SOME DYNAMIC CHARACTERS OF A REACTIVE DIFFUSION EQUATION WITH ENZYME REACTANTS

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Abstract. The kinetics parameters of the models driven by diffusion, the ratio of diffusion coefficients easily induces complex spatial patterns. In this paper, firstly, we study some dynamic characters of a reactive diffusion equation with enzyme reactants without any feedback control terms, obtaining the condition of equilibrium, its stability, the parameter domains of diffusion-driven instability and some bifurcation values; then we study some dynamic characters of a reactive diffusion equation with enzyme reactants with feedback control terms \( m_u \) and \( m_v \), obtaining the condition of equilibrium, its stability, the parameter domains of diffusion-driven instability and some bifurcation values, such as diffusion coefficient \( d_c \), wave-number \( p_c \).

Key Words. Diffusion-driven instability, Reaction and diffusion equation, Wave-number, Enzyme reactants,

1. Introduction

Every single particle can move in a stochastic way when it is moving or reacting. Scientific practices show that the outward diffusion of the matter results in some random movement of single particle. It is called diffusion course when those microcosmic random movements induce some macroscopically regular movement. Diffusion exists among all matters, diffusion phenomena can be seen almost everywhere[1]. Turing (1952) suggested that, under certain conditions, chemicals could react and diffuse in such a way as to produce steady heterogeneous spatial patterns of chemical or morphogenic concentration. Therefore, it is very important to study some dynamic characters of such models. Murray studied a reactive diffusion equation which was given by Schnakenberg (1979) in details. Thomas (1975) considered a special reaction equation involving the substrates oxygen and uric acid. By some dimensionless transformation, this model can be written as

\[
\begin{align*}
{u_t} &= r[a - u - \rho \cdot R(u, v)] + \nabla^2 u \\
{v_t} &= r[\alpha(b - v) - \rho \cdot R(u, v)] + d \nabla^2 v
\end{align*}
\]

where \( \alpha, \rho, a, b, d, k \) are positive constants, and

\[ R(u, v) = \frac{u \cdot v}{1 + u + k \cdot u^2} \]

here, \( k > \frac{1}{4} \). \( d \) is called diffusion coefficient, \( u \) denotes uric acid concentration, \( v \) denotes oxygen concentration[2].
In this paper, firstly, we study some dynamic characters of a reactive diffusion equation with enzyme reactants without any feedback control terms, obtaining the condition of equilibrium, its stability and the parameter domains of diffusion-driven instability and some bifurcation values; secondly, we study some dynamic characters of a reactive diffusion equation with enzyme reactants with feedback control terms \( \mu \) and \( \nu \), obtaining the condition of equilibrium and its stability and the parameter domains of diffusion-driven instability and some bifurcation values, such as diffusion coefficient \( d \), wave-number \( p \) and so on.

2. Diffusion Driven Instability without any control terms

2.1. Equilibrium and Stability.

In order to obtain some conditions for the reactive diffusion equation to be unstable, we study the problem of equilibrium and stability of the model (1) in the absence of diffusion term. Let

\[
\begin{align*}
    f(u, v) &= r[a - u - \frac{\rho uv}{1 + u + ku^2}] \\
    g(u, v) &= r[a(b - v) - \frac{\rho uv}{1 + u + ku^2}]
\end{align*}
\]

Then it is easy to find the equilibrium satisfying the following equation [3]

\[
\begin{align*}
    a - u &= \frac{\rho uv}{1 + u + ku^2} \\
    \alpha(b - v) &= \frac{\rho uv}{1 + u + ku^2}
\end{align*}
\]

Via some calculations, we get a cubic algebraic equation about vector \( u \), i.e.,

\[\alpha ku^3 + (\alpha + \rho - \alpha a)u^2 + (\alpha + \alpha \rho b - \alpha a - \rho a)u - \alpha = 0.\]

From practical point of view, here we only care about the existence of the positive equilibrium. Let

\[f(u) = \alpha ku^3 + (\alpha + \rho - \alpha a)u^2 + (\alpha + \alpha \rho b - \alpha a - \rho a)u - \alpha a\]

then we have

\[\lim_{u \to +\infty} f(u) = +\infty, \quad \lim_{u \to 0^+} f(u) = -\alpha a < 0, \quad \lim_{u \to -\infty} f(u) = -\infty\]

So that we know that the positive roots of equation (3) are likely to be one of the following three cases:

(1) Three identical positive roots;
(2) One positive single root, two identical positive roots;
(3) Three different positive roots.

