NUMERICAL MODELLING AND ANALYSIS OF THE FORWARD AND INVERSE PROBLEMS IN ELECTRICAL CAPACITANCE TOMOGRAPHY

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Abstract. Electrical capacitance tomography (ECT) attempts to reconstruct the internal permittivity distribution of an object. Data for the image reconstruction are capacitance measurements taken at the periphery. There are potential applications for ECT in monitoring multi-phase flow in industrial processes. The image reconstruction of ECT remains a challenging problem. Linear and nonlinear reconstruction methods are studied in this paper. In this paper two-dimensional image reconstruction programs based on standard regularized Gauss Newton (GN) and Broyden Quasi-Newton (BQN) methods are implemented. The forward problem is solved via using finite element method (FEM). In GN the forward problem and the Jacobian matrices are recalculated in each iteration. In BQN an approximated inverse of the Jacobian matrix can be calculated directly. This paper studies a combined GN and BQN algorithm, in which for first few iterations GN is used and then BQN iterations are used for the last few iterations close to the solution. This combined GN and BQN technique is demonstrated by using simulated and experimental ECT data.

Key Words. Electrical capacitance tomography, inverse problems, linear and nonlinear methods, finite element method.

1. Introduction

The objective of Electrical capacitance tomography (ECT) image reconstruction is to find the permittivity distribution from capacitance measurements [13]. There are many potential applications of ECT, for example, oil and air separation [3], [4], [1]. ECT is used to reconstruct dielectric properties of an object. Measured data are the (dc) capacitance matrix between electrodes. Applying electric potential on an excitation electrode produces an electrostatic field penetrating the dielectric material so that an ECT sensors can measure the permittivity or dielectric constant of a sample.

This paper studies the image reconstruction of ECT and a software package that has been implemented for 2D ECT [10]. We have implemented the ECT reconstruction codes in Matlab, including 2D FE forward solver, sensitivity analysis and the inverse solvers. A more detailed information about the ECT reconstruction can be seen in [10]. The sensitivity maps have been generated by using an efficient formulation based on reciprocity property extracted from the FEM results. This paper examines the use of BQN method to the ECT inverse problem. For low contrast
permittivity reconstruction and for some cases regularized linear inverse solvers can be used. In general case even for low permittivity contrast the superposition property does not hold, so linear methods are not adequate. One needs to look for non-linear inverse solvers. The computational time for non-linear solvers are long, so one can compromise by using some approximations to save the computational time. A combined BQN and GN appears to be a good choice.

2. ECT system

A typical ECT sensor [12] comprises an array of conducting plate electrodes, mounted on the outside of a non-conductive pipe, surrounded by an electrical shield. Figure 1 shows an experimental ECT system that has been used for experimental data in this paper. For metal walled vessels, the sensor must be mounted internally, using the metal wall as the electrical shield. Additional components include radial and axial guard electrodes, of which many configurations have been tried to improve the quality of the measurements and hence images. It is not necessary for the electrodes to make physical contact with the specimen. So, ECT can be used on conveyor-lines, or externally mounted to plastic piping to reduce the risk of contamination.

![Figure 1. The PTL ECT system showing sensor, ECT system and host computer, thanks to Process Tomography Ltd (PTL) [8]](image)

3. Forward problems

Forward problems are to simulate the measurement process. Figure 2 shows the electrode arrangement for 8 electrode ECT system. In ECT forward problem, the excitation voltage and the permittivity distribution is given and the objective is to calculate the solution of the interior electric field as well as predict the measurement value of the measurement (collected electric charges) electrodes. Assume that the electric conductivity, internal charges, and magnetic field are negligible. Then, the mathematical model of the forward ECT problem is given by

\[
\nabla \cdot (\varepsilon \nabla \phi) = 0 \quad \text{in} \quad V_d,
\]

where the boundary conditions

\[
\phi = v_k \quad \text{on} \quad E_k,
\]
and

\begin{equation}
\phi = 0 \quad \text{on} \quad \partial V \setminus \cup_k E_k,
\end{equation}

where \( V_d \) is the region containing the field (possibly it is an infinite region), \( \varepsilon \) is dielectric permittivity, \( E_k \) is the \( k \)-th electrode held at the potential \( v_k \), usually attached on the surface of an insulator \( \phi = V \). The electric charge on the \( k \)-th electrode is given by

\begin{equation}
Q_k = \int_{E_k} \varepsilon \frac{\partial \phi}{\partial n} ds,
\end{equation}

where \( n \) is the inward normal on the \( k \)-th electrode. The FEM is a powerful method to solve such a problem. In Figure 3 one can see a FEM mesh of a 2D ECT system. Using Galerkin’s approximation, this boundary value problem results in a linear system of equations,

