DISSIPATION CONTROL OF AN N-SPECIES FOOD CHAIN SYSTEM

LICHUN ZHAO¹,², QINGLING ZHANG², and QICHLANG YANG¹

Abstract. First, the passive control and $H_\infty$ control of a n-species food chain system are studied and some sufficient conditions rendering the closed-loop system globally asymptotically stable at the positive equilibrium point are given. Second, an optimal control law of a n-species food chain system is acquired.

Key Words. Passivity analysis, Dissipation-control, Optimal control, $H_\infty$-control, $L_2$-gain, Food chain system.

1. Introduction

The management of population resources is one of the important control problems for ecosystems. Much attention has been paid to the optimal control of population systems[1~6]. The occurrence of impulse phenomenon arouses some researchers’ interests on the impulse control of population systems[7~10]. Recently, induction control was also applied to the control of population systems[11]. In general, there are different controls exerted on the population systems to investigate their permanence and stability, etc. Since most biological systems in the natural world have nonlinearity, it is this nonlinearity that makes the system in question have dissipative structure, thus Prigogine’s dissipative structure theory plays a key role in dealing with these problems in nonlinear biological systems. As two special cases of dissipation, passivity analysis and $H_\infty$-control are investigated by many researchers[12~17]. In this paper, we will discuss their applications in population dynamic systems.

In the natural world, the relations of populations in ecosystems are very complicated. In these relations, the food chain system is typical. Now consider the following n-species food chain system

$$\begin{align*}
\dot{x}_1 &= x_1(a_{10} - a_{11}x_1 - a_{12}x_2), \quad x_1(0) = x_{10} > 0, \\
\dot{x}_2 &= x_2(-a_{20} + a_{21}x_1 - a_{23}x_3), \quad x_2(0) = x_{20} > 0, \\
& \vdots \\
\dot{x}_{n-1} &= x_{n-1}(-a_{n-1,0} + a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n), \quad x_{n-1}(0) = x_{(n-1)0} > 0, \\
\dot{x}_n &= x_n(-a_{n,0} + a_{n,n-1}x_{n-1}), \quad x_n(0) = x_{n0} > 0,
\end{align*}$$

where $a_{ij}$ is a non-negative constant, $a_{ij} (i = 1, \cdots, n; j = 1, \cdots, n)$ is positive constant, $x_i(t) (i = 1, \cdots, n)$ is the population of the i-th species. In ecology, the first species is said to be the first producer, the second species the first consumer.
and the third species the secondary consumer and so on.

The study for n-species the food chain systems is complete. Some results of their stability and permanence are seen in [1, 2]. The food chain systems with controls are studied in some papers such as [16].

This paper is organized as follows: In section 2, we study the global stability of a n-species chain food system and get stability conditions for it. In section 3, we investigate the $H_{\infty}$ control problem of system (1.1), some sufficient conditions which make the system in question globally stable and permanent are obtained. In section 4, Using passivity control, we study the optimal control problem and get optimal control law which makes the system permanent.

2. Passivity – based control

2.1. Preliminaries. Consider a nonlinear system of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x), \end{cases}$$

with state space $X = \mathbb{R}^n$, set of inputs $U = \mathbb{R}^m$ and set of outputs $Y = \mathbb{R}^p$.

For this system, the following concepts are given.

**Definition 2.1.** [15] System (2.1) is said to be dissipative with supply rate $s(u(t), y(t))$, if there exists a storage function $S : \mathbb{R} \to \mathbb{R}^+$ such that, for all $x(t_0) = x_0 \in X$, $t \geq t_0$ and all the input function $u$, the following inequality is satisfied

$$S(x(t)) \leq S(x(t_0)) + \int_{t_0}^{t} s(u(t), y(t)) dt.$$  

Inequality (2.2) is called dissipation inequality (where $x(t, t_0, x_0)$ is the solution of system (2.1) with initial condition $x_0 \in X$).

