ASYMPTOTIC BEHAVIOR FOR A REACTIVE DIFFUSION EQUATIONS

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Abstract. In this paper we study asymptotic behavior for a reaction diffusion equation with important practical background by using differential equation theory. Firstly we discuss existing condition of steady solution and stability problem. Then, we study diffusion-driven instability and the character of the wave-number for such model. Finally we prove there exists a confined set in the first quadrant. The results of this paper show that an ultimate steady state spatially inhomogeneous solution will emerge. Applying such model into the practical ecology systems, we can get some possible strategies for pest control.

Key Words. Diffusion-driven instability, Reaction diffusion equation, Turning space, wave-number

1. Introduction.

The most wonderful results have been obtained constantly in the most abstract domain of Mathematics, meanwhile abstract mathematics offers powerful modern tools for natural science and social science to solve omnifarious problems. With the development of the natural science, Mathematics is becoming more and more important in biology science, material science and information technology. Many scholars have been attracted to the areas and a lot of results [1-8] have been obtained.

In 1972, scientist Gierer and Meinhardt established a kind of activator(u)-inhibitor(v) model. This model (1) described a kind of predator-prey relationship between two species A and B:

\[ u_t = a - bu + \frac{u^2}{v(1+ku^2)} \]
\[ v_t = u^2 - v \]

Where \( a, b, k \) are all positive numbers. \( u \) and \( v \) denotes the density or level of the species A and B, respectively. Obviously A is the activator and B is the inhibitor. The characteristic of the system embodies a kind of predator-prey ecological model. On the other hand, the two species, which affect each other, may diffuse in such a way as to produce some steady state heterogeneous spatial patterns. Model (2) can describe this relationship between the two species.

\[ u_t = r(a - bu + \frac{u^2}{v(1+ku^2)}) + \nabla^2 u \]
\[ v_t = r(u^2 - v) + d\nabla^2 v \]

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In this paper we mainly discuss diffusion-driven instability of such system. Firstly, we study the parameter domains of diffusion-driven instability and some bifurcation values: diffusion coefficient $d_c$, wave number $m_c$. Secondly, we discuss the existence of a confined set in the first quadrant for the predator-prey model. Through studying, we can forecast whether diffusion of a species will happen in a confined space, in addition, we can get the measure to prevent it from happening.

2. Equilibrium and stability

We can get the equilibrium $(u_0, v_0)$ of system (1) through the equations

\[(3)\quad a - bu + \frac{u^2}{v(1 + ku^2)} = 0\]
\[v_t = u^2 - v = 0\]

From discussion we know $(0,0)$ is not equilibrium. Then we analysis the equation

\[(4)\quad 1 = \frac{1}{(bu - a)(1 + ku^2)}\]

Suppose $u \neq \frac{a}{b}$ the equation (4) is equal to the equation (5)

\[(5)\quad bk^3 - aku^2 + bu - a - 1 = 0\]

Hence we draw the conclusion that if $u$ is solution of the equation (4) then, $(u, u^2)$ is the equilibrium of the system (1).

Let

\[(6)\quad F(u) = bk^3 - aku^2 + bu - a - 1\]
\[(7)\quad F'(u) = 3bk^2 - 2aku + b\]
\[(8)\quad \triangle = 4a^2k^2 - 12b^2k\]

Obviously, when $\triangle \geq 0$, if

\[a \geq b\sqrt{\frac{3}{k}} \text{ or } a \leq -b\sqrt{\frac{3}{k}}\]

we can get

\[(9)\quad u_1 = \frac{ak - \sqrt{-3b^2k + a^2k^2}}{3bk}\]
\[u_2 = \frac{ak + \sqrt{-3b^2k + a^2k^2}}{3bk}\]

furthermore, when $u < u_1$ or $u > u_2$

\[F'(u) > 0\]

when $u_1 < u < u_2$

\[F'(u) < 0\]

