DECENTRALIZED OUTPUT FEEDBACK ROBUST CONTROL FOR A CLASS OF UNCERTAIN NONLINEAR CIRCLE-LINKED LARGE-SCALE COMPOSITE SYSTEMS

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Abstract. This paper proposes a new class of systems—circle-linked systems, and gives a necessary and sufficient condition for the realization of circle-linked systems by using linear algebra theory. For a class of uncertain nonlinear circle-linked large-scale composite systems, we study the problem of decentralized output feedback robust stabilization by employing the Riccati equation approach and the matrix theory. And we design a type of nonlinear decentralized robust output feedback stabilizing controllers for uncertain nonlinear circle-linked large-scale composite systems.

Key Words. Circle-linked Systems; Robust Stabilization; Decentralized Control; Riccati equation; output feedback.

1. Introduction

Many control systems have various connections with nature and practical engineering; therefore it is significant to study the control problems with practical backgrounds. Many researchers devoted themselves to studying control systems and obtained a lot of valuable results [1-5]. It is found that many systems are circle-linked, and of which is cyclical limited. As we know, the ecological system consists of a number of ecological links and the elements in each ecological link cyclically and depend on each other. There are also connections between ecological links, as well as the element in each ecological link. Because of the nature’s complication, the connection between them suffers from uncertain influence. Besides the ecological system, a lot of systems also have the same properties motioned above. According to what we have observed, we put forward circle-linked systems and a class of nonlinear circle-linked composite systems. In the past few years many researchers have obtained some results on the output feedback of nonlinear systems [6-8] which is an important problem in control theory. However, the design method of the output feedback stabilization for nonlinear large-scale composite systems [9-12] is few. The reason consists in both the complexity of the output feedback and the large scale of these systems. In this paper we discuss decentralized robust control problems of nonlinear circle-linked large-scale composite systems and design region decentralized output feedback robust stabilizing controllers by using the Riccati equation method, Lyapunov function and matrix theory.
2. Definition of circle-linked system and problem statement

We introduce the following notation: $A^+$ denotes the Moore-Penrose inverse of the matrix $A$, $\| \cdot \|$ denotes the spectrum norm of a matrix, $A > 0 (A < 0)$ denotes that the $A$ is positive definite (negative definite); $V^+_{\omega}(E)$ is a smooth vector field with dimension $n$ defined on the set of $E$. $\lambda_M(A)$ denotes the maximum singular value of a matrix $A$, $\lambda_{\min}(A)$ denotes the minimum eigenvalue of the matrix $A$. $\lambda_{\max}(A)$ denotes the maximum eigenvalue of the matrix $A$.

Consider the following system:

\[ \dot{x} = Ax \]

**Definition 1.** Suppose $A \in \mathbb{R}^{n \times n}$, let $\alpha_1, \alpha_2, \cdots, \alpha_k (k \leq n)$ be linearly independent in $\mathbb{R}^n$. If there exist $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ which satisfy
\[ A\alpha_k = \lambda_{k-1}\alpha_{k-1}, \ldots, A\alpha_2 = \lambda_1\alpha_1, A\alpha_1 = \lambda_2 \alpha_k \]
then $\alpha_1, \alpha_2, \cdots, \alpha_k$ is called a $k$-order circle-linked eigenvector group of $A$.

**Definition 2.** System (1) is said to be a circle-linked realizable system, if there exists nonsingular invertible transformation, $x = T \tau$ such that

\[ \dot{\tau} = \begin{bmatrix} \square_1 & \square_2 & \cdots & \square_t \end{bmatrix} \tau \]

where

\[ \square_1 = \begin{bmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1} & 0 \\ \lambda_k & 0 & 0 & \cdots & 0 \end{bmatrix} \]

\[ \square_2 = \begin{bmatrix} 0 & \lambda_{k+1} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{k+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{k+k_2} & 0 \\ \lambda_{k+k_2} & 0 & 0 & \cdots & 0 \end{bmatrix} \]

\[ \square_t = \begin{bmatrix} 0 & \lambda_{\sum_{i=1}^{k} k+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_n & 0 & 0 & \cdots & 0 \end{bmatrix} \]

with $k_1 + k_2 + \cdots + k_t = n, \lambda_i, i = 1, 2, \cdots, n$. From both definition 1 and definition 2 above, we have the following lemmas.