**Definition 2.2.** [15] System (2.1) is said to be passive, if it is dissipative with supply rate $u^T y$. System (2.1) is said to be strictly input passive, if there exists a $\sigma > 0$ such that the system is dissipative with supply rate $s(u, y) = u^T y - \sigma \|u\|^2$. System (2.1) is said to be strictly output passive, if there exists a $\epsilon > 0$ such that the system is dissipative with supply rate $s(u, y) = u^T y - \epsilon \|y\|^2$.

**Definition 2.3.** [15] If $u = 0$, $y = 0$, $\forall \ t \geq 0$ imply $\lim_{t \to -\infty} x(t) = 0$, $\forall \ t \geq 0$, then system (2.1) is said to be zero-state detectable.

Differentiate the inequality (2.2) with respect to $t$ at $t_0$, we have

$$S_x(x)f(x) \leq s(u, h(x)), \quad \forall \ x, u$$

where $S_x(x) = \left( \frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n} \right) = (S_1, \ldots, S_n)$.

Inequality (2.3) is called differential dissipation inequality.
Lemma 2.1. \[15\] In system (2.1), for \( \epsilon_1 > 0, \)
\[
\begin{equation}
S_x(x)[f(x) + g(x)u] \leq u^T h(x) - \epsilon_1 h^T(x)h(x), \quad \forall x, u
\end{equation}
\]
is equivalent to
\[
\begin{equation}
S_x(x)f(x) \leq -\epsilon_1 h^T(x)h(x),
\end{equation}
\]
The following inequality is often used
\[
\begin{equation}
S_x(x)[f(x) + g(x)u] \leq \frac{1}{2} \gamma^2 ||u||^2 - \frac{1}{2} ||h(x)||^2, \quad \forall x, u
\end{equation}
\]
where \( \gamma \) is a positive constant.

Lemma 2.2. \[15\] For system
\[
\begin{equation}
\dot{x} = f(x) + g(x)k(x), \quad f(0) = 0, \quad k(0) = 0,
\end{equation}
\]
assume that the equilibrium \( x^* = 0 \) of \( \dot{x} = f(x) \) is asymptotically stable, and there exists a \( C^1 \) function \( S(x) > 0 \), for some \( \epsilon > 0 \), \( S(x) \) is semi-positive at point \( x^* = 0 \) and satisfies
\[
\begin{equation}
S_x(x)[f(x) + g(x)k(x)] \leq -\epsilon ||k(x)||^2,
\end{equation}
\]
then the equilibrium \( x^* = 0 \) of system (2.7) is asymptotically stable.

For system (1.1), the existence and uniqueness conditions of the positive equilibrium of system (1.1) is given in the following lemma.

Lemma 2.3. \[1\] If system (1.1) has a unique isolated positive equilibrium \( x^* = (x_1^*, x_2^*, x_3^*) \), then it is globally asymptotically stable.

Lemma 2.4. For system (1.1), if the conditions in Lemma 2.3 hold, the equilibrium \( x^* = (x_1^*, x_2^*, x_3^*) \) of system (1.1) is globally asymptotically stable.

2.2. Passivity based-control. System (1.1) is equivalent to the following system (\( x^* \) is a positive equilibrium point of system (1.1))
\[
\begin{equation}
\begin{cases}
\dot{x}_1 = x_1[-a_{11}(x_1 - x_1^*) - a_{12}(x_2 - x_2^*)], & x_1(0) = x_{10} > 0, \\
\dot{x}_2 = x_2[a_{21}(x_1 - x_1^*) - a_{23}(x_3 - x_3^*)], & x_2(0) = x_{20} > 0, \\
\vdots & \vdots \\
\dot{x}_n = x_n a_{nn,n-1}(x_{n-1} - x_{n-1}^*), & x_n(0) = x_{n0} > 0.
\end{cases}
\end{equation}
\]

