ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A NONLINEAR SIZE-STRUCTURED POPULATION MODEL

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Abstract. We study a size-structured model which describes the dynamics of \( n \)-subpopulations with nonlinear growth, reproduction and mortality rates. We establish existence and uniqueness results. We also show that there exists a compact global attractor for the trajectories of the dynamical system defined by the solutions of the model. In addition, we consider two open reproduction examples and perform equilibrium analysis for these cases.

Key Words. Size-structured model, existence-uniqueness, compact global attractor, stability analysis

1. Introduction

We study the following initial-boundary value problem that describes the dynamics of \( n \) size-structured subpopulations:

\[
\begin{align*}
(u_i)_t + (g_i(x, P(t))u_i)_x + m_i(x, P(t))u_i &= 0, & 0 < x \leq x_{\text{max}} \leq \infty, & t > 0, \\
g_i(0, P(t))u_i(0, t) &= c_i(t) + \sum_{j=1}^{n} \int_{0}^{x_{\text{max}}} \gamma_{i,j} \beta_j(x, P(t))u_j(x, t)dx, & t > 0, \\
u_i(x, 0) &= u_{i0}(x), & 0 \leq x \leq x_{\text{max}} \leq \infty.
\end{align*}
\]

Here \( u_i(x, t), i = 1, 2, \ldots, n \), is the density of individuals in the \( i \)th subpopulation having size \( x \) at time \( t \). The function \( P(t) = \sum_{i=1}^{n} \int_{0}^{x_{\text{max}}} u_i(x, t)dx \) represents the total number of individuals in the population at time \( t \). The functions \( g_i, m_i \) and \( \beta_i \) denote the growth rate, mortality rate and reproduction rate of an individual in the \( i \)th subpopulation, respectively. These rates depend on the size of the individual \( x \) and the total number of individuals in the population \( P \). The constant parameter \( 0 \leq \gamma_{i,j} \leq 1 \) represents the probability that an individual of the \( j \)th subpopulation will reproduce an individual of the \( i \)th subpopulation. We assume that \( \sum_{j=1}^{n} \gamma_{i,j} = \sum_{i=1}^{n} \gamma_{i,j} = 1, \ i, j = 1, 2, \ldots, n \). The function \( c_i \) is the inflow rate of the \( i \)th subpopulation of zero-size individuals from an external source (e.g., seeds flown by wind).

The model (1.1) is a generalization of several size-structured population models (often referred to as distributed rate models) which have been widely investigated...
in recent years (see [3, 4, 7, 8, 10]). In [1], an implicit finite difference approximation was developed to obtain the existence-uniqueness of weak solutions as well as convergence of the difference approximations for the quasilinear problem given in (1.1) with \( x_{\text{max}} < \infty \). In [2], the contraction mapping argument was used to obtain the existence-uniqueness of solutions when \( x_{\text{max}} = \infty \), and therein the vital rates of each subpopulation depend on the total number of individuals in the population \( P(t) \) but not on the individual’s size \( x \).

The goal of this paper is twofold. On the one hand, we will establish existence and uniqueness results for the general model (1.1) using the contraction mapping argument. On the other hand, we will study the asymptotic behavior of solutions to the model using asymptotic theory of dissipative systems. We will also perform equilibrium analysis for two example problems.

The paper is organized as follows. In section 2, we present local existence and uniqueness results for problem (1.1). In section 3, we establish the continuous dependence on initial conditions and the global existence. In section 4, we show that the solutions of problem (1.1) generate a \( C_0 \)-semigroup and there exists a compact global attractor for the trajectories of the dynamical system defined by the solutions of problem (1.1). In section 5, we consider two open reproduction examples and perform equilibrium analysis.

2. Local existence and uniqueness results

For simplicity, let \( \Omega = [0, x_{\text{max}}) \times [0, \infty) \). In order to carry out our program, we impose the following assumptions on the parameters in problem (1.1):