**Lemma 1.** System (1) is a circle-linked realizable system if and only if there exist the $k_1, k_2, \cdots, k_t$-order circle-linked eigenvector groups of $A$ which constitute a base of $\mathbb{R}^n$.

**Lemma 2.** If $\alpha_1, \alpha_2, \cdots, \alpha_t$ is the $k$-order circle-linked eigenvector group of $A$, then $\alpha_1, \alpha_2, \cdots, \alpha_t$ are all the eigenvector of $A^k$.

**Lemma 3.** Suppose $A \in \mathbb{R}^{n \times n}$. If $\lambda_1, \lambda_2, \cdots, \lambda_s$ are the different eigenvalues of $A^k$, $(k > 1)$, choose arbitrary linearly independent vector groups $\alpha_{11}, \ldots, \alpha_{is}, (i = 1, 2, \cdots, s)$ in $V_{\lambda_i}A^k$, then $\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1r_1}; \cdots; \alpha_{s1}, \alpha_{s2}, \cdots, \alpha_{sr_s}$ are linearly independent.
Lemma 4. Suppose $A \in \mathbb{R}^{n \times n}$. The dimension of the eigensubspace $V^k_\lambda$ which belongs to $\lambda$ of $A^k$ is less than the order of $\lambda$, that is $\dim V^k_\lambda \leq s$ ($s$ is the order of $\lambda$).

Suppose the order of the eigenvector groups are $k_1, k_2, \cdots, k_t$ respectively. Let $k$ be the least common multiple of $k_1, k_2, \cdots, k_t$. Then

Theorem 1. System (1) is a circle-linked realizable system if and only if

1. All the eigenvalues of $A^k$ are in $\mathbb{R}$.
2. For every eigenvalue $\lambda$ of $A^k$, the dimension of $V^k_{\lambda}$ equals to the order of $\lambda$.
3. For every $\lambda$, there exist some $k_i$-order circle-linked eigenvector group which form a base of $V^k_{\lambda}$.

Proof. Theorem 1 can be obtained by lemmas 1-4.

The steps of the algorithm can be summarized as follows:

1) Find the order of all the $k_i$-order circle-linked eigenvector groups of $A$: $k_1, k_2, \cdots, k_t$, and let $k$ be their least common multiple.
2) Find all the eigenvalues of $A^k$.
3) Solve $(\lambda I - A^k) = 0$. Determine the basic set of the solution space of the homogeneous equation $(\lambda I - A^k) = 0$ composed of the $k_i$-order circle-linked eigenvector of $A$.
4) If for every $\lambda$, there exist a circle-linked eigenvector of $A$ which constitutes a basic set of the solution, and rank $(\lambda I - A^k) = n - s$, where $s$ is the order of $\lambda$, then system (1) is a circle-linked realizable system.

From the above steps, we get $T$ whose columns are the solutions, and $T$ is also invertible. Let $x = T\bar{x}$. Then system (1) is transformed to:

$$
\dot{x} = \begin{bmatrix} \Box_1 & & & \\ & \Box_2 & & \\ & & \ddots & \\ & & & \Box_t \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_t \end{bmatrix} v
$$

Consider a system of the form:

$$(4) \quad \dot{x} = Ax + Bu \quad y = Cx$$

Definition 3. For system (4), if there exists nonsingular transformation $x = T\bar{x}$ and state feedback $u = Lx + v$ such that system (4) is transformed to:

$$
\dot{\bar{x}} = \begin{bmatrix} \Box_1 & & & \\ & \Box_2 & & \\ & & \ddots & \\ & & & \Box_t \end{bmatrix} \bar{x} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_t \end{bmatrix} v
$$