Because the producer of a food chain system plays important role in the permanence problem of the system, we track its density. The matrix form of corresponding control system with output \( y_2 \) is
\[
\begin{equation}
\begin{cases}
\dot{x} = f_2(x) + g_2(x)u_2(x), \\
y_2 = h_2(x),
\end{cases}
\end{equation}
\]
Theorem 2.1. For system (2.10), if the conditions in Lemma 2.3 hold and \( \|g_1(x)\| \leq \|g_2(x)\| \), where \( \Delta = 2\alpha \), further if 
\[
\begin{align*}
\Delta &= \sum_{i=1}^{n} a_i (x_i - x_i^*) \\
h &= \frac{a_2}{a_1},
\end{align*}
\]
then the control law 
\[
\dot{u}_1 = -N_1 - \sqrt{\Delta} \frac{a_2}{2\epsilon},
\]
\( \dot{u}_2 = -N_2 + \sqrt{\Delta} \frac{a_2}{2\epsilon} \), 
and 
\[
\dot{u}_3 = -N_3 + \sqrt{\Delta} \frac{a_2}{2\epsilon},
\]
\( \dot{u}_n = -N_n + \sqrt{\Delta} \frac{a_2}{2\epsilon} \),
where \( \Delta = N_1^2 + 4\epsilon a_1 N_1^2 \) asymptotically stabilizes the equilibrium \( x^* = 0 \) of system (2.13). Further, if \( \|u_2(N)\| \neq 0 \), then the equilibrium \( x^* = 0 \) of system

Proof. In system (2.10), consider a storage function
\[
S(x) = \sum_{i=1}^{n} \alpha_i (x_i - x_i^*) \ln x_i,
\]
where 
\[
\alpha_1 = 1, \quad \alpha_2 = \frac{a_2}{a_1}, \quad \alpha_3 = \frac{a_2 a_3}{a_1 a_2}, \quad \ldots, \quad \alpha_n = \frac{a_2 a_3 \cdots a_{n-1} n_n}{a_1 a_2 a_3 \cdots a_{n-1} n_n}.
\]
We show that system (2.10) is dissipative with supply rate 
\[
s(u_2, y_2) = u_2^T y_2 - \epsilon \|y_2\|^2 \quad (\epsilon > 0).
\]
Obviously, \( S(x) \) satisfies the second inequality of system (2.5). And for \( 0 < \epsilon \leq a_1 \), we can get 
\[
-a_1 (x_i - x_i^*)^2 \leq -\epsilon (x_i - x_i^*)^2.
\]
So the first equation of (2.5) holds provided that \( 0 < \epsilon \leq a_1 \). According to Lemma 2.1 and Definition 2.2, we complete the proof of the theorem.

Now, we search the control law which asymptotically stabilizes the equilibrium \( x = x^* \). We can get the following system by substitution \( N = x - x^* \).
\[
\begin{align*}
\dot{N} &= f_2(N) + g_2(N) \dot{u}_2(N), \\
y_2 &= h_2(N),
\end{align*}
\]
where 
\[
f_2(N) = f_2(N + x^*), \quad g_2(N) = g_2(N + x^*), \quad \dot{u}_2(N) = u_2(N + x^*), \quad h_2(N) = h_2(N + x^*).
\]
For system (2.13), we can get the following theorem.

Theorem 2.2. For system (2.13), suppose
(1) The conditions in Theorem 2.1 hold;
(2) there exists a \( \epsilon > 0 \) and \( \epsilon \leq a_1 \); then the control law
\[
\dot{u}_{2k}(N) = \dot{u}_{2k+1}(N) = \cdots = \dot{u}_{2n}(N) = 0,
\]
where \( \Delta = N_1^2 + 4\epsilon a_1 N_1^2 \) asymptotically stabilizes the equilibrium \( N^* = 0 \) of system (2.13). Further, if \( \|u_2(N)\| \neq 0 \), then the equilibrium \( N^* = 0 \) of system
(2.13) is globally asymptotically stable.

Proof Take a storage function as follows

\[
\dot{S}(N) = \sum_{i=1}^{n} \left[ \alpha_i (N_i + x_i^* - x_i^* \ln N_i + x_i^*) \right] - \ln \prod_{i=1}^{n} \alpha_i \left( \frac{e}{N_i} \right)^{N_i},
\]

\[\alpha_i (i = 1, 2, \cdots, n)\] is same as those aforementioned.