\begin{itemize}
  \item[(H1)] \( g_i(x, P) \) is a strictly positive Lipschitz function on \( \Omega \) with constant \( L_{g_i} \), continuously differentiable with respect to \( x \) on \( \Omega \), and \( \lim_{x \to x_{\text{max}}} g_i(x, P) = 0 \) for \( P \in [0, \infty) \), and \( g_i(x, P) \) is uniformly bounded on \( \Omega \) with \( 0 \leq g_i \leq g_M \).
  \item[(H2)] \( m_i(x, P) \) is a nonnegative Lipschitz function on \( \Omega \) with constant \( L_{m_i} \).
  \item[(H3)] \( c_i(t) \) is a nonnegative continuous function and uniformly bounded for \( 0 \leq t < \infty \) with \( 0 \leq c_i \leq c_M \).
  \item[(H4)] \( \beta_i(x, P) \) is a nonnegative Lipschitz function on \( \Omega \) with constant \( L_{\beta_i} \) and uniformly bounded on \( \Omega \) with \( 0 \leq \beta_i \leq \beta_M \).
  \item[(H5)] \( u_{i0} \in L^1(0, x_{\text{max}}) \) and \( u_{i0} \geq 0 \).
\end{itemize}

We begin with the definition of the solution to problem (1.1).

**Definition 2.1.** A nonnegative function \( u(x, t) = (u_1(x, t), u_2(x, t), \ldots, u_n(x, t)) \) on \( [0, x_{\text{max}}) \times [0, T) \), with \( u(\cdot, t) \) integrable is a solution of problem (1.1) provided that \( P(t) = \sum_{i=1}^n \int_0^{x_{\text{max}}} u_i(x, t)dx \) is a continuous function on \( [0, T) \) and for \( i = 1, 2, \ldots, n \), \( u_i(x, t) \) satisfies (1.1)_{2}, (1.1)_{3}, and the equation

\[
Du_i(x, t) = -\tilde{m}_i(x, P(t))u_i(x, t) \quad 0 < x < x_{\text{max}}, \ 0 < t < T
\]

with

\[
Du_i(x, t) = \lim_{h \to 0} \frac{u_i(X_i(t + h; x, t), t + h) - u_i(x, t)}{h},
\]

where \( \tilde{m}_i(x, P(t)) = m_i(x, P(t)) + (g_i)_x(x, P(t)) \) and \( X_i(t; x_0, t_0) \) is the solution of the equation for the characteristic curves given by

\[
\begin{align*}
\frac{d}{dt} x(t) &= g_i(x(t), P(t)) \\
x(t_0) &= x_0.
\end{align*}
\]
From (H1) we know that the function $X_i$ is strictly increasing. Thus a unique inverse function $\tau_i(x; t_0, 0)$ exists. Let $z_i(t) = X_i(t; 0, 0)$, the characteristic through the origin, and

$$B_i(t) = c_i(t) + \sum_{j=1}^{n} \int_{t_0}^{t_{\text{max}}} \gamma_{i,j} \beta_j(x, P(t)) u_j(x, t) dx,$$

the inflow of newborns in the $i$th subpopulation at time $t$. In the following, we reduce problem (1.1) to a system of coupled equations for $P(t)$ and $B_i(t)$ using the method of characteristics.

Integrating (2.1) along the characteristics, we have

$$u_i(x, t) = \frac{B_i(\tau_i(0; x, t))}{g_i(0, P(\tau_i(0; x, t)))} \exp \left( - \int_{\tau_i(0; x, t)}^{t} \tilde{m}_i(X_i(s; 0, \tau_i(s; x, t)), P(s)) ds \right) x < z_i(t),$$

$$u_i(x, t) = u_{i0}(X_i(0; x, t)) \exp \left( - \int_{0}^{t} \tilde{m}_i(X_i(s; x, t), P(s)) ds \right) x \geq z_i(t).$$

Then integrating (2.4) with respect to $x$ and summing over the indices $i = 1, 2, \ldots, n$, we obtain an integral equation for $P(t)$,

$$P(t) = \sum_{i=1}^{n} \left[ \int_{0}^{t_{\tau_i}(t)} \frac{B_i(\tau_i(0; x, t))}{g_i(0, P(\tau_i(0; x, t)))} \exp \left( - \int_{\tau_i(0; x, t)}^{t} \tilde{m}_i(X_i(s; 0, \tau_i(s; x, t)), P(s)) ds \right) dx \right.$$

$$+ \int_{t_{\tau_i}(t)}^{t_{\text{max}}} u_{i0}(X_i(0; x, t)) \exp \left( - \int_{0}^{t} \tilde{m}_i(X_i(s; x, t), P(s)) ds \right) dx \left. \right]$$

$$= \sum_{i=1}^{n} \left[ \int_{0}^{t} B_i(\eta) e^{-\int_{0}^{\eta} m_i(X_i(s; 0, \eta), P(s)) ds} d\eta \right.$$

$$+ \int_{t_{\text{max}}}^{t_{\text{max}}} u_{i0}(\xi) e^{-\int_{0}^{\xi} m_i(X_i(s; \xi, 0), P(s)) ds} d\xi \left. \right].$$