Let \( \Gamma_1 \) and \( \Gamma_2 \) be the isoclines of the model (2), i.e.,

\[
\begin{align*}
    \Gamma_1 : F(u) &= \frac{(a - u)(1 + u + ku^2)}{\rho u} \\
    \Gamma_2 : G(u) &= \frac{\alpha b(1 + u + ku^2)}{\alpha ku^2 + \alpha + \rho u + \alpha}
\end{align*}
\]

Clearly, \( \Gamma_1 \) has no asymptote, the asymptote of \( \Gamma_2 \) is \( v = b \). Because of \( \alpha b(1 + u + ku^2) \neq 0 \), we know \( \Gamma_2 \) has no intersection with the \( u \) axes. And when \( \rho ak - 1 < 0 \), and \( 0 < \rho < 1 \), when \( \alpha < a \), \( F'(u) < 0 \), where

\[F'(u) = \frac{-2ku^3 + (ka - 1)u^2 - a}{\rho a^2}\]
Thus we have the following conclusion:

**Theorem 1:** When \( \frac{1}{4} < k < \frac{1}{2} \), there exists a unique positive equilibrium: \((u_0, v_0)\) with \(u_0 < a, v_0 < b\).

**Proof:** For \( \Gamma_1 \), we know

\[
\lim_{u \to +\infty} f(u) = -\infty, \quad \lim_{u \to 0} f(u) = +\infty
\]

and when \( ka - 1 < 0, F'(u) < 0 \), hence \( F(u) \) is digressive monotonously in \((0, +\infty)[4]\).

Because the shape of \( \Gamma_2, \Gamma_1 \) and \( \Gamma_2 \) have only one intersection \((u_0, v_0)\) in the first quadrant. Clearly, \( v_0 < b \) and \( v_0 = \frac{ma-a}{2} + b \), so \( u_0 < a \). The proof is completed.

We will next discuss stability problem of positive equilibrium. Let

\[
\omega = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}
\]

we can get the community matrix

\[
A = \begin{pmatrix} -1 - \frac{\rho v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} & -\frac{\rho u_0}{1 + u_0 + ku_0^2} \\ -\frac{\rho u_0}{1 + u_0 + ku_0^2} & -\alpha - \frac{\rho u_0}{1 + u_0 + ku_0^2} \end{pmatrix}
\]

Through some computing, we know

(6) \(\text{tr}A = -1 - \frac{\rho v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} - \alpha - \frac{\rho u_0}{1 + u_0 + ku_0^2}\)

(7) \(\det A = \alpha + \frac{\alpha \rho v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} + \frac{\rho u_0}{1 + u_0 + ku_0^2}\)

Clearly, \(\text{tr}A < 0, \det A > 0\) iff

(8) \((1 + \alpha)(1 + u_0 + k^2_0)^2 + \rho v_0 (1 - k^2_0) + \rho u_0 (1 + u_0 + ku_0^2) > 0\)

(9) \(\alpha (1 + u_0 + k^2_0)^2 + \rho u_0 (1 + u_0 + ku_0^2) + \alpha \rho v_0 (1 - k^2_0) > 0\)

Hence we have the following conclusion:

**Theorem 2:**

If \( \alpha \leq 1 \) and \( u_0 < \sqrt{\frac{a}{k}} \), then the equilibrium \((u_0, v_0)\) is locally asymptotically stable.

