CALCULUS OF FUNCTIONAL ANALYSIS BECOMES ELEMENTARY ALGEBRA

QUN LIN

Abstract. Calculus of functional analysis becomes two elementary inequalities, first for the derivative definition and second for the fundamental theorem, without using the limit notion, completeness and the Hahn-Banach theorem.

Key Words. Functional analysis, calculus, elementary inequality.

1. Introduction

Calculus of functional analysis, including the derivative definition and the fundamental theorem (FT), uses the \( \epsilon - \delta \) notion, completeness and the Hahn-Banach theorem. They are to be avoided in this note: Derivative is defined by an elementary inequality with an error bound (see (1) below) and without using \( \epsilon - \delta \) at the beginning, and the FT is then given by a second inequality (see (2) below) which is nothing but the sum of former inequalities themselves (see the proof of (2) below) without using completeness and the Hahn-Banach theorem. Calculus of functional analysis is indeed reduced into an inequality.

2. Derivative definition becomes an elementary inequality

Let \( f \) be an abstract function (producing an abstract curve) defined on an interval \([a, b]\), containing subinterval \([x, x + h]\) and taken values in a linear norm space (without the completeness notion, just an elementary linear algebra). The height variation over \([x, x + h]\), \( f(x + h) - f(x) \), is computed by differential = derivative \( \times \) base = \( f'(x)h \), with error bound \( \epsilon(h)h \): For all \( x + h \) near \( x \),

\[
\| f(x + h) - f(x) - f'(x)h \| \leq \epsilon(h)h
\]

(1)

the height \( f(x + h) - f(x) \) over subinterval \([x, x + h]\) is computed by its linear part, differential \( f'(x)h \), with error \( \epsilon(h)h \)

2000 Mathematics Subject Classification. 00A, 65D and 97U.
where the notation $\epsilon(h)$ depends on the size of the argument increment $h$ but is independent of the argument $x$, and is chosen small so that the derivative $f'$ is uniquely defined.

3. **Fundamental theorem becomes another elementary inequality**

If there exists a function $f'$ and a bound $\epsilon(h)$ satisfying the first inequality, (1):

$$||f(x + h) - f(x) - f'(x)h|| \leq \epsilon(h)h$$

then, adding up these inequalities on each subinterval $[x, x + h]$ gives a second elementary inequality: The total height over $[a, b]$ is computed by Reimann’s sum with an error proportional to $\epsilon(h)$ in the first inequality:

$$||[f(b) - f(a)] - \sum_{x \in \text{nodes}} f'(x)h|| \leq (b - a)\epsilon(h)$$

(2)

or even

$$||[f(b) - f(a)] - \sum_{\xi \in [x, x + h]} f'(\xi)h|| \leq (b - a)\epsilon(h).$$

(3)

When $\epsilon(h)$ is chosen small then the total error, (3), is still small, so the element $f(b) - f(a)$ depends on the end points $a$ and $b$ and is independent of the division for the subintervals, and can then be used to define the definite integral $\int_a^b f'(x)dx$. This is the FT. For a detailed proof of (2), see the following familiar argument in real calculus (e.g. [1–4]).

Let the total interval $[a, b]$ be divided into $n + 1$-equal subintervals $[x_i, x_i + h]$, where $a = x_0 < x_1 < \cdots < x_{n+1} = b, \ h = x_{i+1} - x_i$. 

\[\text{if each error } \approx \epsilon(h)h, \text{ as (1)} \]

\[\text{then total error } \approx \epsilon(h), \text{ as (2)} \]
Use the uniform inequality (1) to each subinterval:
\[
\begin{align*}
f(x_0 + h) - f(x_0) &= f'(x_0)h, \\
f(x_1 + h) - f(x_1) &= f'(x_1)h, \\
&\quad \cdots \\
f(x_n + h) - f(x_n) &= f'(x_n)h, \\
\end{align*}
\]
satisfying
\[
\text{upper } \|\epsilon(h,x)\| \leq \epsilon(h),
\]
and add up. The left sides add to
\[
f(b) - f(a) - \sum_{i=0}^{n} f'(x_i)h
\]
and the right sides add to the total error
\[
\| \sum_{i=0}^{n} \epsilon(h,x_i) \| \leq (b-a)\epsilon(h)
\]
or
\[
\| [f(b) - f(a)] - \sum_{i=0}^{n} f'(x_i)h \| \leq (b-a)\epsilon(h).
\]
This completes the proof of (2). By the same argument, (3) can be proved with more calculation. Notice that the above argument uses only the definition (1) itself without using completeness and the Hahn-Banach theorem.

Notice also that the uniform definition (1) (e.g. [1–4]):
\[
f(x + h) - f(x) - f'(x)h = \epsilon(h,x)h
\]
\[
\text{upper } \|\epsilon(h,x)\| \leq \epsilon(h)
\]
is a small modification from the pointwise definition (see Ljusternik-Sobolev, Chapter 8)
\[
f(x + h) - f(x) - f'(x)h = \epsilon(h,x)h
\]
at fixed \(x\): \(\|\epsilon(h,x)\| \leq \epsilon(h)\)
but it greatly simplifies regular calculus of functional analysis.

Functional analysis teachers might react the uniform condition (1). Indeed, it has already been proved by Vainberg that the uniform condition (1) is equivalent to uniform continuity of the pointwise derivative. Such a uniform continuity is exactly the standard condition in the FT, but using the uniform condition (1) is much more simple and practical, reducing functional analysis into elementary algebra.

This is an elementary approach and radically simplifies the proof of the FT for functional analysis, where functional analysts (see Ljusternik-Sobolev, Chapter 8) persistently used Hahn-Banach’s theorem and so it is tricky, complicated and even not complete. We use such an approach to spread functional analysis since 1990’s.

Acknowledgments

Thanks to Professors Zhangxin Chen, Dan Velleman, Congxin Wu and Hung Hsi Wu for their criticisms. Professor Jingzhong Zhang has made full use of the uniform inequality (1) itself to complete an elementarization of differential calculus very successfully.
References


LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China

E-mail: linq@lsec.cc.ac.cn