EXPONENTIAL STABILITY OF DISCRETE TIME DELAY SYSTEMS WITH NONLINEAR PERTURBATIONS

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Abstract. This paper gives sufficient condition for the exponential stability of discrete delay systems with nonlinear perturbations. This new, delay-dependent condition is derived using approach based on Lyapunov’s direct method. A numerical example has been working out to show the applicability of results derived.

Key Words. Lyapunov stability, discrete time delay systems, nonlinear perturbations

1. Introduction

It is well known that stability and robustness are the basic requirements for controlled systems. In practice, to satisfy the performance specification and to have a good transient response of the system, the controlled system is often designed to possess a stability degree. If the controlled system has a stability degree $\alpha$, we say that the system is exponentially stable. A number of research works for dealing with the exponential stability problem have been presented, e.g. [1-6].

In industrial processes, time-delays often occur in the transmission of information or material between different parts of a system. Chemical processing systems, transportation systems, communication systems, and power systems are typical examples of time-delay systems. Since the presence of time-delay often causes serious deterioration on stability and performance of the system, considerable research has been devoted to control problems of time-delay systems.

The problem of exponential stability testing becomes more complicated than that of a system without time delay and/or uncertainties. Recently, [3-6] addressed the stability degree testing problem for continuous time-delay systems. By testing some stability conditions and repeating the computation, they can estimate the stability degree of linear time-delay systems.

However, up to now, the same problem has been a little treated for discrete time-delay systems [7]. This is mainly due to the fact that such systems can be transformed into augmented systems without delay.

This augmentation of the systems is, however, inappropriate for systems with unknown delays or systems with time-varying delays.

The objective of this paper is to investigate the exponential testing problem for discrete uncertain systems. By using the Lyapunov stability theorem, a new criterion is established to ensure the exponential stability of discrete uncertain time-delay systems. Sufficient condition, in the form of time delayed-dependent criteria, is obtained.

The main result of this paper is extension of the result given in [8-9].
2. Notations and Preliminaries

- \( \mathbb{R} \) Set of real numbers
- \( \mathbb{R}_+ \) Set of non-negative real numbers
- \( \mathbb{T}_+ \) Set of non-negative integers
- \( F > 0 \) Positive definite matrix
- \( \|F\| \) Euclidean matrix norm of \( F \)

A linear, autonomous, multivariable discrete perturbed time-delay system can be represented by the difference equation:

\[
x(k + 1) = \sum_{j=0}^{N} (A_j + \Delta A_j)x(k - h_j) + \sum_{j=0}^{M} f_j(x(k - h_j), k), \quad M \leq N, \quad h_0 = 0
\]

with an associated function of initial state:

\[
x(\theta) = \psi(\theta), \quad \theta \in \{-h_N, -h_N + 1, \ldots, 0\}, \quad h_N = \max h_i
\]

where \( x(k) \in \mathbb{R}^n \) is state vector, \( A_j \in \mathbb{R}^{n \times n} \) is constant matrix and pure system time delays are expressed by integers \( h_j \in \mathbb{T}_+ \). \( M \) and \( N \) are given integers.

The vector \( f_j(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{T}_+ \rightarrow \mathbb{R}^n \) is a nonlinear perturbation which satisfy condition

\[
\|f_j(x(k-h_j), k)\| \leq b_j \|x(k-h_j)\|, \quad b_j \in \mathbb{R}_+
\]

**Lemma 2.1.** The byshev’s inequality [6] holds for any real vector \( v_i \):

\[
\left( \sum_{i=1}^{m} v_i \right)^T \left( \sum_{i=1}^{m} v_i \right) \leq m \sum_{i=1}^{m} v_i^T v_i
\]

**Lemma 2.2.** For any matrices \( W, X, Y \) and \( Z \) with the same dimension \( m \times n \), if

\[
W = X + Y + Z
\]

then for any positive square matrix \( P = P^T > 0 \) of dimension \( n \) and positive constants \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) the following statement is true

\[
W^T PW \leq \left( 1 + \varepsilon_1 + \varepsilon_3^{-1} \right) X^T PX + \left( 1 + \varepsilon_2 + \varepsilon_1^{-1} \right) Y^T PY + \left( 1 + \varepsilon_3 + \varepsilon_2^{-1} \right) Z^T PZ
\]