Obviously, \( \dot{S}(N) > 0 \), \( \dot{S}(0) = 0 \). Denote \( k = \dot{u}_2(N) = \dot{u}_2 \), \( S_x = \dot{S}_N(N) = (\dot{S}_1, \dot{S}_2, \dot{S}_3) \), \( f = \dot{f}_2 \), \( g = \dot{g}_2 \) in (2.8), we get

\[
\begin{align*}
-a_{11} N_1^2 + N_1 \dot{u}_{21} & \leq -\epsilon \dot{u}_{21}^2, \\
0 & \leq -\epsilon \dot{u}_{22}^2, \\
& \vdots \\
0 & \leq -\epsilon \dot{u}_{2n}^2,
\end{align*}
\]

(2.15)

According to given conditions \( \dot{u}_{22}(N) = \cdots = \dot{u}_{2n}(N) \equiv 0 \), we can see that all inequalities are satisfied except the first one in (2.15). Change the first one into

\[
\epsilon \dot{u}_{21}^2 + N_1 \dot{u}_{21} - a_{11} N_1^2 \leq 0.
\]

(2.16)

If \( \dot{u}_{21} \) satisfies

\[
\dot{u}_{21}^{(1)} = \frac{-N_1 - \sqrt{\Delta}}{2\epsilon} \leq \dot{u}_{21} \leq \dot{u}_{21}^{(2)} = \frac{-N_1 + \sqrt{\Delta}}{2\epsilon}
\]

(where \( \Delta = N_1^2 + 4\epsilon a_{11} N_1^2 \)), then inequality (2.16) holds. According to Lemma 2.3, the equilibrium \( N^* = 0 \) of system (2.13) is asymptotically stable.

On the other hand, because \( \dot{u}_2 \) satisfies

\[
\dot{S}_N(N) \dot{f}_2(N) + \dot{g}_2(N) \dot{u}_2 \leq -\epsilon \| \dot{u}_2 \|^2.
\]

If \( \| u_2(N) \| \neq 0 \), and \( \dot{S}(N) \) is proper, function \( \dot{S}(N) \) is a Lyapunov function. Further we can obtain that the equilibrium \( N^* = 0 \) of closed-loop system is globally asymptotically stable. Thus we complete the proof of the theorem.

The global stability of the equilibrium point \( N^* = (0, \cdots, 0) \) of system (2.13) is equivalent to the global stability of the equilibrium point \( x^* = (x_1^*, x_2^*, \cdots, x_n^*) \) of system (2.9), thus we know that the controls in Theorem 2 make the system in question permanent. However, if systems are disturbed by outside signals unknown but bounded in \( L^2 \) space, we may study it by \( H_{\infty} \)-control. This is our next topic.

3. \( H_{\infty} \)-control

3.1. Preliminaries. Consider the following nonlinear system with standard control structure (Figure 1.)

\[
\sum: \quad \begin{cases} 
\dot{x} = f(x, u, d), \\
y = g(x, u, d), \\
z = h(x, u, d),
\end{cases}
\]

where \( u \) is the control input, \( d \) is the disturbance input, \( y \) is the measured output, \( z \) is the controlled output.
Optimal $H_\infty$-control problem is: to design a control law based on $y$ such that the $L_2$-gain, from the disturbance input $d$ to the controlled output $z$, of the closed-loop system is minimum and the closed-loop system is stable in some sense. In fact, it is very difficult to realize the optimal $H_\infty$-control. So, we look for suboptimal $H_\infty$-control, i.e., for a prescribed value $\gamma > 0$, to find a controller such that the $L_2$-gain of the closed-loop system is not larger than $\gamma > 0$. And then $\gamma > 0$ is decreased continuously to achieve the optimal $H_\infty$-control.

Suppose that there is an external disturbance input, and all state-vector can be measured, i.e., $y = x$ in system (2.1), thus a simpler form of system (3.1) is

$$\begin{cases}
\dot{x} = f(x) + b(x)u + g(x)d, \\
z = \begin{pmatrix} x \\ u \end{pmatrix},
\end{cases}$$

where all functions belong to $C^k (k \geq 2)$.