**Proof:** Clearly, when \( \alpha \leq 1 \) holds, we have \(\text{tr}A < 0\). For \( u_0 < \sqrt{\frac{a}{k}}, \det A > 0 \), so the conclusion is established.

### 2.2. Diffusion Driven Instability in one space dimension.

In the following, we only discuss instability in one space dimension, namely

(10) \[
\begin{cases}
u_t = r[a - u - \frac{\rho a v}{1 + u + k u^2}] + u_{xx} \\
v_t = r[b - v - \frac{\rho a v}{1 + u + k u^2}] + v_{xx}
\end{cases}
\]

By a coordinate transform of the model of (10)

\[
\omega = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}
\]
we obtain \( \omega_t = rA\omega + D\nabla^2\omega \).

Here
\[
D = \begin{pmatrix}
1 & 0 \\
0 & d
\end{pmatrix}
\]

(11)

\[
A = \begin{pmatrix}
-1 - \frac{\rho_n(1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} & -\frac{\rho_n}{1 + u_0 + ku_0^2} \\
\rho_n(1 - ku_0^2) & -\alpha - \frac{\rho_n}{1 + u_0 + ku_0^2}
\end{pmatrix}
\]

Let
\[
\omega(r, t) = \sum_p C_p e^{i\lambda p} \omega_p(r)
\]
we get
\[
(\lambda - rA + DK^2)\omega_p = 0
\]
where \( p \) is called wave-number. Because \( \omega_p \neq 0 \), we have
\[
(12) \quad \text{det}(\lambda I - rA + Dp^2) = 0
\]
Certainly, \( \lambda \) is the function of the wave-number \( p \). So when \( Re(\lambda(p)) > 0 \), the stable equilibrium \((u_0, v_0)\) is unstable, it will bring about some spatial diffusion phenomenon.

From (12), we can get
(13) \quad \lambda^2 + \lambda[p^2(1 + d) - rtrA] + h(p^2) = 0

Therefore
\[
h(p^2) = dp^4 - r \left[ d(-1 - \frac{\rho_n(1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2}) - \alpha - \frac{\rho_n}{1 + u_0 + ku_0^2} \right] p^2 + r^2 \text{det}(A)
\]

If \( trA < 0 \), and because \( p^2(1 + d) > 0 \) for an arbitrary \( p \), we have
\[
p^2(1 + d) - rtrA > 0
\]
so \( Re(\lambda(p)) > 0 \) if \( hp^2 < 0 \) for some \( p \). If \( h_{min} < 0 \), then there exists \( p_1^2, p_2^2 \), such that \( h(p^2) < 0 \) when \( p_1^2 < p^2 < p_2^2 \).

According to the above discussion, we have the following conclusion:

**Theorem 3:** If the following condition is satisfied
\[
(14) \quad [d(-1 - \frac{\rho_n(1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2}) - \alpha - \frac{\rho_n}{1 + u_0 + ku_0^2}] > 4d \cdot \text{det}(A)
\]
then there are a series of \( p \) to make the equilibrium of system (10) is unstable.

Proof: when (14) is satisfied, we have
\[
h_{min} = r^2 \text{det}(A) - \frac{r^2}{4d} [d(-1 - \frac{\rho_n(1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2}) - \alpha - \frac{\rho_n}{1 + u_0 + ku_0^2}]^2 < 0
\]
so there exist such \( p^2 \neq 0 \) that \( hp^2 < 0 \) is right. The signs of the two roots of the equation
\[
\lambda^2 + \lambda[p^2(1 + d) - r \cdot trA] + h(p^2) = 0
\]
is thus different, then the equilibrium is unstable. The proof is completed.