Then substituting (2.4) in the definition of $B_i(t)$ and using the same changes of variable as those used in (2.5), we find an integral equation for $B_i(t)$,

$$B_i(t) = c_i(t) + \sum_{j=1}^{n} \left[ \int_{0}^{t} \gamma_{i,j} \beta_j(X_j(t; 0, \eta), P(t)) B_j(\eta) e^{-\int_{0}^{\eta} m_j(X_j(s; 0, \eta), P(s)) ds} d\eta \right.$$

$$+ \int_{0}^{t_{\text{max}}} \gamma_{i,j} \beta_j(X_j(t; \xi, 0), P(t)) u_{j0}(\xi) e^{-\int_{0}^{\xi} m_j(X_j(s; \xi, 0), P(s)) ds} d\xi \left. \right].$$

On the one hand, if $P(t)$ and $B_i(t)$ are nonnegative continuous solutions of (2.5)-(2.6), then $u(x, t)$ defined by (2.4) is a solution of (1.1). On the other hand, if $u(x, t)$ is a solution of (1.1), then $P(t)$ and $B_i(t)$ are nonnegative continuous solutions of (2.5)-(2.6). Therefore, in order to obtain the existence and uniqueness results for problem (1.1), we only need to study the solvability of the system of integral equations (2.5)-(2.6). To this end, for $K > ||u_0||_{L^1} = \sum_{i=1}^{n} \int_{t_0}^{t_{\text{max}}} |u_{i0}(x)| dx$, let

$$S_{T,K} = \{ f(t) \in C[0, T] : f(0) = ||u_0||_{L^1}, 0 \leq f(t) \leq K \}. \quad \text{For each } P \in S_{T,K},$$

$$B_i(t) \in C[0, T] \text{ be the unique nonnegative solution of (2.6), and we define the}$$
operator $\mathcal{P} : \mathcal{S}_{T,K} \to C[0,T]$ in such a way that $\mathcal{P}(P)(t)$ is the right hand side of (2.5) for these $P(t)$ and $B_i(t)$.

The following results can be established using similar techniques as those in [2] and [5].

**Lemma 2.2.** Suppose that hypotheses (H1)-(H5) hold. Then there exists a value $T > 0$ for which $\mathcal{P}$ has a unique fixed point.

**Theorem 2.3.** Suppose that hypotheses (H1)-(H5) hold. Then there exists a value $T > 0$ such that problem (1.1) has a unique solution up to time $T$.

### 3. Continuous dependence on initial conditions and global existence

In this section, we establish the continuous dependence on initial conditions and the global existence of the solution. First, we show that the fixed point of the operator $\mathcal{P}$ associated with an initial condition depends continuously on this initial condition.

**Lemma 3.1.** Let $P^n_i(t)$ and $P^*_i(t)$ be the fixed points associated with the initial conditions $u_0$ and $v_0$, respectively. Then

$$|P^n_i(t) - P^*_i(t)| \leq \frac{nL \beta M^n \beta M t + 1}{1 - L} \|u_0 - v_0\|_i,$$

where $L$ is the contraction constant of the operator $\mathcal{P}$.

By means of (3.1), we can establish the following continuous dependence on initial conditions result using a similar argument as that in [5].

**Theorem 3.2.** Let $u_k(x,t) = (u_{k1}(x,t), u_{k2}(x,t), \ldots, u_{kn}(x,t))$, $k = 1, 2$, be the solution of problem (1.1) in the sense of definition 2.1 corresponding to the initial condition $u_{10} = (u_{101}, u_{102}, \ldots, u_{10n})$. Then, for fixed $t > 0$, integrable initial condition $u_{10}$, any $\epsilon > 0$ and $1 \leq i \leq n$, there exists $\delta = \delta(\epsilon, u_{10}, t) > 0$ such that if $\|u_{10} - u_{20}\|_i < \delta$ then $\|u_{11}(\cdot, t) - u_{21}(\cdot, t)\|_i < \epsilon$.

Next, we establish a bound on the total population $P(t)$. This bound will be useful for showing the global existence of the solution. The arguments are similar to those used in [5].