**Proof.**

\[
\begin{align*}
W^T PW - & \left( 1 + \varepsilon_1 + \varepsilon_3^{-1} \right) X^T PX - \left( 1 + \varepsilon_2 + \varepsilon_1^{-1} \right) Y^T PY - \left( 1 + \varepsilon_3 + \varepsilon_2^{-1} \right) Z^T PZ \\
= & - \varepsilon_1 X^T PX + X^T PY + Y^T PX - \varepsilon_1^{-1} Y^T PY \\
- & \varepsilon_2 Y^T PY + Y^T PZ + Z^T PY - \varepsilon_2^{-1} Z^T PZ \\
- & \varepsilon_3 Z^T PZ + Z^T PX + X^T PZ - \varepsilon_3^{-1} X^T PX \\
= & - \left( \varepsilon_1^{1/2} X - \varepsilon_1^{-1/2} Y \right)^T P \left( \varepsilon_1^{1/2} X - \varepsilon_1^{-1/2} Y \right) \\
- & \left( \varepsilon_2^{1/2} Y - \varepsilon_2^{-1/2} Z \right)^T P \left( \varepsilon_2^{1/2} Y - \varepsilon_2^{-1/2} Z \right) \\
- & \left( \varepsilon_3^{1/2} Z - \varepsilon_3^{-1/2} X \right)^T P \left( \varepsilon_3^{1/2} Z - \varepsilon_3^{-1/2} X \right) \leq 0
\end{align*}
\]
3. Main results

**Definition 3.1.** The system (2.1) is said to have a stability degree $\alpha$ (or to be exponentially stable), with $\alpha > 1$, being real positive scalar, if the state of system given, (2.1) can be written as:

$$x(k) = \alpha^{-k}p(k)$$  \hspace{1cm} (3.1)

and the system governing the state $p(k)$ is globally asymptotically stable.

In this case, the parameter $\alpha$ is called the convergence rate (see [10] for continuous case).

**Theorem 3.1.** System (2.1) is asymptotically stable if:

$$\|A_0\| + \left\| N \sum_{j=1}^{N} A_j^T A_j \right\|^{\frac{1}{2}} + \left( (M + 1) \sum_{j=0}^{M} b_j^2 \right)^{\frac{1}{2}} < 1$$  \hspace{1cm} (3.2)

**Proof.** Let the Lyapunov functional be

$$V(x_k) = \beta x^T(k) x(k) + N \sum_{j=1}^{N} \sum_{l=1}^{h_j} x^T(k-l) S_j x(k-l)$$  \hspace{1cm} (3.3)

where

$$x_k = x(k + \theta), \theta \in \{-h_N, -h_N + 1, \ldots, 0\}$$  \hspace{1cm} (3.4)

The forward difference along the solutions of system (2.1) is

$$\Delta V(x_k) = \beta \left[ A_0 x(k) + \sum_{j=1}^{N} A_j x(k-h_j) + \sum_{j=0}^{M} f_j (x(k-h_j), k) \right]^T$$

(3.5)

$$+ N \sum_{j=1}^{N} \sum_{l=1}^{h_j} \left[ x^T(k-l+1) S_j x(k-l+1) - x^T(k-l) S_j x(k-l) \right]$$

$$- \beta x^T(k) x(k)$$

Applying 2.2 on (3.5), one can get

$$\Delta V(x_k) \leq \beta \left( 1 + \varepsilon_1 + \varepsilon_3^{-1} \right) x^T(k) A_0^T A_0 x(k)$$

$$+ \beta \left( 1 + \varepsilon_2 + \varepsilon_1^{-1} \right) \left( \sum_{j=1}^{N} A_j x(k-h_j) \right)^T \left( \sum_{j=1}^{N} A_j x(k-h_j) \right)$$

(3.6)

$$+ \beta \left( 1 + \varepsilon_3 + \varepsilon_2^{-1} \right) \left( \sum_{j=0}^{M} f_j (x(k-h_j), k) \right)^T \left( \sum_{j=0}^{M} f_j (x(k-h_j), k) \right)$$

$$+ N \sum_{j=1}^{N} x^T(k) S_j x(k) - N \sum_{j=1}^{N} x^T(k-h_j) S_j x(k-h_j)$$

$$- \beta x^T(k) x(k)$$
Based on the 2.1 follows

$$
\Delta V (x_k) \leq x^T (k) \left( \beta (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - \beta I_n + N \sum_{j=1}^{N} S_j \right) x (k) \\
+ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) N \sum_{j=1}^{N} x^T (k - h_j) A_j^T A_j x (k - h_j) \\
+ \beta (1 + \varepsilon_3 + \varepsilon_2^{-1}) (M + 1) \sum_{j=0}^{M} ||f_j (x (k - h_j), k)||^2 \\
- N \sum_{j=1}^{N} x^T (k - h_j) S_j x (k - h_j)
$$

