FURTHER RESULTS ON ASYMTOTIC STABILITY OF LINEAR DISCRETE TIME DELAY AUTONOMOUS SYSTEMS

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Abstract. Our aim, in this communication, is to improve the existing results concerning asymptotic stability of a particular class of linear discrete time delay systems. In that sense, this paper offers a new sufficient condition for delay-dependent asymptotic stability of systems of the form \( x(k+1) = A_0x(k) + A_1x(k-h) \). A methodology used here is based on Lyapunov approach. Numerical computations are presented for illustration and comparison with the previous results.

Key Words. Discrete time delay systems, asymptotic stability, sufficient conditions

1. Introduction

In the existing stability criteria, two ways of approach have been mainly adopted. Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides simple algebraic conditions.

A considerable interest has been permanently shown in the problem of asymptotic stability of time delay systems. It is obvious that there are much more published papers in the area of continuous than discrete time delay systems. Certainly, one of the basic reasons for that lies in the fact that discrete time delay systems are of finite dimensions, so the equivalent systems of considerably high order can be easily built [3]. This paper overcomes this need and the whole procedure is taken on the basic system matrices, what enables one to check system stability without any particular effort.

For the sake of brevity, in the sequel, we present a short overview of different results only in the area of linear discrete time delay systems with particular contributions in the field of system stability investigations. Namely, [4] was the first to pay attention to this class of systems solving a synthesis problem for control of the systems governed by linear differential-difference equations. It has been shown, in the same paper, that such systems are equivalent to infinite dimensional difference equations whose matrix elements can be readily calculated by recursive formulas. Some results, concerning stability in the sense of Lyapunov, were also derived. A more general discussion concerning different aspects of continuous and discrete time delay systems can be found, also, in [4] with particular attention to optimal control. Several sufficient conditions for asymptotic stability of linear discrete-delay systems were presented in [9]. Since these conditions are independent of delay and possess simple forms, they provide useful tools to check system stability at the first stage.

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The study of stabilization problem for general decentralized large-scale linear continuous and discrete time delay systems using local feedback controllers were presented in [6]. The local feedback control was assumed to be memory less. In that sense, the sufficient stabilization conditions were established. The problem of delays in interconnections, for the same class of systems, was studied later in [7]. The paper of [12] presents some new sufficient conditions for robust and D-stability of discrete-delay perturbed systems. It has been shown that these results are less conservative than those reported in literature, particularly to [9].

Based on a derived algebraic inequality a criterion to guarantee the robust stabilization and state estimation for perturbed discrete-time delay large scale systems was proposed in [13]. That criterion is independent of time delay and does not need the solution of Lyapunov or Riccati equation.

Paper of [10] extends some of the basic results in the area of Lyapunov (asymptotic) to linear, discrete, time invariant time-delay systems. These results are given in the form of only sufficient conditions and represent other generalization of some previous ones or completely new results. In the latter case it is easy to show that, in most cases, these results are less conservative than those in the existing literature. The paper of [11] presents some new sufficient conditions for asymptotic stability of a particular class of linear perturbed time-delay systems with multiple delays. The results obtained there, compared to the corresponding in the current literature, show definite improvements and are less restrictive. Since these conditions are independent of delay and are in simple forms, they provide an easy way to investigate system asymptotic stability and robustness.

The practical and finite time stability of linear discrete time delay systems were, for the first time, investigated in paper of [2]. Some further improvements of the results of [9] in the area of Lyapunov stability were also given in the before mentioned paper.

2. Notations and preliminaries

\[ \mathbb{R} \] Real vector space

\[ F = (f_{ij}) \in \mathbb{R}^{n \times n} \] Real matrix

\[ F^T \] Transpose of matrix \( F \)

\[ F > 0 \] Positive definite matrix

\[ F \geq 0 \] Positive semi definite matrix

\[ \lambda(F) \] Eigenvalue of matrix \( F \)

\[ \sigma(F) = \|F\| \] Singular value of matrix

\[ \|F\| = \sqrt{\lambda_{\text{max}}(F^TF)} \] Euclidean matrix norm of \( F \)

A linear, autonomous, multivariable discrete time-delay system can be represented by the difference equation

\[ x(k + 1) = A_0 x(k) + A_1 x(k - h), \tag{2.1} \]

with an associated function of initial state

\[ x(\theta) = \psi(\theta), \quad \theta \in \{-h, -h+1, \ldots, 0\}. \tag{2.2} \]

The equation (2.1) is referred to as homogenous or the unforced state equation. \( x(k) \in \mathbb{R}^n \) is a state vector and \( A \in \mathbb{R}^{n \times n} \), is a constant matrix of appropriate dimension, and pure system time delay is expressed by integers \( h = 1, 2, \ldots \).