where $\Box_i \in \mathbb{R}^{k_i \times k_i}, B_i \in \mathbb{R}^{k_i \times m_i}, C \in \mathbb{R}^{r \times k_i}, \sum_{i=1}^{t} k_i = n, \sum_{i=1}^{t} m_i = m \geq t, \sum_{i=1}^{t} r_i = r \geq t$. Then system (4) is said feedback circle linked block decoupling. $(k_1, k_2, \cdots, k_t), (m_1, m_2, \cdots, m_t), (r_1, r_2, \cdots, r_t)$ are the indexes of blocks. Especially, if $t = m = r$, i.e., $B_i \in \mathbb{R}^{k_i \times 1}, C_i \in \mathbb{R}^{1 \times k_i}, (i = 1, 2, \cdots, m)$, then system (4) is termed feedback circle linked linear decoupling. If $L = 0$ the above corresponding
definitions are termed circle-linked block decoupling and circle-linked linear decoupling, respectively.

**Remark 1.** Definition 3 is similar to the notation of state decoupling without feedback in paper [14]. The nonsingular transform $T$ can be get via the method of paper [14], using the feedback, the circle linked linear decoupling can be realized.

**Theorem 2.** If system (1) can realize circle-linked system, the order of the circle-linked eigenvector group of $A$ are $k_1, k_2, \ldots, k_t$. The corresponding circle-linked eigenvector groups are $\alpha_1, \alpha_2, \ldots, \alpha_{k_i}; \alpha_{k_1+1}, \alpha_{k_1+2}, \ldots, \alpha_{k_1+k_2}; \ldots; \alpha_{\sum_{i=1}^{t-1} k_i+1}, \ldots, \alpha_{n}$, respectively. Then, system (4) can realize circle-linked block decoupling if and only if

1) There exists a group of positive integers $m_1, m_2, \ldots, m_t, \sum_{i=1}^{t} m_i = m$, such that each column of the first $m_1$ of $B$ is the linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{k_1};$ each column vector from column $m_1+1$ to column $m_1+m_2$ of $B$ is a linear combination of $\alpha_{k_1+1}, \alpha_{k_1+2}, \ldots, \alpha_{k_1+k_2}$; each column vector from column $\sum_{i=1}^{t-1} m_i + 1$ to column $m$ of $B$ is a linear combine of $\alpha_{\sum_{i=1}^{t-1} k_i+1}, \ldots, \alpha_{n}$.

2) There exists a group of positive integers $r_1, r_2, \ldots, r_t, \sum_{i=1}^{t} r_i = r$, such that each row vector of the first $r_1$ rows of $C$ verticals with other circle linked eigenvector of $A$ except for $\alpha_1, \alpha_2, \ldots, \alpha_{k_1}$; each row vector from $r_1+1$ to $r_1+r_2$ of $C$ verticals with other circle-linked eigenvector of $A$ except for $\alpha_{k_1+1}, \alpha_{k_1+2}, \ldots, \alpha_{k_1+k_2}; \ldots; \alpha_{\sum_{i=1}^{t-1} k_i+1}, \ldots, \alpha_{n}$.

Consider the following nonlinear circle-linked large-scale composite system:

$$
\dot{x}_i = Ax_i + f_i(x_i) + \Delta f_i(x_i) + B(u_i + \Delta g_i(x_i)) + \sum_{j=1, j \neq i}^{N} (H_{ij}(x_j) + \Delta H_{ij}(x_j))
$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $f_i(x_i)$ is the nonlinear part of i-th subsystem; $\Delta f_i(x_i)$ and $\Delta g_i(x_i)$ are non-matching continuous uncertain items and matching continuous uncertain items of the i-th subsystem, respectively; $\sum_{j=1, j \neq i}^{N} H_{ij}(x_j)$ are interconnections, $\sum_{j=1, j \neq i}^{N} \Delta H_{ij}(x_j)$ are uncertain interconnections; $f_i(x_i) \in V^w_n(\Omega_i), H_{ij}(x_j) \in V^w_n(\Omega_j)$, here $\Omega_i$ is a neighborhood of $x_i$; $f_i(0) = \Delta f_i(0) = H_{ij}(0) = 0; i, j = 1, 2, \ldots, N, i \neq j$.