(3.7)

$$
\Delta V (x_k) \leq x^T (k) \left( \beta (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - \beta I_n + N \sum_{j=1}^{N} S_j \right) x (k) \\
+ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) N \sum_{j=1}^{N} x^T (k - h_j) A_j^T A_j x (k - h_j) \\
+ \beta (1 + \varepsilon_3 + \varepsilon_2^{-1}) (M + 1) \sum_{j=0}^{M} x^T (k - h_j) b_j^T x (k - h_j) \\
- N \sum_{j=1}^{N} x^T (k - h_j) S_j x (k - h_j)
$$

(3.8)

or:

$$
\Delta V (x_k) \leq x^T (k) \left[ \beta (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - \beta I_n \\
+ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) (M + 1) b_j^T I_n + N \sum_{j=1}^{N} S_j \right] x (k) \\
+ \sum_{j=1}^{N} \left\{ x^T (k - h_j) \left[ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) N A_j^T A_j \\
+ \beta (1 + \varepsilon_3 + \varepsilon_2^{-1}) (M + 1) b_j^T I_n - NS_j \right] x (k - h_j) \right\}
$$

(3.9)

$$
\dot{b}_j = \begin{cases} 
  b_j, & j \leq M \\
  0, & j > M
\end{cases}
$$

(3.10)

If one adopt:

$$
NS_j = \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) N A_j^T A_j \\
+ \beta (1 + \varepsilon_3 + \varepsilon_2^{-1}) (M + 1) b_j^T I_n
$$

(3.11)

then:

$$
\Delta V (x_k) \leq x^T (k) \left\{ \beta (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - \beta I_n \\
+ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) (M + 1) b_j^T I_n \\
+ \sum_{j=1}^{N} \left[ \beta (1 + \varepsilon_3 + \varepsilon_2^{-1}) (M + 1) \hat{b}_j^2 I_n \\
+ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) N A_j^T A_j \right] x (k) \right\}
$$

(3.12)

or:

$$
\Delta V (x_k) \leq x^T (k) \left\{ \beta (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - \beta I_n \\
+ \beta (1 + \varepsilon_2 + \varepsilon_1^{-1}) (M + 1) \sum_{j=0}^{M} b_j^2 I_n \\
+ \beta (1 + \varepsilon_3 + \varepsilon_2^{-1}) N \sum_{j=1}^{N} A_j^T A_j \right\} x (k)
$$

(3.13)
Let:

\[ \alpha \left( 1 + \varepsilon_2 + \varepsilon_1^{-1} \right) = 1 \]  

then:

\[
\Delta V(x_k) \leq x^T(k) \left\{ \frac{\varepsilon_1}{1 + \varepsilon_1 + \varepsilon_2^2} \left[ (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - I 
+ (1 + \varepsilon_3 + \varepsilon_2^{-1}) b^2 I \right] + N \sum_{j=1}^{N} A_j^T A_j \right\} x(k)
\]

\[ \Delta V(x_k) \leq x^T(k) \left\{ \frac{\varepsilon_1}{1 + \varepsilon_1 + \varepsilon_2^2} \left[ (1 + \varepsilon_1 + \varepsilon_3^{-1}) A_0^T A_0 - I 
+ (1 + \varepsilon_3 + \varepsilon_2^{-1}) b^2 I \right] + N \sum_{j=1}^{N} A_j^T A_j \right\} x(k)
\]

\[ = x^T(k) \left\{ g(\varepsilon_1, \varepsilon_2, \varepsilon_3) + N \left\| \sum_{j=1}^{N} A_j^T A_j \right\| \right\} x(k) \]

where \(g(\varepsilon_1, \varepsilon_2, \varepsilon_3)\) is defined by:

\[ g(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{\varepsilon_1}{1 + \varepsilon_1 + \varepsilon_2^2} \left[ (1 + \varepsilon_1 + \varepsilon_3^{-1}) a^2 
+ (1 + \varepsilon_3 + \varepsilon_2^{-1}) b^2 - 1 \right] \]

and:

\[ a = \| A_0 \|, \quad b^2 = (M + 1) \sum_{j=0}^{M} b_j^2 \]

Take the partial derivative of (3.16) with respect to \(\varepsilon_1, \varepsilon_2, \varepsilon_3\), respectively, and set them to be zero, we get three equalities containing \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\).