Let \( V : \mathbb{R}^n \to \mathbb{R} \), so that \( V(x) \) is bounded for and for which \( \|x\| \) is also bounded
3. Main results

Theorem 3.1. If for any given matrix \( Q = Q^T > 0 \) there exists matrix \( P = P^T > 0 \) such that the following matrix equation is fulfilled.

\[
2A_0^T P A_0 + 2A_1^T P A_1 - P = -Q,
\]
then, system (2.1) is asymptotically stable.

Proof. Let the Lyapunov functional be

\[
V(x_k) = x^T(k) P x(k) + \sum_{j=1}^{h} x^T(k-j) S x(k-j),
\]

(3.2)

\[ P = P^T > 0, \quad S = S^T \geq 0, \]

The forward difference along the solutions of system (2.1) is

\[
\Delta V(x_k) = x^T(k) \left( A_0^T P A_0 - P + S \right) x(k)
\]

(3.3)

\[
\Delta V(x_k) = x^T(k) \left( 2A_0^T P A_0 + 2A_1^T P A_1 - P \right) x(k)
\]

(3.4)

\[
\Delta V(x_k) = x^T(k) \left( 2A_0^T P A_0 + 2A_1^T P A_1 - P \right) x(k)
\]

or

\[
\Delta V(x_k) = x^T(k) \left( 2A_0^T P A_0 + 2A_1^T P A_1 - P \right) x(k)
\]

(3.5)

(3.6)

Then, it is obvious that

\[
\Delta V(x_k) \leq x^T(k) \left( 2A_0^T P A_0 + 2A_1^T P A_1 - P \right) x(k).
\]

If for any given matrix \( Q = Q^T > 0 \) there exists matrix \( P = P^T > 0 \) being solution of (3.1), then certainly follows

\[
V(x_k) > 0, \quad \Delta V(x_k) < 0, \quad \forall x_k \neq 0,
\]

so, system (2.1) is asymptotically stable. Q.E.D.

Corollary 3.1. The system (2.1) is asymptotically stable, independent of delay, if

\[
\frac{\sigma^2_{\text{max}}(A_1)}{2\sigma^2_{\text{max}}(P^2)} < \frac{\lambda_{\min}(2Q-P)}{2\sigma^2_{\text{max}}(P^2)},
\]

(3.9)

where, for any given matrix \( Q = Q^T > 0 \) there exists matrix \( P = P^T > 0 \) being the solution of the following Lyapunov matrix equation

\[
A^T_0 P A_0 - P = -Q.
\]

(3.10)
Proof
On the basis of Theorem (3.1), system (2.1) is asymptotically stable if the following condition is satisfied

\[(3.11) \quad 2A_0^T PA_0 + 2A_1^T PA_1 - P < 0.\]

This demand can be divided into two particular stability conditions:

(i): For any given matrix \( Q = Q^T > 0 \) there exist a matrix \( P = P^T > 0 \) being solution of the Lyapunov matrix equation (3.10),

or:

(ii): Matrix \( P = P^T > 0 \) satisfies the following condition

\[(3.12) \quad 2A_1^T PA_1 - (2Q - P) < 0.\]

First condition is equivalent to the demand that \( A_0 \) is discrete stable matrix. If the answer is positive, one has only to check the condition, (3.12), for the system (2.1) to be asymptotically stable.

If matrix \( A_0 \) is discrete stable then from (3.10) follows \( P - Q \geq 0 \) and from (3.12) follows \( 2Q - P > 0 \).

So, it is obvious

\[(3.13) \quad 0 < 2Q - P \leq Q.\]

This inequality will be satisfied if and only if

\[(3.14) \quad \lambda_{\text{min}} (2Q - P) > 0,\]

\[(3.15) \quad \lambda_{\text{max}} (2Q - P) \leq \lambda_{\text{min}} (Q),\]

[1].

Inequality (3.12) is equivalent to the following condition

\[(3.16) \quad \lambda_i (2A_1^T PA_1 - (2Q - P)) < 0.\]

Since all matrices in (3.16) are real and symmetric, then using Weyls inequality, [1], it follows

\[
\lambda_i (2A_1^T PA_1 + P - 2Q) \leq \lambda_{\text{max}} (2A_1^T PA_1 + P - 2Q) \\
\leq \lambda_{\text{max}} (2A_1^T PA_1) - \lambda_{\text{min}} (2Q - P) \\
= 2\lambda_{\text{max}} (A_1^T P^T P A_1) - \lambda_{\text{min}} (2Q - P) \\
= 2\sigma_{\text{max}} (P^T P A_1) - \lambda_{\text{min}} (2Q - P) \\
\leq 2\sigma_{\text{max}}^2 (A_1) \sigma_{\text{max}} (P^T) - \lambda_{\text{min}} (2Q - P). 
\]

So, if condition

\[(3.18) \quad 2\sigma_{\text{max}}^2 (A_1) \sigma_{\text{max}} (P^T) - \lambda_{\text{min}} (2Q - P) < 0,\]

is satisfied, system (2.1) will be asymptotically stable.