**Lemma 5** ([13]) If $\phi(x) \in V^w_n(\Omega)$ and $\phi(0) = 0$, then there exists a smooth function matrix $R(x)$ on $\Omega$ such that $\phi(x) = R(x)x$.

3. Main results and proof

**Assumption 1.** The nominal subsystem of the i-th subsystem of system (6) is a circle-linked block decoupling and indexes of blocks are $(k_1, k_2, \ldots, k_t)$, $(m_1, m_2, \ldots, m_t)$, $(r_1, r_2, \ldots, r_t)$. 

According to Assumption 1, there exists a nonsingular transformation \( x_i = T x_i \) such that system (6) is transformed to:

\[
\begin{bmatrix}
\dot{x}_{i1} \\
\dot{x}_{i2} \\
\vdots \\
\dot{x}_{it}
\end{bmatrix} =
\begin{bmatrix}
\emptyset_1 \\
\emptyset_2 \\
\vdots \\
\emptyset_t
\end{bmatrix} +
\begin{bmatrix}
\Delta f_{i1}(x_i) \\
\Delta f_{i2}(x_i) \\
\vdots \\
\Delta f_{it}(x_i)
\end{bmatrix} +
T^{-1} f_i(x_i) +
\begin{bmatrix}
\Delta g_{i1}(x_i) \\
\Delta g_{i2}(x_i) \\
\vdots \\
\Delta g_{it}(x_i)
\end{bmatrix}
\]

\[
T^{-1} \sum_{j=1, j \neq i}^{N} (H_{ij}(x_j) + \Delta H_{ij}(x_j))
\]

where \( T^{-1} \Delta f_i(x_i) = (\Delta f_i(x_i))_{txi}, \Delta g_i(x_i) = (\Delta g_i(x_i))_{txi}, T x_i = (x_i)_{txi}, y_i = (y_i)_{txi}, i = 1, 2, \ldots, N. \)

**Assumption 2.** \((\emptyset_i, B_i)\) is controllable, \((\emptyset_i, B_i)\) is detectable \((l = 1, 2, \ldots, t)\).

From Assumption 2, for any positive definite matrices \( Q_l \in \mathbb{R}^{k_l \times k_l} \) and \( R_l \in \mathbb{R}^{n_l \times m_l} \), the following Riccati equation

\[
\square_l^T P_l + P_l \square_l = -P_l B_l R_l^{-1} B_l^T P_l + Q_l = 0
\]

has a unique positive definite solution \( P_l \). Let

\[
P_l = \text{diag}(P, P, \ldots, P), \quad Q_l = \text{diag}(Q_1, Q_2, \ldots, Q_t)
\]

**Assumption 3.** \( \|\Delta g_i(x_i)\| \leq \rho_l(y_i, l), \|\Delta f_i(x_i)\| \leq \eta_l(y_i, l)\phi_l(\|x_i\|), l = 1, 2, \ldots, t, i = 1, 2, \ldots, N; \|\Delta H_{ij}(x_j)\| \leq \beta_{ij}\|x_j\|, i, j = 1, 2, \ldots, N, i \neq j \), where \( \eta_l(\cdot) \geq 0, \rho_l(\cdot) \) are given continuous functions, and \( \frac{\eta_l(y_i, l)}{\|B_l^T P_l C_l^T y_i\|} \) is enable-continuous function.

if \( B_l^T P_l C_l^T y_i = 0, \) then \( \eta_l(y_i, l) = 0. \)

**Assumption 4.** There exists nonsingular matrix \( F_l \), such that \( B_l^T P_l = F_l C_l(l = 1, 2, \ldots, t) \) where \( P_l \) satisfies (8).