**Lemma 3.3.** Let $u(x,t)$ a solution of (1.1) up to time $T$. Then $P(t)$ satisfies the following bound

$$P(t) \leq \left(\|u_0\|_i + \frac{n \beta M}{\beta M} \right) e^{\beta M t} - \frac{n \beta M}{\beta M} \quad \text{for} \quad t \in [0,T].$$

**Theorem 3.4.** Suppose that hypotheses (H1)-(H5) hold. Then problem (1.1) has a unique solution for all positive time.

### 4. Existence of a compact global attractor

In this section, we assume that $c_i(t) \equiv c_i$ (constant) and establish the existence of a compact global attractor for the trajectories of the dynamical system defined by the solutions of (1.1).

Let $X = \{ u = (u_1, u_2, \ldots, u_n) : u_i(x) \geq 0 \text{ a.e. and } u_i(x) \in L^1(0, x_{\text{max}}), i = 1, 2, \ldots, n \}$ be the closed subset of the Banach space $\times_{i=1}^n L^1(0, x_{\text{max}})$ with norm $\|u\|_i = \sum_{i=1}^n \int_0^{x_{\text{max}}} |u_i(x)| \, dx$ and the family of maps $\{ S(t) : X \to X, \ t \in [0, \infty) \}$ defined by $S(t)u_0 = u(x,t)$, where $u_0 = (u_{10}, u_{20}, \ldots, u_{n0})$ and $u = (u_1, u_2, \ldots, u_n)$ are the
Let on time. Thus, uous dependence on initial conditions, we can establish the continuous dependence of orbits of bounded sets are bounded.

Lemma 4.1. Let $S(t)$ be a $C_0$ semigroup such that $S(t) = U(t) + W(t)$. If $U(t)$ is compact and there exists a continuous function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $k(t, r) \to 0$ as $t \to \infty$ and $\|W(t)\Phi\| \leq k(t, r)$ if $\|\Phi\| \leq r$, then $S(t)$ is asymptotically smooth.

Lemma 4.2. An asymptotically smooth $C_0$-semigroup defined in a complete metric space that is point dissipative, and for which orbits of bounded sets are bounded, has a compact global attractor.

Using the above results and similar arguments as in [5], we are able to establish the following results concerning the existence of a compact global attractor for the dynamical system defined by the solutions of (1.1).

Clearly, we have $S(0)u_0 = u(x, 0) = u_0$, and the existence-uniqueness of the solution of (1.1) in the sense of Definition 2.1 implies the semigroup property $S(t)u_0 = S(t - r)S(r)u_0$. Furthermore, using a similar argument as that of continuous dependence on initial conditions, we can establish the continuous dependence on time. Thus, $S(t)$ is a $C_0$-semigroup on $X$.

Next, we show that $S(t)$ is asymptotically smooth and point dissipative. To this end, let $S(t) = U(t) + W(t)$ such that

$$
U(t)\Phi = (U(t)\Phi)_1, (U(t)\Phi)_2, \ldots, (U(t)\Phi)_{i+1}, . . . , (U(t)\Phi)_n,
$$

$$
W(t)\Phi = (W(t)\Phi)_1, (W(t)\Phi)_2, \ldots, (W(t)\Phi)_{i+1}, . . . , (W(t)\Phi)_n
$$

with

$$
(U(t)\Phi)_i = \begin{cases} u_i(x, t) & \text{a.e. } x \leq z_i(t) \\ 0 & \text{a.e. } x > z_i(t) \end{cases}
$$

and

$$
(W(t)\Phi)_i = \begin{cases} 0 & \text{a.e. } x \leq z_i(t) \\ u_i(x, t) & \text{a.e. } x > z_i(t) \end{cases}
$$

for all initial distribution $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_{i+1}, \ldots, \Phi_n) \in X$. Here $z_i(t) = X_i(t; 0, 0)$ is the characteristic through the origin as before.

Lemma 4.3. Suppose that hypotheses (H1)-(H5) hold. Then $U(t)$ is compact.

Lemma 4.4. If $W(t)$ is the operator defined by (4.2) and $\mu = \inf_{(x, P) \in \Omega, 1 \leq i \leq n} m_i(x, P)$ is strictly positive, then there exists a continuous function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(t, r) \to 0$ as $t \to \infty$ and $\|W(t)\Phi\|_{L^1} \leq k(t, r)$ for $\|\Phi\|_{L^1} \leq r$.