According to Theorem 7.1.1 in [15], for system (2.2), the control $u$ such that the $L_2$-gain, from the disturbance input $d$ to the controlled output $z$, is not larger than $\gamma > 0$ was found by solving $P \geq 0$ in the following Hamilton–Jacobi inequality

$$\text{(HJIa): } P_x f(x) + \frac{1}{2} P_x \frac{1}{\gamma^2} g(x) g^T(x) - b(x) b^T(x) P^T_x + \frac{1}{2} x^T_x x \leq 0,$$

where $P_x = (P_1, P_2, P_3)$.

3.2. $H_\infty$-control. The matrix form of the control system of system (1.1) with disturbance is described as

$$\begin{cases}
\dot{x}_3 = f_3(x) + b_3(x)u_3(x) + g_3(x)d_3(x), \\
y_3 = \begin{pmatrix} h_3(x) \\ u_3(x) \end{pmatrix},
\end{cases}$$

where $x = x_3 \in \mathbb{R}^n$, $y_3 = y_3 \in \mathbb{R}^s$, $u_3 \in \mathbb{R}^m$, $d_3 \in \mathbb{R}^r$.

$$f_3(x) = \begin{pmatrix}
x_1(a_{10} - a_{11} x_1 - a_{12} x_2) \\
x_2(-a_{20} + a_{21} x_1 - a_{22} x_3) \\
\vdots \\
x_{n-1}(-a_{n-1,0} + a_{n-1,n-2} x_{n-2} - a_{n-1,n} x_n) \\
x_n(-a_{n,0} + a_{n,n-1} x_{n-1})
\end{pmatrix},$$
Theorem 3.1. The global stability of the equilibrium point $N^* = (0, 0, \cdots, 0)$ of system (3.3) is equivalent to the global stability of the equilibrium point $x^*$ of system (3.4).

Using Theorem 7.1.1 and Proposition 7.1.2 in [15], we look for the control law which guarantee the stability of original system.

Theorem 3.2. In system (3.4), suppose that system (1.1) has a unique positive equilibrium $x^*$ and for $\gamma > 1$,

(i) If $0 < a_{11} < \frac{1}{2}$, take $b_{31}(N + x^*) = \sqrt{1 + (1 - 2a_{11})(N_1 + x_1^*)^2}$,

$\quad b_{3i}(N + x^*) = \frac{\alpha_i^2 + (N_i + x_i^*)^2}{\alpha_i}$ where $\alpha_i (i = 1, 2, \cdots, n)$ is the same as aforementioned, let a control law be

$$u_3(N + x^*) = -b_3^T(N + x^*)P_N^T(N + x^*)$$

$$= \begin{pmatrix}
-\frac{\gamma N_1}{N_1 + x_1^*} \sqrt{1 + (1 - 2a_{11})(N_1 + x_1^*)^2} \\
-\frac{\gamma N_2}{N_2 + x_2^*} \sqrt{\alpha_2^2 + (N_2 + x_2^*)^2} \\
\vdots \\
-\frac{\gamma N_n}{N_n + x_n^*} \sqrt{\alpha_n^2 + (N_n + x_n^*)^2}
\end{pmatrix}.$$
(ii) If \( a_{i1} \geq \frac{1}{2} \), take \( b_{31}(N + x^*) = 1 \), \( b_{3i}(N + x^*) = \sqrt{a_i^2 + (N_2 + x_2^*)^2}/\alpha_i \) (\( i = 1, 2, \cdots, n \)), let a control law be

\[
\mathbf{u}_3(N + x^*) = -b_3^T(N + x^*)P_N^T(N + x^*)
\]

Then the \( L_2 \) - gain of the closed - loop of system (3.9), from \( d \) to \( z \), is not larger than \( \gamma \).

