2$. Using (65) we get

$$\gamma_k = \gamma_0 = 10.1 - 7.97 - 0.32 - 0.074 0.17 0.27 0.17 - 0.073 - 0.32 - 0.21$$

Solving (36), we obtain $B$, the matrix of coefficients $\beta_k$. In the following, each interval, $[x_k, x_{k+1}]$ is divided into subintervals $[y_j, y_{j+1}]$, where

$$x_k = y_0 < y_1 < y_2 < y_3 < y_4 = x_{k+1}, \quad k \in \{1, 2, \ldots, 7\}$$

$$y_{j+1} - y_j = h/4, \quad j \in \{0, 1, 2, 3\}, \quad m = 4 \quad \text{and} \quad u_{k,0} = u_{k-1,4}, \quad k \in \{2, \ldots, 8\}.$$
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Table 1

<table>
<thead>
<tr>
<th>k/j</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.002</td>
<td>0.012</td>
<td>0.021</td>
<td>0.03</td>
<td>0.04</td>
</tr>
<tr>
<td>1</td>
<td>0.04</td>
<td>0.062</td>
<td>0.084</td>
<td>0.106</td>
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<tr>
<td>2</td>
<td>0.128</td>
<td>0.151</td>
<td>0.173</td>
<td>0.193</td>
<td>0.217</td>
</tr>
<tr>
<td>3</td>
<td>0.217</td>
<td>0.226</td>
<td>0.235</td>
<td>0.244</td>
<td>0.253</td>
</tr>
<tr>
<td>4</td>
<td>0.253</td>
<td>0.244</td>
<td>0.235</td>
<td>0.226</td>
<td>0.217</td>
</tr>
<tr>
<td>5</td>
<td>0.217</td>
<td>0.19</td>
<td>0.17</td>
<td>0.151</td>
<td>0.13</td>
</tr>
<tr>
<td>6</td>
<td>0.13</td>
<td>0.107</td>
<td>0.085</td>
<td>0.063</td>
<td>0.041</td>
</tr>
<tr>
<td>7</td>
<td>0.041</td>
<td>0.031</td>
<td>0.022</td>
<td>0.012</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Table 2 shows the values of \( \tilde{v}_{kj} \), calculated with the help of the relations (37):

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>-0.15</td>
<td>-0.15</td>
<td>-0.15</td>
<td>-0.25</td>
</tr>
<tr>
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<td>-0.35</td>
<td>-0.35</td>
<td>-0.35</td>
<td>-0.35</td>
</tr>
<tr>
<td>2</td>
<td>-0.35</td>
<td>-0.35</td>
<td>-0.35</td>
<td>-0.35</td>
<td>-0.25</td>
</tr>
<tr>
<td>3</td>
<td>-0.25</td>
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<td>-0.15</td>
<td>-0.15</td>
<td>-0.001</td>
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<tr>
<td>4</td>
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<td>0.145</td>
<td>0.145</td>
<td>0.145</td>
<td>0.248</td>
</tr>
<tr>
<td>5</td>
<td>0.248</td>
<td>0.352</td>
<td>0.352</td>
<td>0.352</td>
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<tr>
<td>6</td>
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<td>0.354</td>
<td>0.354</td>
<td>0.354</td>
<td>0.253</td>
</tr>
<tr>
<td>7</td>
<td>0.253</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.302</td>
</tr>
</tbody>
</table>

Table 3 shows the values for different values of \( \mu \).

Table 3

<table>
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<th>k</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{u} )</td>
<td>-0.002</td>
<td>0.007</td>
<td>0.031</td>
<td>0.055</td>
<td>0.065</td>
<td>0.055</td>
<td>0.031</td>
<td>0.0074</td>
<td>-0.002</td>
</tr>
<tr>
<td>( \tilde{v} )</td>
<td>-0.078</td>
<td>-0.033</td>
<td>-0.047</td>
<td>-0.034</td>
<td>-0.0004</td>
<td>0.034</td>
<td>0.048</td>
<td>0.034</td>
<td>0.078</td>
</tr>
<tr>
<td>( \phi )</td>
<td>-0.08</td>
<td>-0.026</td>
<td>-0.016</td>
<td>0.021</td>
<td>0.064</td>
<td>0.09</td>
<td>0.08</td>
<td>0.041</td>
<td>0.075</td>
</tr>
<tr>
<td>( u e )</td>
<td>0</td>
<td>0.009</td>
<td>0.031</td>
<td>0.053</td>
<td>0.062</td>
<td>0.053</td>
<td>0.031</td>
<td>0.0092</td>
<td>0</td>
</tr>
<tr>
<td>( \tilde{\varepsilon} / 10^{-3} )</td>
<td>-2.8</td>
<td>-1.47</td>
<td>-1.21</td>
<td>1.44</td>
<td>2.3</td>
<td>1.7</td>
<td>-2.8</td>
<td>-1.7</td>
<td>-2.2</td>
</tr>
</tbody>
</table>

The approximate values of the solution of equation (14), where \( g \) is definite by (66), were compared with these obtained by exact solution:

\[
u(x, \mu) = \mu^2 \sin(\pi x)\]

Finally, the solution \( \phi_{kj} \) of the boundary problem (64)-(66) for \( \mu_l = 1/2 \) was computed from (40). Hence

\[\phi_{kj} = \tilde{u}_{kj} + \tilde{v}_{kj}, \quad k \in \{0, 1, \ldots, 8\}, \quad j \in \{0, 1, 2, 3, 4\}.\]

Table 3 shows the values of \( \tilde{u}(x_k, \mu_l), \tilde{v}(x_k, \mu_l), \phi^+(x_k, \mu_l), k \in \{1, 2, \ldots, 8\} \) for different values of \( \mu \). The numerical solution \( \tilde{u} \) of the boundary problem (14)-(15) has been compared with the exact solution \( u e \) and an estimation of the correspond approximations:

\[\tilde{\varepsilon} = \tilde{u} - u e\]

have been given.
5. Conclusions

In the section 3 we found an evaluation of the error for the approximate solution \( \tilde{u} \). According with (42), in the calculations reported in Tables 1-6, it should be observed that for \( \mu \) fixed, we get

\[
\tilde{\varepsilon} = \tilde{\varepsilon}(x_i, \mu_l) = U(x_j, \mu_l) - \tilde{u}(x_j, \mu_l)
\]

and further, we have

\[
f(x, \mu_l) = g(x, \mu_l), \quad \forall x \in D_1, \forall \mu_l \in \Delta_2.
\]

Let us consider for our numerical example, the norm

\[
\|g\|_\infty = \inf \{K | g(x, \mu) \leq K, \forall (x, \mu) \in \Delta_1 \times \Delta_2 \}
\]

which leads to the value: \( \|g\|_\infty = 1.4 \). Now, the inequality (63) becomes

\[
\int_0^1 \left[ \left( \mu \frac{d(U - \tilde{u})}{dx}, \mu \frac{d(U - \tilde{u})}{dx} \right) + \mu(U - \tilde{u})^2 \right] \bigg|_{z=0} + \mu(U - \tilde{u})^2 \bigg|_{z=H} + \frac{1}{2} (u - \tilde{u}, u - \tilde{u}) \bigg] d\mu \leq 1.4C_1 h.
\]

Thus, for \( h \to 0 \), the error tends to zero. From the numerical results, we have observed that the error \( \tilde{\varepsilon} \) attains maximum value for \( x = 1/2 \) and \( \mu = 1/2 \).
References


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