Corollary 4.1. Under the hypotheses of lemma 4.4, the semigroup $S(t)$ associated with the solution of problem (1.1) is asymptotically smooth.

To prove the next theorem we will need an additional assumption:

(H6) There exists $P^{\infty}$ such that

$$
\sup_{(x, P) \in [0, x_{\max}) \times [P^{\infty}, \infty), 1 \leq i \leq n} \{\beta_i(x, P) - m_i(x, P)\} = -K < 0.
$$

Theorem 4.5. Assume that hypotheses (H1)-(H6) hold. Let $S(t)$ be the semigroup associated with the solution of problem (1.1). Then $S(t)$ is point dissipative, and orbits of bounded sets are bounded.
For the next result we need the following assumption.

\[(H7) \quad \mu = \inf_{(x,P) \in [x_0,x_{\max}) \times [0,\infty), 1 \leq i \leq n} \{m_i(x,P)\} > 0 \text{ for some } x_0 \in [0,x_{\max}).\]

**Theorem 4.6.** Under the assumptions (H1)-(H7), the \(C_0\)-semigroup \(S(t)\) defined by the solution of (1.1) has a compact global attractor.

**5. Examples**

In this section, we choose \(n = 2\) and perform equilibrium analysis for two examples with open reproduction (i.e., individuals in the \(i\)th subpopulation may also reproduce individuals in the \(j\)th subpopulation). To this end, for \(i = 1,2\), we impose additional assumptions on the parameters as follows:

\[(H8) \quad m_i \text{ and } \beta_i \text{ are increasing and nonincreasing functions of } P, \text{ respectively.}\]

Furthermore, there exists a unique positive \(P_1^*\) such that \(\beta_i(P_1^*) = m_i(P_1^*)\).

\[(H9) \quad \text{There is no external inflow of newborns, i.e., } c_i = 0.\]

**5.1. Example A.** In the first example we assume that \(m_i\) and \(\beta_i\) depend on the total population \(P\) but not on the individual’s size. We also assume \(x_{\max} = \infty\).

Then problem (1.1) takes the form:

\[
(u_t)_i + (g_i(P(t))u_i)_x + m_i(P(t))u_i = 0 \quad \text{ for } 0 < x < \infty, \quad t > 0,
\]

\[
(5.1) \quad g_i(P(t))u_i(0,t) = \sum_{j=1}^{2} \int_{0}^{\infty} \gamma_{i,j} \beta_j(P(t))u_j(x,t)dx \quad \text{ for } t > 0,
\]

\[
u_i(x,0) = u_{i0}(x) \quad \text{ for } 0 \leq x < \infty.
\]

Integrating (5.1) \(_1\) with respect to \(x\) and using the boundary condition (5.1) \(_2\), we obtain the following system of differential equations:

\[
(5.2) \quad P'_i = \gamma_{i,1} \beta_1(P)P_1 + \gamma_{i,2} \beta_2(P)P_2 - m_i(P)P_i := F_i(P_1, P_2), \quad i = 1, 2
\]

with \(P = P_1 + P_2\).

Clearly, \((0,0)\) is the only boundary equilibrium point for system (5.2). Furthermore, since the Jacobian evaluated at \((0,0)\) is given by

\[
J(0,0) = \begin{bmatrix}
\gamma_{1,1} \beta_1(0) - m_1(0) & \gamma_{1,2} \beta_2(0) \\
\gamma_{2,1} \beta_1(0) & \gamma_{2,2} \beta_2(0) - m_2(0)
\end{bmatrix},
\]

one can see that \((0,0)\) is either unstable or a saddle point. When it is a saddle point, one can show that the stable manifold is not in \(\mathbb{R}_+^2\). In fact, one can easily prove that given any positive initial condition \(\liminf_{t \to \infty} P(t) \geq \delta\) for some positive constant \(\delta\).

We next show the unique existence of the positive equilibrium of (5.2) when \(m_i(P)\) and \(\beta_i(P)\) satisfy \(m_i(0) < \gamma_{i,i} \beta_i(0)\).