**Proof** Let

\[
P(N + x^*) = \sum_{i=1}^{n} \alpha_i(N_i + x_i^* - x_i^* \ln(N_i + x_i^*))
\]

Thus

\[
\frac{\partial P(N + x^*)}{\partial N_i} = \frac{\gamma N_i \alpha_i}{N_i + x_i^*}, \quad i = 1, 2, \cdots, n.
\]

Denote \( P_N(N + x^*) = (\frac{\partial P(N + x^*)}{\partial N_1}, \cdots, \frac{\partial P(N + x^*)}{\partial N_n})^T = (P_1, P_2, \cdots, P_n)^T \) and substitute \( f_i(N + x^*), P_N(N + x^*), b_3(N + x^*), g_3(N + x^*), b_3(N + x^*) \) into (HJJa), we get

\[
-2a_{11}N_1(N_1 + x_1^*)P_1 - 2a_{12}N_2(N_1 + x_1^*)P_1 + (\frac{1}{\gamma^2} - b_{31}^2(N + x^*))P_1^2 + N_1^2
\]

\[
+2a_{21}N_1(N_2 + x_2^*)P_2 - 2a_{23}N_3(N_2 + x_2^*)P_2 + (\frac{1}{\gamma^2} - b_{32}^2(N + x^*))P_2^2 + N_2^2
\]

\[
+2a_{32}N_2(N_3 + x_3^*)P_3 - 2a_{34}N_4(N_3 + x_3^*)P_3 + (\frac{1}{\gamma^2} - b_{33}^2(N + x^*))P_3^2 + N_3^2
\]

\[
\vdots
\]

\[
+2a_{n-1,n-2}N_{n-2}(N_{n-1} + x_{n-1}^*)P_{n-1} - 2a_{n-1,n}N_n(N_{n-1} + x_{n-1}^*)P_{n-1}
\]

\[
+(\frac{1}{\gamma^2} - b_{3,n-1}^2(N + x^*))P_{n-1}^2 + N_{n-1}^2
\]

\[
+2a_{n-1,n}N_{n-1}(N_n + x_n^*)P_n + (\frac{1}{\gamma^2} - b_{3,n}^2(N + x^*))P_n^2 + N_n^2 \leq 0.
\]

Then, we have

\[
-b_{31}^2(N + x^*)\left(\frac{\gamma N_1}{N_1 + x_1^*}\right)^2 + N_1^2 + \left(\frac{N_1}{N_1 + x_1^*}\right)^2 - 2a_{11}N_1^2 \gamma
\]

\[
-b_{32}^2(N + x^*)\left(\frac{\gamma N_2\alpha_2}{N_1 + x_1^*}\right)^2 + N_2^2 + \left(\frac{N_2\alpha_2}{N_2 + x_2^*}\right)^2
\]

\[
\vdots
\]