\begin{equation}
K \Phi = b,
\end{equation}

where \( K \) is the discrete representation of the operator \( \nabla \cdot \varepsilon \nabla \), and \( b \) is the boundary condition term. A Dirichlet boundary condition was applied for the excitation and sensing electrodes. The external boundary (screen) is left unsigned (Neumann zero) as it approximates perfect shielding of the screen. The linear system of equations can be solved by using preconditioned conjugate gradient (PCG) method. An algebraic multigrid (AMG) solver [9] is used as a preconditioner for conjugate gradient, so that the forward problem can be solved in an efficient way. In practice an inter-electrode shielding prevents this singularity of the electric field. Figure 4 shows the electric field intensity, the singularity of electric field in the edge of excitation electrode. The neighboring sensing electrode can be seen.

The FEM model has been validated with the experimental test data [10]. Figure 5 shows an experimental test data collected from a test example versus the results from the forward solver. In the test example there is a plastic ring (permittivity 1.8) and the measured data and simulated ones show a good agreement. Figure

![Figure 2. ECT sensor arrangement and geometry](image)
3.1. Sensitivity map. The liberalization is well known for full Maxwell’s equations [11]. For a perturbed region, we have

\[
\frac{dQ_{ij}}{d\epsilon} \delta \epsilon = \int_{\Omega} \delta \epsilon E_i \cdot E_j \, dv,
\]

where $\Omega$ is the perturbed region, and $E_i$ and $E_j$ are the electric fields (the solutions of the forward problems) when electrodes $i$ and $j$ are excited. This sensitivity
formula results in an efficient method for the assembly of the Jacobian matrix. Figure 7 shows the sensitivity map for two opposite electrodes (electrodes 1-5) for the empty tube. The sensitivity map may change as the background changes. If the change in the background permittivity distribution is not very high, the sensitivity map remains almost the same. Figure 9 shows the sensitivity map between two opposite electrodes when the background is the permittivity distribution of Figure 8 which includes ring and a rod with permittivity 8 near to the wall. High permittivity inclusions change the pattern of the electrostatic field between two electrodes. The change of the electric field pattern changes the sensitivity map. The change of the sensitivity map with the background permittivity distribution means the requirement of updating the sensitivity map (Jacobian matrix) in image reconstruction iterations. The ill-posed inverse permittivity problem is sensitive to the accuracy of the computed Jacobian matrix. If the Jacobian has some error, that may cause a large error to the image.
4. The inverse problem

The inverse problem in ECT is to reconstruct the permittivity map of the interior given the capacitance data from exterior electrodes [5]. This is an ill posed and nonlinear problem, and therefore hard to solve with noisy measurement data and error in simulation of the forward model. The prior knowledge of the permittivity distributions may help to solve the problem to obtain an acceptable approximated solution. Prior knowledge is also a key to choose the reconstruction scheme as there is no unique and reliable method to solve the inverse problem. The expected information from an ECT image also determines which method must be used. For example, sometimes an indication for the approximate place of an object inside the pipe is enough. However, in another case, it might be important to create an

\[ \text{Figure 6. Percentage error between measured and simulated capacitances for the model of Figure 5} \]

\[ \text{Figure 7. Sensitivity plot for two opposite electrodes for free space} \]
accurate image of the shape of anomalies and sometimes the absolute value of the permittivity distribution is required.

4.1. Linear methods. Linear reconstruction relies on the fact that for small changes, the capacitance can be approximated in a linear fashion with the permittivity, which may be expressed by using the Jacobian matrix $J$ as

\begin{equation}
J(\epsilon - \epsilon_0) \approx F(\epsilon) - C_m,
\end{equation}

where $C_m$ is the vector of capacitance measurements and the forward solution $F(\epsilon)$ is the predicted capacitance from the forward model with permittivity $\epsilon$, and $\epsilon_0$ is the background permittivity. This could be interpreted as seeking either a difference image from the difference between two sets of measurement data, or it could be a step in a non-linear iterative algorithm in which the capacitance difference
is taken between calculated and measured data. If the number of unknowns is smaller or bigger than the number of the measurements, then the matrix $J$ is not square. In such a case we can use the Moore-Penrose generalized inverse, however we must also consider the stability of the solution. In particular, measurement noise and computational error that occur during the forward modelling means that the perturbations in object properties that can be reconstructed have also to be big enough, in order to create sufficient signal changes above the noise and computation errors. Mathematically, this is described as ill posedness of the inverse problem. The minimization of misfit between data and model is difficult, because small errors in the measurements or simulations can lead to large errors in the solution. For this reason, naturally, some assumptions and a solution are required. We use the linear approximation:

\begin{equation}
\delta C = J \delta \epsilon,
\end{equation}

if $J$ is not square

\begin{equation}
\delta C = J^+ \delta \epsilon,
\end{equation}

where $J^+ = (J^T J)^{-1} J^T$ in under determined case and $J^+ = (J J^T)^{-1} (J^T)$ for over-determined case. The ill-posed problem means that the condition number of the $(J J^T)$ is very high. Big change on the $\epsilon$ makes a small change on the measurements or $F(\epsilon)$. In order to overcome the ill posedness one needs to use regularization methods.

This regularization would mean making the linear system better conditioned. However, it does not necessarily mean having a solution that is acceptable. An acceptable solution can be achieved by considering the realistic situation in the measurement as well as the material side. In measurement side we would like to include the reality of the hardware noise and any other sources of errors on either $C_m$ or $F(\epsilon)$. In the parameter side a good initial guess is a good regularization which means $\| \epsilon - \epsilon_0 \|$ is small.

A powerful set of techniques dealing with sets of equations or matrices that are either singular or numerically very close to singular, which is the so-called singular value decomposition (SVD). SVD allows one to diagnose the problems in a given matrix and provides numerical answer as well. In Figure 10 one can see the plot of the eigenvalues from singular value decomposition for 8 electrodes ECT system with 28 independent measurements.

Because of linear dependence in the data measurement there is a sudden fall in eigenvalues. But normally we choose those measurements which are theoretically independent, as the linearly dependent do not create any extra information. The slope of the eigenvalue plot in first 28 points shows the ill-posedness and in ideal case it must be a flat line.

Any $m \times n$ matrix with $m \geq n$ can be written as the product of an $m \times n$ column-orthogonal matrix $U$, an $n \times n$ diagonal matrix with positive or zero elements, and the transpose of an $n \times n$ orthogonal matrix $V$. SVD is a decomposition of matrix $J = U \Sigma V^T$ where $\text{diag}(\Sigma)$ are the singular values of $J$ in order and $J^+ = U \Sigma^{-1} V^T$. Doing $\sigma_{\text{new}} = \frac{\sigma}{(\sigma^2 + \alpha^2)}$ is the same as doing Tikhonov regularization. Choosing smaller $\alpha$ allows us to recover more details from the image that are higher spatial frequency data.
4.2. Nonlinear inverse problem. In principle, for the inverse permittivity problem the aim is to obtain a stable solution $\epsilon^*$ which minimizes the residual error [7]

\[ f(\epsilon) = \frac{1}{2} (F(\epsilon) - C)^T (F(\epsilon) - C) = \frac{1}{2} \| F(\epsilon) - C \|^2, \]

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the non-linear forward operator in a problem with $n$ parameters (voxels) and $m$ measurements, and $C \in \mathbb{R}^m$ is the vector of capacitance measurements for a fixed set of applied voltage. For clarity in the sequel let $D(\epsilon) = (F(\epsilon) - C)$. As $F(\epsilon)$ is analytic, the Taylor series of $D(\epsilon)$ is

\[ D(\epsilon + h) = D(\epsilon) + D'(\epsilon)h + \frac{1}{2} D''(\epsilon)h^2 + O(h^3). \]

In an initial simplistic attempt to minimize $f(\epsilon)$ one can follow a linear least squares approach from which the Gauss–Newton iteration for well-posed problems is derived by seeking a step $h$ for which $D(\epsilon + h) \approx 0$. Neglecting second order terms and above

\[ D(\epsilon) = -D'(\epsilon)h, \]

\[ h_{NR} = -(D'(\epsilon))^{-1}D(\epsilon) = F'(\epsilon)^{-1}(C - F(\epsilon)), \]

we finally arrive to the Gauss–Newton iterative solution

\[ \epsilon_{k+1} = \epsilon_k + F'(\epsilon_k)^{-1}(C - F(\epsilon)). \]

Here $F'(\epsilon_k)$ is the Jacobian matrix. The gradient $\nabla f = f'(\epsilon)^T$ and the Hessian $Hf = f''(\epsilon)$ of the residual $f$ at $\epsilon$ are

\[ \nabla f = D'(\epsilon)^T D(\epsilon) = F'(\epsilon)^T (F(\epsilon) - C), \]

\[ Hf = D'(\epsilon)^T D'(\epsilon) + D''(\epsilon)D(\epsilon) = F'(\epsilon)^T F''(\epsilon) + \sum_i F''_i(\epsilon)(F_i(\epsilon) - C_i). \]
As soon as a minimum is approached, the second derivative terms become negligible, therefore it can be assumed that

\[(15) \sum_i F_i''(\epsilon)(F_i(\epsilon) - C_i) \simeq 0.\]