From (3.18) finally follows (3.9).

If (3.10) is not satisfied this Corollary (3.1) can not be used in further stability system investigation.
Having in mind the inequality (3.14), the right-side of (3.10) is always positive. If this is not true, condition (3.10) has no any sense, since its left-side is always positive and less than right-side. This ends the Corollary (3.1). Q.E.D.

**Corollary 3.2.** The system (2.1) is asymptotically stable, independent of delay, if

\[
\sigma_\text{max}^2(A_1) < \frac{\lambda_\text{min}(Q)}{2\sigma_\text{max}^2(P^{\frac{1}{2}})},
\]

where, for any given matrix \( Q = Q^T > 0 \) there exists matrix \( P = P^T > 0 \) being the solution of the following Lyapunov matrix equation

\[
2A_0^TPA_0 - P = -Q. \tag{3.20}
\]

**Proof.** Following the idea of Corollary (3.1), the demand given by (3.11) can be divided into two particular stability conditions:

(i): For any given matrix \( Q = Q^T > 0 \) there exists a matrix \( P = P^T > 0 \) being solution of the Lyapunov matrix equation (3.20), or:

(ii): Matrix \( P = P^T > 0 \) satisfies the following condition

\[
2A_0^TPA_1 - Q < 0. \tag{3.21}
\]

It is obvious that if condition (3.19) is valid, condition given by (3.21) is also satisfied. In the sequel the rest of the proof is identical to that given in Corollary (3.1) and is omitted here for the sake of brevity. Q.E.D.

**Conclusion 3.1** In preceding Corollaries the basic demand, given by (3.1) has been divided into two parts. First condition given by (3.10) and (3.20) is eliminate, and if one of these conditions is satisfied, then one of the conditions given by (2.1) or (3.19) is used to clarify the final asymptotic system stability conclusion.

Conditions (3.10) and (3.20) are satisfied if and only if \( A_0 \) or \( \sqrt{2}A_0 \) are discrete stable matrices, respectively. Then it follows that Corollary (3.2) is more restrictive then Corollary (3.1) in view of first condition.

However, this is true only if

\[
\frac{1}{\sqrt{2}} \leq |\lambda_i(A_0)| < 1. \tag{3.22}
\]

Let

\[
|\lambda_i(A_0)| < \frac{1}{\sqrt{2}}, \tag{3.23}
\]

be satisfied. Moreover let us use the same matrix in Corollary (3.1) and (3.2). If all eigenvalues of matrix \( A_0 \) satisfy condition \( |\lambda_i(A_0)| < \frac{1}{\sqrt{2}} \).

Let us now take the same matrix \( Q \), when we use Corollary (3.1) and (3.2) in system stability investigation. Then from (3.13) follows

\[
\lambda_\text{min}(2Q - P) \leq \lambda_\text{max}(2Q - P) \leq \lambda_\text{min}(Q), \tag{3.24}
\]

e.g., we may say that minimal eigenvalue of matrix \( Q \) will never be less than minimal eigenvalue of matrix \( 2Q - P \).

When we have \( P = Q \), then one should use sign of equality in (3.22). This may be
when $A_0$ is null matrix, (3.9).
It follows that condition given by (3.19-3.20) are less restrictiven than conditions
given by (3.9-3.10).
Since from Theorem (3.1) follows Corollary (3.1) and (3.2), these conditions are
less restrictive than those given by the conditions of Corollary (3.1) and (3.2).

4. Illustrative example

To illustrate the superiority of results presented, we develop numerical example
Example 1. Let us consider a discrete delay system described by
\[ x(k + 1) = A_0 x(k) + \varphi A_1 x(k - h), \]
where $\varphi$ is adjustable parameter and scalar parameter $a$ takes the following values:
$-0.15$ and $0.50$.
The delay - independent asymptotic stability conditions are obtained in terms of
$\varphi$ and are summarized in Table 4.1 and compared with some other given in the
current literature: [9], [10], [12].

5. Conclusion

In this paper, new sufficient conditions for delay - independent asymptotic stability
are presented.
It is demonstrated that these results are less restrictive than some present in the
current literature.

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