Then solving the three equations we obtain:

\[ \varepsilon_1 = \frac{1-b-a}{a}, \quad \varepsilon_2 = \frac{b}{1-b-a}, \quad \varepsilon_3 = \frac{a}{b} \]

and:

\[ g_{\min}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = -(1-b-a)^2 \]

If:

\[ \Delta V(x_k) \leq x^T(k) \left\{ g_{\min}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + N \left\| \sum_{j=1}^{N} A_j^T A_j \right\| \right\} \]

is satisfied, then (3.15) is hold too. Further:

\[
\Delta V(x_k) \leq x^T(k) \left[ -\left( 1 - \sqrt{M+1} \left( \sum_{j=0}^{M} b_j^2 \right)^{\frac{1}{2}} - \| A_0 \| \right)^2 
+ N \left\| \sum_{j=1}^{N} A_j^T A_j \right\| \right] x(k)
\]

If condition (3.2) is satisfied then \(\Delta V(x_k) < 0, \forall x_k \neq 0\) and system under consideration is asymptotically stable. \(\square\)
Theorem 3.2. System (2.1) is exponential stable if
\[ \alpha \|A_0\| + \sqrt{N} \sum_{j=1}^{N} \alpha^{2(h_j+1)} \|A_j^T A_j\| + \sqrt{(M + 1)} \sum_{j=0}^{M} \alpha^{2(h_j+1)} b_j^2 < 1 \] (3.22)

where \( \alpha \) is stability degree.

Proof. Let us define a new variable:
\[ p(k) = \alpha^k x(k) \] (3.23)

Then from (2.1) follows:
\[ p(k+1) = \alpha A_0 p(k) + \sum_{j=1}^{N} A_j \alpha^{h_j+1} p(k-h_j) + \sum_{j=0}^{M} \alpha^{k+1} f_j (\alpha^{-(k-h_j)} p(k-h_j), k) \] (3.24)

Also, one can define:
\[ \hat{A}_j = \alpha^{h_j+1} A_j \]
\[ \hat{f}_j (\alpha^{-(k-h_j)} p(k-h_j), k) = \alpha^{k+1} f_j (\alpha^{-(k-h_j)} p(k-h_j), k) \] (3.25)

Then:
\[ \| \hat{f}_j (\alpha^{-(k-h_j)} p(k-h_j), k) \| \leq \alpha^{h_j+1} b_j \| p(k-h_j) \| \]
\[ = \hat{b}_j \| p(k-h_j) \| \]
\[ p(k+1) = \hat{A}_0 p(k) + \sum_{j=1}^{N} \hat{A}_j p(k-h_j) + \sum_{j=0}^{M} \hat{f}_j (\alpha^{-(k-h_j)} p(k-h_j), k) \] (3.27)

Applying the basic result of 3.1 on (3.27), one can get
\[ \alpha \|A_0\| + \sqrt{N} \sum_{j=1}^{N} \|A_j^T \hat{A}_j\| + \sqrt{(M + 1)} \sum_{j=0}^{M} \hat{b}_j^2 < 1 \] (3.28)

Then from (3.28) follows stability condition, given by (3.22).

4. Numerical Examples

Example 1. Consider the discrete time delay system:
\[ x(k+1) = \sum_{j=0}^{2} A_j x(k-h_j) + \sum_{j=0}^{1} f_j (x(k-h_j), k) \]

where:
\[ A_0 = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & -0.15 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{pmatrix} \]
\[ A_2 = \begin{pmatrix} 0.1 & 0.2 \\ 0.05 & 0.1 \end{pmatrix}, \quad \| f_0 (x(k), k) \| \leq 0.1 \| x(k) \| \]

\[ \| f_1 (x(k - h_1), k) \| \leq 0.1 \| x(k - h_1) \|, \quad h_1 = 1, \quad h_2 = 2 \]

Applying the 3.1 follows:

\[ \| A_0 \| + \sqrt{2 \| A_1^T A_1 + A_2^T A_2 \|} + \sqrt{2 (0.1^2 + 0.1^2)} = 0.8833 < 1 \]

Therefore, the system under consideration with known delays is asymptotically stable.

In the case when the system (2.1) are not contain the nonlinear perturbations \( f_0 (\cdot, \cdot) = f_2 (\cdot, \cdot) = 0 \), we have:

\[ \| A_0 \| + \sqrt{2 \| A_1^T A_1 + A_2^T A_2 \|} = 0.6833 \]

This condition is less conservatively then condition

\[ \| A_0 \| + \| A_1 \| + \| A_2 \| = 0.7272 < 1 \]

proposed by [8].

Applying the 3.2 for given pure time delays follows: \( \alpha = 1.0385 \).

It also, follows that poles of discrete time delay system under consideration, under the presence of given perturbations, lie within the circle, centered in origin in \( z \) plane, with radius being \( r = 1/1.0385 = 0.9629 \).

5. Conclusion

New sufficient condition has been derived to guarantee exponential stability for linear delay systems with nonlinear perturbations. This delay-dependent condition is derived using approach based on Lyapunov’s direct method. A numerical example has been working out to show the applicability of results derived.

6. References


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