If \((\bar{P}_1, \bar{P}_2)\) is a positive equilibrium of (5.2), then we have

\[
(5.3) \quad \gamma_{1,1} \beta_1(\bar{P}) \bar{P}_1 + \gamma_{1,2} \beta_2(\bar{P}) \bar{P}_2 - m_i(\bar{P}) \bar{P}_i = 0,
\]

which implies

\[
(5.4) \quad \prod_{i=1}^{2} \left[\gamma_{i,i} \beta_i(\bar{P}) - m_i(\bar{P})\right] = \gamma_{1,2} \gamma_{2,1} \beta_1(\bar{P}) \beta_2(\bar{P})
\]

and

\[
(5.5) \quad m_i(\bar{P}) = \gamma_{i,i} \beta_i(\bar{P}).
\]
Let \( h_i(P) = \frac{m_i(P)}{\beta_i(P)} \). Then (5.4) becomes

\[ (5.6) \quad H(P) = \gamma_{1,2} \gamma_{2,1} \]

with

\[ H(P) = \prod_{i=1}^{2} [h_i(P) - \gamma_{i,i}] . \]

On the one hand, since \( m_i \) and \( \beta_i \) are increasing and nonincreasing functions of \( P \), respectively, we have that \( h_i \) is monotonically increasing. Noticing that \( m_i(0) < \gamma_{i,i} \beta_i(0) \), \( h_i^{-1}(\gamma_{i,i}) \) exists and is unique. Let \( \hat{P} = h_i^{-1}(\gamma_{i,i}) \), then \( H(\hat{P}) = 0 \). On the other hand, using (5.5) one can see \( P > \hat{P} \). Without loss of generality, we may assume \( \hat{P}_2 = \max\{\hat{P}_1, \hat{P}_2\} \). Thus we have \( \hat{P} > \hat{P}_2 \).

Furthermore, we find that

\[ H'(P) = \sum_{i,j=1, i \neq j}^{2} h_i'(P) [h_j(P) - \gamma_{j,j}] > 0 \quad \text{for} \quad P > \hat{P}_2. \]

Then \( H(\hat{P}_2) = 0 \) implies that \( H(P) = \gamma_{1,2} \gamma_{2,1} \) has a unique positive solution \( P \) satisfying \( P > \hat{P}_2 \). Thus (5.3) has a unique positive solution \( \hat{P} \), and consequently, (5.2) has a unique positive equilibrium \((\hat{P}_1, \hat{P}_2)\).

Moreover, we can show that the unique positive equilibrium \((\hat{P}_1, \hat{P}_2)\) is locally asymptotically stable. For this purpose, evaluating the Jacobian matrix at the equilibrium \((\hat{P}_1, \hat{P}_2)\), we obtain

\[ J(\hat{P}_1, \hat{P}_2) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \]

where

\[
\begin{align*}
J_{11} &= \gamma_{1,1} \beta_1(\hat{P}) + \gamma_{1,2} \beta_2(\hat{P}) - m_1(\hat{P}) - \gamma_{1,1} \beta_1,
J_{12} &= \gamma_{1,1} \beta_1(\hat{P}) + \gamma_{1,2} \beta_2(\hat{P}) - m_1(\hat{P}) + \gamma_{1,2} \beta_2,
J_{21} &= \gamma_{2,1} \beta_1(\hat{P}) + \gamma_{2,2} \beta_2(\hat{P}) - m_2(\hat{P}) + \gamma_{2,2} \beta_2,
J_{22} &= \gamma_{2,2} \beta_2(\hat{P}) + \gamma_{2,2} \beta_2(\hat{P}) - m_2(\hat{P}) - \gamma_{2,2} \beta_2.
\end{align*}
\]

Then we find

\[
\text{Tr}(J) = \sum_{i,j=1, i \neq j}^{2} [\gamma_{i,i} \beta'_i(P) \hat{P}_i + \gamma_{i,j} \beta'_j(P) \hat{P}_j - m_i'(P) \hat{P}_i] \\
+ \sum_{i=1}^{2} [\gamma_{i,i} \beta_i(P) - m_i(P)] < 0
\]

and

\[
\text{det}(J) = \sum_{i,j=1, i \neq j}^{2} [\gamma_{i,i} \beta'_i(P) \hat{P}_i + \gamma_{i,j} \beta'_j(P) \hat{P}_j - m_i'(P) \hat{P}_i] \\
\times [\gamma_{j,j} \beta_j(P) - m_j(P) - \gamma_{j,i} \beta_i(P)] \\
+ \prod_{i=1}^{2} [\gamma_{i,i} \beta_i(P) - m_i(P)] - \gamma_{1,2} \gamma_{2,1} \beta_1(\hat{P}) \beta_2(\hat{P}).
\]
Noticing that \( \gamma_{i,j}\beta_i(\bar{P}) < m_i(\bar{P}) < m_i(\bar{P}) + \gamma_{i,j}\beta_j(\bar{P}) \), we have

\[
\det(J) > \prod_{i=1}^{2} [\gamma_{i,i}\beta_i(\bar{P}) - m_i(\bar{P})] - \gamma_{1,2}\gamma_{2,1}\beta_1(\bar{P})\beta_2(\bar{P}) = 0.
\]

Therefore, the unique positive equilibrium \((\bar{P}_1, \bar{P}_2)\) is locally asymptotically stable.