The parameter values of making \( h_{min} = 0 \) have very important significance. We call it the bifurcation values[6]. They satisfy
\[
(15) \quad \text{det}(A) = \frac{1}{4d} \left[ d(-1 - \frac{\rho_n(1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2}) - \alpha - \frac{\rho_n}{1 + u_0 + ku_0^2} \right]^2
\]
again
\[ det(A) = \alpha + \frac{\alpha \varphi v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} + \frac{\rho u_0}{1 + u_0 + ku_0^2} \]
We obtain the equation, which is satisfied by bifurcation \( d_c > 1 \) of diffusion coefficient ratio:
\[ d_c^2 f_u^2 + 2(f_c g_u - f_u g_c) d_c + g_c^2 = 0 \]
here
\[ f_u = -1 - \frac{\rho v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2}, \quad f_v = -\frac{\rho u_0}{1 + u_0 + ku_0^2}, \]
\[ g_u = -\frac{\rho v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2}, \quad g_v = -\alpha - \frac{\rho u_0}{1 + u_0 + ku_0^2}. \]
Therefore, the bifurcation \( p_c \) of the wave-number is given by the following formula
\[ p_c^2 = r \cdot \frac{d_c f_u + g_v}{2 d_c} = r \left[ \frac{det(A)}{d_c} \right]^{1/2} = r \left[ \frac{f_u g_u - f_v g_v}{d_c} \right]^{1/2} \]
Clearly, we have
\[ d_c = \frac{-2 (f_c g_u - f_u g_v) \pm \sqrt{16 f_c^2 g_u^2 - 16 f_c g_u f_u g_v}}{2 f_u^2} \]
Let
\[ \Delta = -16 \rho^2 u_0 v_0 (1 - ku_0^2) (1 + u_0 + ku_0^2)^5 \left[ \alpha - \frac{\alpha \varphi v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} - \frac{\rho u_0}{1 + u_0 + ku_0^2} \right]. \]

**Theorem 4:**
(a) when \( \alpha > \frac{\alpha \varphi v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} + \frac{\rho u_0}{1 + u_0 + ku_0^2}, 1 - ku_0^2 > 0 \) \[ holds, there does not exist any bifurcation of diffusion coefficient ratio; \]
(b) when \( 1 - ku_0^2 = 0, \Delta = 0, \) there exists a bifurcation of diffusion coefficient ratio:
\[ d_c = \frac{\alpha (2 \sqrt{k} + 1) + \rho}{(2 \sqrt{k} + 1)} \]
simultaneously, there exist two bifurcations of critical wave-number
\[ p_c^2 = r \]
(c) when \( \alpha < \frac{\alpha \varphi v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} + \frac{\rho u_0}{1 + u_0 + ku_0^2}, 1 - ku_0^2 > 0 \) \[ holds, there exist two bifurcations \( d_c \) of diffusion coefficient ratio; \]
(d) when \( 1 - ku_0^2 < 0 \) and
\[ \alpha - \frac{\alpha \varphi v_0 (1 - ku_0^2)}{(1 + u_0 + ku_0^2)^2} \geq \frac{\rho u_0}{1 + u_0 + ku_0^2}, \]
there also exist two bifurcations \( d_c \) of diffusion coefficient ratio.

**Proof:** For (a), when the hypothesis holds, we have \( \Delta < 0 \), so that the equation about \( d_c \) has no real solutions, which means there does not exist bifurcation of diffusion coefficient ratio. For (b), the equation about \( d_c \) has a root. For (c), when given conditions are satisfied, \( \Delta > 0 \), hence there exist two bifurcations \( d_c \) of diffusion coefficient ratio. In a similar way we can prove (d) is true.
2.3. A Numerical Simulation.

Let \( a = \frac{3}{4}, \alpha = \frac{3}{4}, k = 1, \rho = 2, b = 3 \) then equation (2) becomes

\[
\begin{align*}
    u_t &= \frac{3}{4} - u - \frac{2uv}{1 + u + u^2} \\
    v_t &= \frac{3}{4}(3 - v) - \frac{2uv}{1 + u + u^2}
\end{align*}
\]

The equilibrium of (16) should satisfy the following equation

\[
12u^3 + 35u^2 + 51u - 9 = 0
\]

Using Mathematica, we can obtain the real solution of (17) which is 

\[ u_0 = 0.158332 \]

because

\[ v_0 = b - \frac{a - u_0}{\alpha} \]

we have

\[ v_0 = 2.21111 \]

Substitution of \( u_0 = 0.158332, v_0 = 2.21111 \) into (8) and (9) shows that the conditions of (8) and (9) are satisfied. So the equilibrium \( (0.158332, 2.21111) \) is locally asymptotically stable. For an arbitrary real number \( d \), we have

\[ 25.79182d^2 - 1.28913d + 1.035485 > 0 \]

So we know from Theorem 3 that there is a series of wave-numbers \( p \) to make the system (10) unstable.