If $\triangle < 0$, if

\[0 \leq a < b\sqrt{\frac{3}{k}}\]

then

\[F'(u) > 0\]
On the other hand, because $a, b, k$ are all bigger than 0, we have

$$0 < u_1 < u_2 < \frac{a}{b}$$

Hence, we can get the conclusion that

**Lemma 1**: If $a > b\sqrt{\frac{3}{k}}$ and $u > \frac{a}{b}$, then $F'(u) > 0$

**Proof**

When $a > b\sqrt{\frac{3}{k}}$, we have

$$0 < u_1 < u_2 < \frac{a}{b}$$

hence, $u > \frac{a}{b}$, $F'(u) > 0$

**Lemma 2**: Function $F(u)$ there and there is only one zero point $u_0$, $u_0 > \frac{a}{b}$ in $(\frac{a}{b}, +\infty)$

**Proof**

If $0 \leq a < b\sqrt{\frac{3}{k}}$, $F'(u) > 0$. If $a \geq b\sqrt{\frac{3}{k}}$, $F'(u) > 0$, when $u > \frac{a}{b}$

On the other hand

$$F(a/b) = -1, \lim_{u \to +\infty} F(u) = +\infty$$

Then, we draw the conclusion

**Theorem 1**: If $a > 0, b > 0, c > 0$, the system (1) there and there is only one positive equilibrium $(u_0, v_0)$, and $u_0 > \frac{a}{b}$

From the **Theorem 1**, we know there must exists a positive equilibrium for system (1), but under what condition the system is stable? As everyone know the system (1) is stable if and only if

$$(10) \quad tr(A) < 0, \quad det(A) > 0$$

where

$$A = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}_{(u_0, v_0)}$$

in this matrix

$$(11) \quad f_u = -b + \frac{2(bu_0 - a)^2}{u_0}, f_v = \frac{-u_0^2}{v_0^2(1 + ku_0^2)}, g_u = 2u_0, g_v = -1$$

then, we get

$$tr(A) < 0 \implies a > bu_0 - \sqrt{\frac{u_0(b + 1)}{2}}$$

$$det(A) > 0 \implies bu_0 - 1 + \sqrt{1 + 2bu_0} < a < bu_0$$

In the end we draw the conclusion
Theorem 2: If parameters of the system (1) satisfy
\[ bu_0 - \sqrt{\frac{u_0(b + 1)}{2}} < a < bu_0, \text{ when } (b > \frac{u_0 - 2}{2} \text{ and } u_0 \geq 1) \] 
\[ or \]
\[ bu_0 - \frac{1 + \sqrt{1 + 2u_0}}{2} < a < bu_0, \text{ when } (b \leq \frac{u_0 - 2}{2} \text{ and } u_0 \geq 1) \]
the positive equilibrium \((u_0, u_0^2)\) is stable.

3. Parameter domains of Turing instability.

Let
\[ w = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} \]
Where \((u_0, v_0)\) is the positive equilibrium of the system (1). Then we have
\[ w_t = rA w + D \nabla^2 w, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \]
Where
\[ A = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}_{(u_0, v_0)} \]
is the linearized matrix of the system (1) at equilibrium \((u_0, v_0)\). By using variable separation method \(^{[10]}\), we have
\[ \nabla^2 W + m^2 W = 0, \quad (n \cdot \nabla)W = 0 \]
Hence, the eigenvalues of linearized system (2) satisfy the equation (15)
\[ |\lambda I - rA| + Dm^2 \] = 0
Therefore we know that the model (2) is stable in the absence of diffusion if and only if
\[ tr(A) < 0, \quad detA > 0 \]
Form (15) we get the equation (16)
\[ \lambda^2 + \lambda[m^2(1 + d) - r(f_u + g_v)] \] + \[ h(m^2) = 0 \]
Here
\[ h(m^2) = dm^4 - r(df_u + g_v)m^2 + r^2 detA \]
Through some calculation, we know if
\[ df_u + g_v > 0 \]
\[ (df_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0 \]
the system (2) with diffusion is unstable.
Finally we get the parameter domains of Turing instability.
\[ \Omega_1 = \{(a, b, d) | a, b, d \text{ satisfied with } (10)\} \]
\[ \Omega_1 = \{(a, b, d) | a, b, d \text{ satisfied with } (18)\} \]
Then the Turing space is $\Omega_1 \cap \Omega_2$. Now we discuss the domain size of the wave number $m$.