We now eliminate the existence of periodic solutions using Dulac’s criterion as follows: Let \( B = \frac{1}{P_1 P_2} \) for any \((P_1, P_2) \in \mathbb{R}_+^2 \), then

\[
\frac{\partial(BF_1)}{\partial P_1} + \frac{\partial(BF_2)}{\partial P_2} = \frac{\gamma_{1,1}\beta_1'}{P_2} + \frac{\gamma_{1,2}\beta_2'}{P_1} - \frac{m'_1}{P_2} + \frac{\gamma_{2,1}\beta_1'}{P_2} - \frac{\gamma_{2,2}\beta_2'}{P_1} - \frac{m'_2}{P_1} < 0.
\]

Hence, from the Poincaré-Bendixson trichotomy it follows that the interior equilibrium is globally asymptotically stable.

5.2. Example B. In the second example, we assume that \( x_{\text{max}} = 1 \) and the ingested food is allocated among maintenance, individual growth and reproduction. In particular, we assume that a fraction \( k_i \in (0, 1) \) of ingested food is channelled to growth and maintenance, and a fraction \( (1 - k_i) \) to reproduction. We also assume that for an individual of size \( x \) in the \( i \)th subpopulation, \( k_if_i(P)x \) is the rate at which maintenance needs energy and \( k_if_i(P)(1 - x) \) is what remains for growth. Thus we have the following sub-models for the growth and reproduction rates for each subpopulation

\[
g_i(x, P) = g_i^0 k_i f_i(P)(1 - x) \quad \text{and} \quad \beta_i(x, P) = \beta_i^0 (1 - k_i) f_i(P)x,
\]

where \( g_i^0 \) and \( \beta_i^0 \) are positive constants. In addition, we assume that the mortality rate for an individual in the \( i \)th subpopulation is given by \( m_i(P) \). Thus problem (1.1) becomes

\[
(u_i)_t + g_i^0 k_i f_i(P)(1 - x)u_i + m_i(P)u_i = 0 \quad 0 < x \leq 1, \quad t > 0,
\]

\[
(5.7) \quad g_i^0 k_i u_i(0, t) = \sum_{j=1}^{2} \gamma_{i,j}\beta_j^0 (1 - k_j) \int_0^1 xu_j(x, t)dx \quad t > 0,
\]

\[
u_i(x, 0) = u_{i0}(x) \quad 0 \leq x \leq 1.
\]

Integrating (5.7) and multiplying (5.7) by \( x \) and integrating once again, we have the following system of differential equations:

\[
(5.8) \quad P'_1 = \sum_{j=1}^{2} \gamma_{i,j}\beta_j^0 (1 - k_j) f_j(P)Q_j - m_i(P)P_i,
\]

\[
Q'_i = g_i^0 k_i f_i(P)(P_i - Q_i) - m_i(P)Q_i,
\]

where \( P_i = \int_0^1 u_i(x, t)dx \) and \( Q_i = \int_0^1 xu_i(x, t)dx \).

We show that (5.8) has a unique positive equilibrium when \( m_i(P) \) and \( f_i(P) \) satisfy \( m_i(0) < \alpha_i f_i(0) \) with

\[
\alpha_i = \frac{-g_i^0 k_i + \sqrt{(g_i^0 k_i)^2 + 4g_i^0 k_i \gamma_{i,i} \beta_i^0 (1 - k_i)}}{2}.
\]
Suppose that \((\bar{P}_1, \bar{P}_2, \bar{Q}_1, \bar{Q}_2)\) is a positive equilibrium of (5.8). Then we have

\[
\sum_{j=1}^{2} \gamma_{i,j} \beta_i^0 (1 - k_j) f_j(P) Q_j - m_i(P) P_i = 0,
\]

\[
g_i^0 k_i f_i(P)(P_i - \bar{Q}_i) - m_i(P) \bar{Q}_i = 0.
\]