3. Diffusion Driven Instability with feedback control terms

Now we discuss the dynamic characters of equation (2) after adding feedback control terms \( mu \) and \( nv \):

3.1. Equilibrium and Stability.

In order to obtain some conditions for the reactive diffusion equation to be unstable, we study the problem of equilibrium and stability of the model (1) with diffusion. Let

\[
\begin{align*}
    f(u, v) &= r[a + (m - 1)u - \frac{\rho uv}{1 + u + ku^2}] \\
    g(u, v) &= r[ab + (n - \alpha)v - \frac{\rho uv}{1 + u + ku^2}]
\end{align*}
\]

Then it is easy to find the equilibrium satisfying the following equation[3]

\[
\begin{align*}
    a + (m - 1)u &= \frac{\rho uv}{1 + u + ku^2} \\
    ab + (n - \alpha)v &= \frac{\rho uv}{1 + u + ku^2}
\end{align*}
\]

By some calculations, we get a cubic algebraic equation about the vector \( u \):

\[
k(n - \alpha)(m - 1)u^3 + [(n - \alpha)ak - (m - 1)\rho]u^2 + [(n - \alpha)(m - 1) + (n - \alpha)a - \rho(a - ab)]u + (n - \alpha)a = 0
\]

where \( \alpha \neq n, m \neq 1 \). From a practical point of view, here we only care about the existence of the positive equilibrium. Let

\[
f(u) = k(n - \alpha)(m - 1)u^3 + [(n - \alpha)ak - (m - 1)\rho]u^2 + [(n - \alpha)(m - 1) + (n - \alpha)a - \rho(a - ab)]u + (n - \alpha)a
\]

and if \( n < \alpha, m < 1 \), then we have

\[
\lim_{u \to +\infty} f(u) = +\infty, \quad \lim_{u \to -\alpha} f(u) = (n - \alpha)a < 0, \quad \lim_{n \to -\infty} f(u) = -\infty
\]

So we know that the positive roots of equation (3) are likely to be one of the following three cases:
(1) Three identical positive roots;
(2) One positive single root, two identical positive roots;
(3) Three different positive roots.

Let $\Gamma_1$ and $\Gamma_2$ are the isoclines of the model (2)

\begin{align*}
\Gamma_1 : F(u) &= \frac{(a + (m - 1)u)(1 + u + ku^2)}{\rho u} \\
\Gamma_2 : G(u) &= \frac{ab(1 + u + ku^2)}{(\alpha - n)ku^2 + (\alpha + \rho - n)u + (\alpha - n)}
\end{align*}

Clearly, $\Gamma_1$ has no asymptote, and the asymptote of $\Gamma_2$ is $v = -\frac{ab}{n - \alpha}$. Because of $ab(1 + u + ku^2) \neq 0$, we know $\Gamma_2$ has no intersection with the $u$ axes. And when $(m - 1 + \rho ak) < 0$, and $0 < \rho < 1$, when $\alpha < a$, $F'(u) < 0$, where

$$F'(u) = \frac{k(m - 1)(\rho + 1)u^3 + (m - 1 + ka\rho)u^2 + [\rho(\alpha - a) + (m - 1)(1 - \rho)]u - \rho a}{\rho^2 u^2}$$

Thus we have the following conclusion:

**Theorem 5**: When $\alpha = n$, $m \neq 1$ and $(ab - a)(m - 1) > 0$, there exists a unique positive equilibrium:

$$u_0 = \frac{ab - a}{m - 1}, \quad v_0 = \frac{ab[1 + \frac{ab - a}{m - 1} + k(\frac{ab - a^2}{m - 1})]}{\rho \cdot \frac{ab - a}{m - 1}}$$

When $\alpha < m - 1 + \rho ak < 0$, there exists a unique positive equilibrium $(u_0, v_0)$, and $0 < u_0 < \frac{a}{1 - m}, 0 < v_0 < \frac{ab}{\alpha - n}$.