**Assumption 5.** \( W^T + W(W(x)) = W_{ij}(x_j)_{N \times N} \) is a positive definite matrix in the reign \( \Omega \), where

\[
W_{ij} = \begin{cases} 
\lambda_{\min}(Q) - 2\lambda_M(P^{T-1} M_i(x_i)T), & i = j \\
-2(\lambda_M(P^{T-1} N_{ij}(x_j)T) + \beta_{ij}\lambda_M(P^{T-1}\|T\|)), & i \neq j
\end{cases}
\]

By lemma 5, there exist smooth function matrices \( M_i(x_i) \) and \( N_{ij}(x_j) \), such that \( f_i(x_i) = M_i(x_i)x_i \) and \( H_{ij}(x_j) = N_{ij}(x_j)x_j, 1 \leq i, j \leq N, i \neq j. \)

**Theorem 3.** If system (6) satisfies Assumption 1-5, then system (6) has decentralized robust stabilization controllers.

**Proof.** Design controllers

\[
u_i = K C_l^T y_i + u_1^i(y_i) + u_2^i(y_i), 1 \leq i, j \leq N, i \neq j
\]
Let \( \varepsilon > \maxsup \) where

\[
\begin{align*}
V & = \begin{bmatrix}
u_{i1}^T(y_i) & u_{i2}^T(y_i) & \ldots & u_{i\lambda}^T(y_i)
\end{bmatrix}, \quad u_i^a(y_i) = \begin{bmatrix}
u_{i1}^a(y_i) & u_{i2}^a(y_i) & \ldots & u_{i\lambda}^a(y_i)
\end{bmatrix}
\end{align*}
\]

\( \lambda \) is an undetermined positive number,
\( K = \text{diag}(K_1, K_2, \ldots, K_T); \)
\( T = \begin{bmatrix} -P_i & \cdots & -P_i \end{bmatrix} \begin{bmatrix} C_1^- \cdots C_T^- \end{bmatrix} \)

Then the closed-loop system composed of system (6) and (10) are

\[
\begin{align*}
s_i = T^{-1}f_i(x_i) + T^{-1}\Delta f_i(x_i) + B_i[y_i - K\bar{C}^-y_i]
\end{align*}
\]

\( i = 1, 2, \ldots, N \)

Construct a Lyapunov function for system (13) as

\[
V = \sum_{i=1}^{N} \sum_{i=1}^{T} \pi_i^T P_i \pi_i = \sum_{i=1}^{N} \pi_i^T P_i \pi_i
\]

The derivation of the Lyapunov function \( V \) with respect to time \( t \) is

\[
\dot{V} = -\sum_{i=1}^{N} \pi_i^T Q_i \pi_i + \sum_{i=1}^{N} 2\pi_i^T P_i P_i^{-1} f_i(x_i) + \sum_{i=1}^{N} 2\pi_i^T P_i P_i^{-1} \sum_{j=1,j \neq i}^{N} (H_{ij} + \Delta H_{ij})
\]

\[
+ 2\sum_{i=1}^{N} \sum_{i=1}^{N} \pi_i^T P_i B_i [u_{i\lambda}^a(y_i) + \Delta g_{ii}(y_i)] + 2\sum_{i=1}^{N} \pi_i^T P_i [\Delta f_{ii}(y_i) + B_i u_{i\lambda}^a(y_i)]
\]

if \( \dot{y}_i = 0 \), by (11) and (12), we get

\[
\begin{align*}
\dot{\pi}_i^T P_i B_i [u_{i\lambda}^a(y_i) + \Delta g_{ii}(y_i)] & \leq 0
\end{align*}
\]

Let \( \varepsilon > \maxsup_{x_i \in \Omega_i, i=1,2,\ldots,N} [f_i(x_i)[\pi_i]] \), then