From above discussion, we know only when (18) is fulfilled, $h(m^2)$ can be negative. Now suppose (18) is fulfilled, then it follows that

$$m_1^2 < m^2 < m_2^2$$

Where

$$m_1^2 = r\frac{(df_u + g_v) - [(df_u + g_v)^2 - 4d|A|]^{1/2}}{2d}$$

$$m_2^2 = r\frac{(df_u + g_v) + [(df_u + g_v)^2 - 4d|A|]^{1/2}}{2d}$$

Then $h(m^2) < 0$. From (17) we have

$$h_{\min} = r^2|A| - \frac{(df_u + g_v)^2}{4d}$$

Obviously $h_{\min}, m_1^2, m_2^2$ are all functions of bifurcation coefficient $d$, and

$$\frac{dh_{\min}}{dd} = 2r^2(df_u + g_v)f_u(-4d) - r^2(df_u + g_v)^2(-4)$$

$$= \frac{(df_u + g_v)(g_v - df_u)4r^2}{16d^2}$$

Hence $h_{\min}$ decreases with $d$ increasing when $d > g_v/f_u$; $h_{\min}$ increases with $d$ increasing when $d < g_v/f_u$. So $h_{\min}$ will get to the maximum about $d$ when $d = g_v/f_u$.

Suppose that when $d = d_c$, $h_{\min} = 0$, then $d_c$ satisfies

$$d_c^2f_u^2 + 2(2f_vg_u - f_u g_v)d_c + g_v^2 = 0$$

thereafter, we know $d = d_c$ is a bifurcation of the diffusion coefficient and

$$d_c = \frac{-2(2f_vg_u - f_u g_v) \pm (4(2f_vg_u - f_u g_v)^2 - 4f_u^2g_v^2)^{1/2}}{2f_u^2}$$

Accordingly

$$m_c^2 = r\frac{d_c f_u + g_v}{2d_c}$$

In addition, after some calculation, we get

$$\lim_{d \to +\infty} m_1^2 = 0 \quad \lim_{d \to +\infty} m_2^2 = rf_u$$

By such analysis, we know the increasing of $d$ will intensify the instability of this diffusion system.

4. Confined set in the first quadrant

From Smoller[11], we know that if a confined set exists for the model (2) in the absence of diffusion, the same set would also contain the solutions when diffusion is included. So that we want to prove existence of a confined set in the first quadrant for the model (2) in the absence of diffusion.

Let $\Gamma_1$ and $\Gamma_2$ be the Null clines of system (2)

$$\Gamma_1: \quad v = \frac{u^2}{(bu - a)(1 + ku^2)}$$

$$\Gamma_2: \quad v = u^2$$
Then, constructing four lines \( L_1, \ L_2, \ L_3, \ L_4 \), where

\[
L_1 : \quad u = \delta \\
L_2 : \quad u = \sqrt{\frac{\delta^2}{(b\delta - a)(1 + k\delta^2)}} \\
L_3 : \quad u = \delta^2 \\
L_4 : \quad u = \frac{(b\delta - a)(1 + k\delta^2)}{\delta^2}
\]

in which \( \delta \) is a positive number such that \( b\delta - a > 0 \)

**Lemma 3:** There exists a set \( \Psi_1 = \{ \delta | a/b < \delta < \varepsilon_1 \} \), in which \( \delta \) is a positive number such that \((b\delta - a)(1 + k\delta^2) < 1 \) and \( b\delta - a > 0 \).