Proof: when $\alpha = n, m \neq 1, u_0 = \frac{ab - a}{m - 1}, v_0 = \frac{ab(1 + u + ku^2)}{\rho u}$.

we have

$$v_0 = \frac{\alpha b[1 + \frac{\alpha b - a}{m - 1} + k(\frac{\alpha b - a^2}{m - 1})]}{\rho \cdot \frac{\alpha b - a}{m - 1}}$$

For $\alpha < m < 1 - \rho ak < 1$,

$$\lim_{u \to +\infty} f(u) = -\infty, \quad \lim_{u \to 0^+} f(u) = +\infty$$

and $F'(u) < 0$, hence $F(u)$ is digressive monotonously in $(0, +\infty)[4]$.

Because the shape of $\Gamma_2$, $\Gamma_1$ and $\Gamma_2$ have only one intersection $(u_0, v_0)$ in the first quadrant.

Clearly, $0 < v_0 < \frac{ab}{\alpha - n}$ and $v_0 = \frac{a + (m - 1)u - ab}{n - \alpha}$, so $0 < u_0 < \frac{a}{1 - m}$. The proof is completed.
We will discuss the stability problem of positive equilibrium. Let
\[ \omega = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} \]
we can get the community matrix
\[ A = \begin{pmatrix} m - 1 - \frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2} & -\frac{\rho u_0}{1+u_0+k u_0^2} \\ -\frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2} & n - \alpha - \frac{\rho u_0}{1+u_0+k u_0^2} \end{pmatrix} \]
Through some computations, we know
\[ trA = m - 1 - \frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2} + n - \alpha - \frac{\rho u_0}{1+u_0+k u_0^2} \]
Clearly, \( trA < 0, detA > 0 \) iff
\[ (m-1+n-\alpha)(1+u_0+k u_0^2)^2 + \rho v_0(1-k u_0^2) + \rho u_0(1+u_0+k u_0^2) > 0 \]
\[ (m-1+n-\alpha)(1+u_0+k u_0^2)^2 + \rho v_0(1-k u_0^2) + \rho u_0(1+u_0+k u_0^2) > 0 \]
Hence we have the following conclusion:
\[ \text{Theorem 6: If (8) holds and } u_0 < \sqrt{\frac{1}{k}}, \text{ then the equilibrium } (u_0, v_0) \text{ is local asymptotically stable.} \]
Proof: Clearly, when (8) holds, we have \( trA < 0 \). For \( u_0 < \sqrt{\frac{1}{k}}, detA > 0 \), so the conclusion is true.

3.2. Some bifurcation values for diffusion-driven instability. In the following, using the same way of (2.2), we get the following condition:

\[ det(A) = \frac{1}{4d} \cdot \left[ d(m - 1 - \frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2}) + n - \alpha - \frac{\rho u_0}{1+u_0+k u_0^2} \right]^2 \]
again
\[ det(A) = (m - 1)(n - \alpha) - (n - \alpha) \cdot \frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2} - (m - 1) \cdot \frac{\rho u_0}{1+u_0+k u_0^2} \]
By (26) and (27), we obtain the following equation, which satisfies the bifurcation \( d_c(d_c > 1) \) of diffusion coefficient ratio:
\[ d_c^2 f_u^2 + 2(2f_v g_u - f_u g_v) d_c + g_v^2 = 0 \]
here
\[ f_u = m - 1 - \frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2}, \quad f_v = -\frac{\rho u_0}{1+u_0+k u_0^2}, \]
\[ g_u = -\frac{\rho v_0(1-k u_0^2)}{(1+u_0+k u_0^2)^2}, \quad g_v = n - \alpha - \frac{\rho u_0}{1+u_0+k u_0^2}. \]
Therefore, the bifurcation \( p_c \) of the wave-number is given by the following formula
\[ p_c^2 = r \cdot \frac{d_c f_u + g_v}{2d_c} = r \left[ \frac{det(A)}{d_c} \right]^{1/2} = r \left[ \frac{f_u g_v - f_v g_u}{d_c} \right]^{1/2} \]
Clearly, we have
\[ dc = \frac{-2(f v g_u - f_u g_v) \pm \sqrt{16 f^2 g_u^2 - 16 f_v g_u f_u g_v}}{2 f^2} \]
if we let
\[ \Delta = -\frac{16 \rho^2 u_0 v_0 (1 - k u_0^2)}{(1 + u_0 + k u_0^2)^3} [(m - 1)(n - \alpha) + (n - \alpha) \cdot \frac{\rho v_0 (1 - k u_0^2)}{(1 + u_0 + k u_0^2)^2} + \frac{(m - 1) \rho u_0}{1 + u_0 + k u_0^2}] . \]

**Theorem 7:**
(a) when
\[ (m - 1)(n - \alpha) > (\alpha - n) \cdot \frac{\rho v_0 (1 - k u_0^2)}{(1 + u_0 + k u_0^2)^2} + \frac{(m - 1) \rho u_0}{1 + u_0 + k u_0^2} \cdot (1 - k u_0^2) > 0 \]
holds, there does not exist any bifurcation of diffusion coefficient ratio;
(b) when \( 1 - k u_0^2 = 0, \Delta = 0, \) there exists a bifurcation of diffusion coefficient ratio:
\[ dc = \frac{(\alpha - n)(2 \sqrt{k} + 1) + \rho}{(1 - m)(2 \sqrt{k} + 1)} \]
at the same time, there exist two bifurcations of critical wave-number
\[ p_c^2 = \tau (1 - m); \]
(c) when
\[ (m - 1)(n - \alpha) < (\alpha - n) \cdot \frac{\rho v_0 (1 - k u_0^2)}{(1 + u_0 + k u_0^2)^2} + \frac{(m - 1) \rho u_0}{1 + u_0 + k u_0^2} \cdot (1 - k u_0^2) > 0 \]
holds, there exist two bifurcations \( d_c \) of the diffusion coefficient ratio;
(d) when \( 1 - k u_0^2 < 0 \) and
\[ (m - 1)(n - \alpha) + (n - \alpha) \cdot \frac{\rho v_0 (1 - k u_0^2)}{(1 + u_0 + k u_0^2)^2} > \frac{(1 - m) \rho u_0}{1 + u_0 + k u_0^2} \]
there also exist two bifurcations \( d_c \) of the diffusion coefficient ratio.

Proof: For (a), when the hypotheses holds, we have \( \Delta < 0 \), so the equation about \( d_c \) has no real solutions, which means there are no bifurcation points of the diffusion coefficient ratio. For (b), the equation about \( d_c \) has a root. For (c), when given conditions are satisfied, \( \Delta > 0 \), hence there exist two bifurcations \( d_c \) of diffusion coefficient ratio. In a similar way we can prove (d) is true.

### 4. Conclusion

In this paper, firstly, we studied some dynamic characters of a reactive diffusion equation with enzyme reactants without any feedback control terms, and obtained the condition of equilibrium, its stability and the parameter domains of diffusion-driven instability and some bifurcation values; we also studied some dynamic characters of a reactive diffusion equation with enzyme reactants with feedback control terms \( m u \) and \( n v \), obtaining the condition of equilibrium, its stability and the parameter domains of diffusion-driven instability and some bifurcation values, such as diffusion coefficient \( d_c \), wave-number \( p_c \).

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References


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