\[
-b_{3n}^2(N + x^*)\left(\frac{\gamma N_n\alpha_n}{N_n + x_n^*}\right)^2 + N_n^2 + \left(\frac{N_n\alpha_n}{N_n + x_n^*}\right)^2 \leq 0.
\]
If the following inequalities hold
\[
\begin{align*}
-b_{31}^2(N + x^*)\gamma^2\left(\frac{N_1}{N_1 + x_1^*}\right)^2 + N_1^2 + \left(\frac{N_1}{N_1 + x_1^*}\right)^2 - 2a_{11}N_1^2\gamma & \leq 0, \\
-b_{32}^2(N + x^*)\gamma^2\left(\frac{N_2}{N_2 + x_2^*}\right)^2 + N_2^2 + \left(\frac{N_2}{N_2 + x_2^*}\right)^2 - \gamma & \leq 0, \\
& \vdots \\
-b_{3n}^2(N + x^*)\gamma^2\left(\frac{N_n}{N_n + x_n^*}\right)^2 + N_n^2 + \left(\frac{N_n}{N_n + x_n^*}\right)^2 & \leq 0,
\end{align*}
\]
(3.8)
then the inequalities in (3.7) hold. According to the inequalities in (3.8), we have
\[
\begin{align*}
b_{31}^2(N + x^*)\gamma^2 + 2a_{11}(N_1 + x_1^*)^2\gamma - [1 + (N_1 + x_1^*)^2] & \geq 0, \\
b_{32}^2(N + x^*)\gamma^2\alpha_2 - [\alpha_2^2 + (N_2 + x_2^*)^2] & \geq 0, \\
& \vdots \\
b_{3n}^2(N + x^*)\gamma^2\alpha_n - [\alpha_n^2 + (N_n + x_n^*)^2] & \geq 0.
\end{align*}
\]
(3.9)
If \(\gamma > 1\), and \(b_{31}(N + x^*), b_{32}(N + x^*), \ldots, b_{3n}(N + x^*)\) satisfy
\[
\begin{align*}
\gamma > 1 & \geq \frac{-a_{11}(N_1 + x_1^*)^2 + \sqrt{[a_{11}(N_1 + x_1^*)^2]^2 + [1 + (N_1 + x_1^*)^2]b_{31}^2(N + x^*)}}{b_{31}^2(N + x^*)}, \\
\gamma > 1 & \geq \frac{\sqrt{\alpha_2^2 + (N_2 + x_2^*)^2}}{b_{32}(N + x^*)\alpha_2}, \\
& \vdots \\
\gamma > 1 & \geq \frac{\sqrt{\alpha_n^2 + (N_n + x_n^*)^2}}{b_{3n}(N + x^*)\alpha_n}.
\end{align*}
\]
(3.10)
Let
\[
\begin{align*}
b_{31}(N + x^*) & = \begin{cases} 
= \sqrt{1 + (1 - 2a_{11})(N_1 + x_1^*)^2}, & a_{11} < \frac{1}{2} \\
= 1, & a_{11} \geq \frac{1}{2}
\end{cases}, \\
b_{32}(N + x^*) & = \frac{\sqrt{\alpha_2^2 + (N_2 + x_2^*)^2}}{\alpha_2}, \\
& \vdots \\
b_{3n}(N + x^*) & = \frac{\sqrt{\alpha_n^2 + (N_n + x_n^*)^2}}{\alpha_n}.
\end{align*}
\]
(3.11)
According to theorem 7.1.1 in [15], we reach the conclusion. Thus complete the proof of the theorem.

**Theorem 3.3.** System (2.4) without output disturbance is
\[
\begin{align*}
\dot{N} & = f_3(N + x^*) + b_3(N + x^*)u_3(N + x^*) \quad u_3 \in \mathbb{R}^m, \quad d_3 \in \mathbb{R}^r, \\
y_3 & = \begin{pmatrix} h_3(N + x^*) \\ u_3(N + x^*) \end{pmatrix} \quad 0 \leq x \leq X, \quad y_3 \in \mathbb{R}^s.
\end{align*}
\]
(3.12)
where \( f_3(N + x^*) \), \( b_3(N + x^*) \), \( u_3(N + x^*) \), \( h_3(N + x^*) \) are expressed as the previous. If \( x^* \) is a unique equilibrium of system (1.1), for \( \gamma > 0 \), the control law
\[
u_3(N + x^*) = -b_3^T(N + x^*)P_{33}^T(N + x^*)
\]
globally asymptotically stabilizes the equilibrium \( N^* = 0 \) of system
\[
(4.1) \dot{N} = f_3(N + x^*) - b_3(N + x^*)b_3^T(N + x^*)P_{33}^T(N + x^*).
\]

The proof of the theorem is the same as that of proposition 7.1.2 in [15], so we omit it. Further, we can get the following corollary.

**Corollary 3.1.** If the conditions in Theorem 4.3 hold, then the control law \( (3.13) \) makes the original system \( (3.3) \) corresponding to the closed-loop system \( (3.14) \) permanent.

**4. Optimal control**

Consider the following system
\[
\dot{x} = f(x) + g_2(x)u, \quad f(0) = 0, \quad x(0) = x_0,
\]

**Lemma 4.1.** [15] For system \( (4.1) \), given the following performance
\[
(4.2) \quad \min \int_0^\infty (\|u(t)\|^2 + l(x(t)))dt,
\]
where \( l \geq 0 \) is a cost function and \( l(0) = 0 \).