For simplicity, let \(\beta_{i,j} = \gamma_{i,j} \beta_i^0 (1 - k_j)\) and \(g_i = g_i^0 k_i\). System (5.9) takes the form

\[
\sum_{j=1}^{2} \beta_{i,j} f_j(P) Q_j - m_i(P) \bar{P}_i = 0,
\]

\[
g_i f_i(P)(\bar{P}_i - \bar{Q}_i) - m_i(P) \bar{Q}_i = 0.
\]

Let \(h_i(P) = \frac{m_i(P)}{f_i(P)}\). From (5.10) we find

\[
\bar{P}_i = \left( \frac{h_i(P)}{g_i} + 1 \right) \bar{Q}_i.
\]

It then follows from (5.10) that

\[
\sum_{j=1}^{2} \beta_{i,j} f_j(P) \bar{Q}_j - m_i(P) \left( \frac{h_i(P)}{g_i} + 1 \right) \bar{Q}_i = 0,
\]

which implies

\[
\sum_{i=1}^{2} \left[ \beta_{i,j} f_i(P) - m_i(P) \left( \frac{h_i(P)}{g_i} + 1 \right) \right] = \beta_{1,2} \beta_{2,1} f_1(P) f_2(P)
\]

and

\[
\beta_{i,j} f_i(P) < m_i(P) \left( \frac{h_i(P)}{g_i} + 1 \right).
\]

Let \(\varphi_i(P) = \beta_{i,j} f_i(P) - m_i(P) \left( \frac{h_i(P)}{g_i} + 1 \right)\). Then (5.13) and (5.14) become

\[
\prod_{i=1}^{2} \frac{\varphi_i(P)}{f_i(P)} = \beta_{1,2} \beta_{2,1}
\]

and

\[
h_i^2(P) + g_i h_i(P) - g_i \beta_{i,j} > 0,
\]

respectively. Since \(\beta_i\) is a nonincreasing function of \(P\), \(f_i\) is a nonincreasing function of \(P\). Moreover, since \(m_i\) is an increasing function of \(P\), \(h_i\) is monotonically increasing. Then it follows from \(m_i(0) < \alpha_i f_i(0)\) that \(h_i^{-1}(\alpha_i)\) exists and is unique. Let \(\bar{P}_i = h_i^{-1}(\alpha_i)\). Without loss of generality, we may assume that \(\bar{P}_2 = \max\{\bar{P}_1, \bar{P}_2\}\). Then we find that \(P > \bar{P}_2\), \(\varphi_i(\bar{P}_2) = 0\) and \(\varphi_i(P) < 0\) for \(P > \bar{P}_2\).

Let \(I(P) = \prod_{i=1}^{2} \frac{\varphi_i(P)}{f_i(P)}\). Then \(I'(P) = \frac{\left( \varphi_1 \varphi_2 + \varphi_1 \varphi_2 \right) f_1 f_2 - \varphi_1 \varphi_2 \left( f_1 f_2 + f_1 f_2 \right)}{f_1^2 f_2^2} \). By the monotonicity of \(m_i\) and \(f_i\), \(\varphi_i(P) < 0\), and thus \(I'(P) > 0\) for \(P > \bar{P}_2\). Meanwhile, \(\varphi_2(\bar{P}_2) = 0\) implies \(I(\bar{P}_2) = 0\). Hence, \(I(P) = \beta_{1,2} \beta_{2,1}\) has a unique solution \(P\) satisfying \(P > \bar{P}_2\), which means that (5.12) has a unique solution \(P\).

By (5.12), \(\frac{\bar{Q}_1}{\bar{Q}_2}\) is unique. Then by (5.11), \(\frac{\bar{P}_1}{\bar{P}_2}\) is unique, and thus \(\bar{P}_1\) and \(\bar{P}_2\) are
unique. Again from (5.11) one can see that $\bar{Q}_1$ and $\bar{Q}_2$ are unique. Therefore, (5.8) has a unique positive equilibrium $(\bar{P}_1, \bar{P}_2, \bar{Q}_1, \bar{Q}_2)$.

Although at this stage we cannot carry out stability analysis for example B, our numerous numerical results indicate that the unique positive equilibrium is globally stable. Our future efforts will focus on studying the stability of this unique positive equilibrium.

References