\[
\begin{align*}
\dot{\pi}_i^T P_i [\Delta f_{ii}(y_i) + B_i u_{i\lambda}^a(y_i)] & \leq 0
\end{align*}
\]
if \(B_i^T P_i C_i^T y_{d(t)} = 0\), then
\[
\begin{align*}
\mathbf{P}^T_i B_i (u_i^T(y_i) + \Delta \theta_i(x_i)) &= 0 \\
\mathbf{P}^T_i B_i (\Delta f_i(x_i) + B_i u_i^y(y_i)) \\
&= \mathbf{P}^T_i \Delta f_i(x_i) \\
&\leq \lambda_{\text{max}}(P_i) \|\Delta f_i(x_i)\| \\
&\leq \lambda_{\text{max}}(P_i) \|\theta_i(y_i, t)\| \|\varphi_i(\|x_i\|)\| \\
&= 0 \\
\dot{V} &\leq -\sum_{i=1}^{N} \lambda_{\text{min}}(Q) - 2\lambda_M(P^T \Omega (x_i) T) \|\mathbf{P}^T_i\|^2 \\
&\leq \sum_{j=1}^{N} 2(\lambda_M(P^T \Omega (x_j) T) + \beta_{ij} \lambda_M(P^T \Omega) \|T\| \|\mathbf{P}^T_i\| \|\mathbf{P}^T_j\|)
\end{align*}
\]
where \(Y = ([\|\mathbf{P}^T_i\|, \|\mathbf{P}^T_j\|, \cdots, \|\mathbf{P}^T_i\|])^T\). Since \(W^T + W > 0\) in \(\Omega\), then \(\dot{V} < 0\) in \(\Omega\).

4. Numeric example

We consider the following uncertain nonlinear circle-linked large-scale composite system

\[
x_1 = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} (u_1 + \theta_1(x_{11} + 2x_{12})e^{-t} \sin x_{13} + \theta_2(x_{13} + 2x_{14})e^{-t} \sin(x_{14} \theta_1) + \frac{1}{\mathbf{w}} \begin{bmatrix} x_{11} \sin x_{12} \\ 0 \\ 0 \\ x_{11} x_{12} \end{bmatrix}
+ \frac{1}{\mathbf{w}} \begin{bmatrix} x_{21}^2 \\ 0 \\ x_{21} x_{23} \\ 0 \end{bmatrix} + \Delta H_{12}(x_2))
\]

\[
y_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} x_1
\]

\[
x_2 = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} (u_2 + \theta_1(x_{21} + 2x_{22})e^{-t} \cos(x_{24} \theta_1) + \frac{1}{\mathbf{w}} \begin{bmatrix} x_{21} \cos x_{22} \\ 0 \\ x_{21} x_{23} \\ 0 \end{bmatrix}
+ \frac{1}{\mathbf{w}} \begin{bmatrix} x_{11}^2 \\ 0 \\ x_{11} x_{13} \\ 0 \end{bmatrix} + \Delta H_{21}(x_1))
\]

\[
y_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} x_2
\]

where \(x_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4}), i = 1, 2, \|\Delta H_{12}(x_2)\| \leq 0.01 \|x_2\|, \|\Delta H_{21}(x_1)\| \leq 0.01 \|x_1\|, (\theta_1, \theta_2) \in \Omega = \{\theta_1, \theta_2 \| \theta_1 < 2, |\theta_2| < 1\}.$$
By Theorem 1, we have

\[
T = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

let

\[
x = T\tilde{x}; \tilde{x} = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix} = T^{-1}x
\]

then the system is converted to:

\[
\begin{bmatrix}
\dot{x}_{11} \\
\dot{x}_{12} \\
\dot{x}_{13} \\
\dot{x}_{14}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix} (u_1 + \theta_1 (x_{11} + 2x_{12})^2 e^{-t} \sin(x_{13} \theta_1) + 2(x_{13} + 2x_{14})^2 e^{-t} \sin(x_{14} \theta_1)) + \Delta H_1(x_1)
\]

\[
y_1 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 2
\end{bmatrix} \tilde{x}_1
\]

\[
\begin{bmatrix}
\dot{x}_{21} \\
\dot{x}_{22} \\
\dot{x}_{23} \\
\dot{x}_{24}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix} (u_2 + \theta_1 (x_{21} + 2x_{22})^2 e^{-t} \sin(x_{13} \theta_1) + 2(x_{23} + 2x_{24})^2 e^{-t} \sin(x_{14} \theta_1)) + \Delta H_2(x_2)
\]

\[
y_2 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 2
\end{bmatrix} \tilde{x}_2
\]

The system is linear circle-linked decoupling.

\[
\Box_1 = \begin{bmatrix}
0 & 1 \\
2 & 0
\end{bmatrix}, \Box_2 = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, B_1 = B_2 = \begin{bmatrix}
0 & 1
\end{bmatrix}, C_1 = \begin{bmatrix}
1 & 1
\end{bmatrix}, C_2 = \begin{bmatrix}
1 & 2
\end{bmatrix}
\]

\((\Box_i, B_i)\) is controllable, \((\Box_i, C_i)\) is detectable \(i = 1, 2\).