If system \( x = f(x), \quad y = l(x) \) is zero-state detectable and there exists a non-negative solution \( V(x) \) to the following equation
\[
(4.3) \quad V_x(x)f(x) - \frac{1}{2}V_x(x)g_2(x)g_2^T(x)V_x^T(x) + \frac{1}{2}l(x) = 0, \quad V(0) = 0,
\]
then control \( u = -\alpha(x) = -g^T(x)V_x^T(x) \) is the optimal control law of system \( (4.1) \) with performance \( (4.2) \).

By substitution \( N = x - x^* \), system (1.1) is changed into
\[
(4.4) \quad \dot{N} = f_4(N).
\]
The corresponding control system is
\[
(4.5) \quad \dot{N} = f_4(N) + g_4(N)u_4(N),
\]
where
\[
f_4(N) = \begin{pmatrix}
-(a_{11}N_1 + a_{12}N_2)(N_1 + x_1^*) \\
(a_{21}N_1 - a_{23}N_3)(N_2 + x_2^*) \\
\vdots \\
(a_{n-1,n-2}N_{n-2} - a_{n-1,n}N_n)(N_{n-1} + x_{n-1}^*) \\
(a_{n,n-1}N_{n-1}(N_n + x_n^*)
\end{pmatrix},
\]
\[
g_4(N) = \text{diag}\{g_{41}(N)\}, \quad u_4(N) = (u_{41}(N), \ldots, u_{4n}(N))^T.
\]

Using the result of Lemma 4.1, we discuss the optimal control problem of system \( (4.5) \) with the following performance:
\[
(4.6) \quad \min \int_0^\infty \|u_4(N)\|^2 + \sum_{i=1}^n p_iN_i^2(t)dt
\]
where $p_i > 0 (i = 1, 2, \cdots, n)$ is the unit cost of the $i$-th population in system (4.5), respectively. Thus the second term of performance is the total cost. The first term is the cost of the control. Next we will describe the optimal problem with system (4.5) and performance (4.6).

First, we look for the non-negative solution of equation (4.4). Let $V(x) = V_1(x_1, x_2, \cdots, x_n)$ where $V_i = \sum_{i=1}^{n} \alpha_i [N_i + x_i^* \ln(N_i + x_i^*) - \ln(e^{x_i^*})]$. Let

$$-2a_{11}N_1^2 - \frac{\alpha_1^2 N_1^2}{(N_1 + x_1^*)^2} g^2_1(N) - p_1 N_1^2$$

$$-2a_{22}N_2^2 \frac{\alpha_2^2 N_2^2}{(N_2 + x_2^*)^2} g^2_2(N) - p_2 N_2^2$$

$$\vdots$$

$$-2a_{nn}N_n^2 - \frac{\alpha_n^2 N_n^2}{(N_n + x_n^*)^2} g^2_n(N) - p_n N_n^2 = 0$$

(4.7)

Thus, we can obtain the following control law

$$u_{41} = -\sqrt{p_1 - 2a_{11}} \frac{N_1 + x_1^*}{\alpha_1},$$

$$u_{42} = -\sqrt{p_2} \frac{N_2 + x_2^*}{\alpha_2},$$

$$\vdots$$

$$u_{4n} = -\sqrt{p_n} \frac{N_n + x_n^*}{\alpha_n}.$$ 

(4.9)

According to Theorem 4.1, we know that the control law (4.5) is the optimal control law of system (4.5) with performance (4.6) and it makes system (4.5) asymptotically stable.

From the discussion above, the following theorem are obtained.

**Theorem 4.1.** For system (4.5) with performance (4.6), the control law (4.10) is the optimal control law and asymptotically stabilizes the equilibrium $N^* = 0$ of it.
In short, the optimal control problem investigated by passivity analysis, is more convenient than that investigated by maximum principle.

5. Conclusions

Because most biological systems have dissipative structure, dissipative analysis plays an important role in biological control theory. In this paper, we use dissipative analysis to investigate passivity control, $H_{\infty}$ control and optimal control and get some control laws which make systems in question stable. These give control theory more access to biology systems, and at the same time, biological systems provide wider space for the applications of control theories.

References


1,2 Anshan Normal University, Anshan 114005 P. R. China

2 College of Sciences. Northeastern University, Shenyang 110006 P.R. China