**Proof.** It is well known that, there exists a right neighborhood of \( a/b \), written as \( \Psi_1 = \{ \delta | a/b < \delta < \varepsilon_1 \} \), in which \((b\delta - a)(1 + k\delta^2) < 1 \)

Then we know \( L_1 \) in on the left of \( L_2 \) and \( L_3 \) is under \( L_4 \)

Obviously, \( \Gamma_1 \) intersects \( L_3 \) at point \((u_1, \delta^2)\), in which \( u_1 \) is the zero point of \( F(u) \)

\[
F(u) = bk\delta^2 u^3 - (ak\delta^2 + 1)u^2 + b\delta^2 u - a\delta^2 \\
= bk\delta^2 ((u^3 - \frac{a}{b} u^2 + \frac{1}{k} u - \frac{a}{bk}) - \frac{1}{bk\delta^2} u^2)
\]

**Lemma 4:** There exists a \( p_0 \) which is independent from \( \delta \) and \( p_0 \) satisfy \( 0 < u_1 < p_0 \)

**Proof.** Because \( \delta > a/b \), we know that

\[
F(u) > T(u) = bk\delta^2 ((u^3 - \frac{a}{b} u^2 + \frac{1}{k} u - \frac{a}{bk}) - \frac{b}{a^2 k} u^2)
\]

Since

\[
\lim_{u \to +\infty} ((u^3 - \frac{a}{b} u^2 + \frac{1}{k} u - \frac{a}{bk}) - \frac{b}{a^2 k} u^2) = +\infty
\]

there must exist a \( p_0 \) which is independent from \( \delta \), and \( F(p_0) > T(p_0) > 0 \)

Lemma 4 is proved

**Theorem 3:** There exists a \( \delta \) which satisfy \((b\delta - a)(1 + k\delta^2) < 1 \) and \( L_2 \) is on the right of \( \Gamma_1 \)

**Proof.** Construct a function

\[
G(\delta) = bk p_0^2 \delta^3 - (ak p_0^2 + 1)\delta^2 + bp_0^2 \delta - a p_0^2
\]

Obviously

\[
G(\frac{a}{b}) = -\frac{a^2}{b^2} < 0 \quad \lim_{\delta \to +\infty} G(\delta) = +\infty
\]

there must exist a right neighborhood of \( a/b \), written as \( \Psi_1 = \{ \delta | a/b < \delta < \varepsilon_2 \} \), in which \( G(\delta) < 0 \)

Now let \( \Psi = \{ \delta | a/b < \delta < \min(\varepsilon_1, \varepsilon_2) \} = \Psi_1 \cap \Psi_2 \)

We get a \( \delta \) from \( \Psi \), which satisfy

\[
p_0 < \sqrt{\delta^2/(b\delta - a)(1 + k\delta^2)}
\]

In addition, through Lemma 1 and Lemma 2 we get

\[
u_1 < \sqrt{\delta^2/(b\delta - a)(1 + k\delta^2)}
\]
We complete the proof of Theorem 3.

In the end, let’s analyze the sign of $f(u, v)$ and $g(u, v)$ on each side of null clines.

\[ f(u, v) = a - bu + \frac{u^2}{v(1 + ku^2)} \]
\[ g(u, v) = u^2 - v \]

we can draw a conclusion.

\[ \vec{n}_i \cdot (f(u, v), g(u, v)) < 0 \quad (i = 1, 2, 3, 4) \]

in which $\vec{n}_i$ is unit vector which is vertical to $L_i$

$\vec{n}_1 = \{-1, 0\} \quad \vec{n}_2 = \{1, 0\} \quad \vec{n}_3 = \{0, -1\} \quad \vec{n}_4 = \{0, 1\}$

Now we draw the conclusion that these four lines: $L_1, \quad L_2, \quad L_3, \quad L_4$ form the boundary of a confined set from which we know a spatially inhomogeneous solution will emerge.

5. Conclusion.

With the fast development of natural science, how to apply science theory into practice is becoming an important aspect. Since the last century, there have been a lot of famous scientists who applied mathematic method into ecological research, and got many brilliant fruits [1-8].

In this paper we study the underlying mechanism, which can generate complex spatial patterns and kinetics for a predator-prey model spatial patterns term, discuss diffusion-driven instability and some bifurcation values: diffusion coefficient $d_c$, wave number $m_c$. In the end we prove that there exists a confined set in the first quadrant.

The result of this paper can be applied into practice for the pest control.

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