Choose \(Q_1 = \begin{bmatrix}
8 & 2 \\
2 & 12
\end{bmatrix}, Q_2 = \begin{bmatrix}
8 & 2 \\
2 & 12
\end{bmatrix}, R_1 = \frac{1}{4}I, R_2 = \frac{1}{6}I\). Solving the Riccati equation (9), we get
After elementary calculation, we get the following results

\[ M_1(x_1) = \frac{1}{36} \begin{bmatrix} x_{11} \sin x_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2(x_2) = \frac{1}{36} \begin{bmatrix} x_{21} \cos x_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_{21} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ N_{12}(x_2) = \frac{1}{36} \begin{bmatrix} x_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_{21} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_{21}(x_2) = \frac{1}{36} \begin{bmatrix} x_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \rho_1(y_i, t) = y_{i1}^2 e^{-t}, \quad \rho_2(y_i, t) = 2y_{i2}^2 e^{-t}, \quad i = 1, 2. \]

\[ C_i^+ = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad C_i^- = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} \]

\[ \eta_1(y_i, t) \phi_1(||x_i||) = 2y_{i1}^2 e^{-||x_i||-12}, \quad \eta_2(y_i, t) \phi_2(||x_i||) = 2y_{i2}^2 e^{-||x_i||-12}, \quad i = 1, 2. \]

\[ W(x) = \begin{bmatrix} 7.1715 - L_1 & -1.0438||x_{21}|| - 0.4128 \\ -1.0438||x_{11}|| - 0.4128 & 7.1715 - L_2 \end{bmatrix} \]

where

\[ L_1 = \frac{1}{36} \left( 208||x_{11} \sin x_{12}||^2 + 145||x_{11}||^2 \right)^{\frac{1}{2}}, \quad L_2 = \frac{1}{36} \left( 208||x_{21} \cos x_{22}||^2 + 145||x_{21}||^2 \right)^{\frac{1}{2}}. \]

Choose \( \Omega = \{ x_i; ||x_i|| < 3, i = 1, 2; j = 1, 2, 3, 4 , \varepsilon = 6e^{-6} \}, \) then \( W^T(x) + W(x) \) is positive definite in \( \Omega. \) So the decentralization output feedback controllers are

\[ u_i = -\begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix} y_i - \begin{bmatrix} y_{i1}^2 e^{-t} \text{sgn}(y_{i1}) \\ 2y_{i2}^2 e^{-t} \text{sgn}(y_{i2}) \end{bmatrix} - \begin{bmatrix} \zeta_i \\ \zeta_i \end{bmatrix}, \quad i = 1, 2. \]

where \( \zeta_{i1} = 126.9906 e^{-6} y_{i1}, \quad \zeta_{i2} = 290.376 e^{-6} y_{i2}, \quad i = 1, 2. \) Sgn is the sign function.

We can obtain the following state responses:

**Conclusion**

This paper considered circle-linked systems and a class of uncertain nonlinear circle-linked large-scale composite systems, and studied the stabilization problems for those systems by using the Lyapunov stability theory and the matrix theory. For a class of uncertain nonlinear circle-linked large-scale composite systems, we designed a type of nonlinear decentralized output feedback robust controllers. Finally, a numerical example is given to illustrate our results.

**References**


Figure 1. Response of the state variable $x(t)$ for the system.

Figure 2. Response of the state variable $x_1(t)$ for subsystem 2.


Figure 3. Response of the state variable $x_2(t)$ for subsystem 2.


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