Under this assumption we can reformulate the residual as

\[(16) f(\epsilon + h) = f(\epsilon) + f'(\epsilon)h + \frac{1}{2}f''(\epsilon)h^2,\]

\[(17) f(\epsilon + h) = f(\epsilon) + (D'(\epsilon)T D(\epsilon))h + \frac{1}{2}(D'(\epsilon)T D'(\epsilon))h^2,\]

\[(18) f(\epsilon + h) = f(\epsilon) + F'(\epsilon)T(F(\epsilon) - C)h + \frac{1}{2}F'(\epsilon)T F'(\epsilon)h^2.\]

Setting the gradient to zero yields

\[(19) h \cdot \nabla f = F'(\epsilon)T (F(\epsilon) - C) + hF'(\epsilon)T F'(\epsilon) = 0\]

from which the step \(h\) is derived as follows:

\[(20) h = (F'(\epsilon)T F'(\epsilon))^{-1}F'(\epsilon)T (C - F(\epsilon)) = F'(\epsilon)\dagger (C - F(\epsilon)),\]

where \(F'(\epsilon)\dagger\) is the so called Moore–Penrose generalized inverse of \(F'(\epsilon)\).

In some ECT applications, the permittivity changes are high. An example of high contrast ECT application is the separation of oil/water. For high contrast ECT problem, the linear method fails to solve the inverse problem properly. A more general approach to nonlinear inverse problem in ECT is regularized GN method. GN method has quadratic convergence rate near the solution and it is accurate in the absence of noise. The image reconstruction in ECT using regularized GN is to optimize

\[(21) \|C_m - F(\epsilon)\|^2 + \alpha^2 \|L\epsilon\|^2.\]

The update formula in each iteration is

\[(22) \delta \epsilon_{n+1} = (J_n^T J_n + \alpha^2 L^T L)^{-1} J_n^T ((C_m - F(\epsilon_n)) - \alpha^2 L^T L \epsilon_n).\]

For \(n = 1\), this is a linear reconstruction. Here \(J_n\) is the Jacobian matrix calculated with the permittivity \(\epsilon_n\), \(C_m\) is the vector of capacitance measurements, and the forward solution \(F(\epsilon_n)\) is the predicted capacitance from the forward model with permittivity \(\epsilon_n\). The matrix \(L\) is the regularization matrix which penalizes extreme changes in permittivity removing the instability in the reconstruction, at the cost of producing artificially smooth images.
4.3. Broyden Quasi–Newton. With the aim of speeding up the nonlinear algorithm BQN method has been studied. A previous study in combining nonlinear and semi-linear steps was conducted earlier [6]. In BQN technique one needs to solve the forward problem in each step and the inverse of the Jacobian matrix can be updated with direct formula as follows.

Assume that $F$ is the boundary capacitance measurement obtained by using finite element method and $C_m$ is the measurement capacitance. Let’s define $D = F - C_m$, $\gamma_i = D_{i+1} - D_i$, and $\delta\epsilon_i = \epsilon_{i+1} - \epsilon_i$. The permittivity distribution that best describes the actual permittivity is the one that makes $D = 0$. The solution for this particular permittivity can be found by using the iterative equation,

$$\epsilon_{i+1} = \epsilon_i - H_i D_i,$$

We start with an initial guess, and update the solution for each iteration. $H_i$ is an approximation of the inverse of the Jacobian matrix related to permittivity distribution $\epsilon_i$. Instead of calculating the Jacobian matrix and solving a linear system of equations in each iteration, the matrix $H$ can be updated with $O(n^2)$ operations, where $n$ is the number of pixels of the image as follows:

$$H_{i+1} = H_i + \frac{(\delta\epsilon_i - H_i \gamma_i)\delta\epsilon_i^T H_i}{\delta\epsilon_i^T H_i \gamma_i}.$$

It has been shown that the method has super-linear convergence [2] and that the set of matrices $H_i$, $i = 1, 2, 3, \ldots$, converges to $H^*$ (the inverse of the Jacobian matrix at the point $x^*$ where $D = 0$ is satisfied). The initial guess for the BQN is important to the convergence. If the algorithm starts close to the solution the BQN converges fast. If the initial guess is far from the solution, the BQN may not converge. This study recommends to use a combined GN and BQN for high contrast permittivity problem, where the first few steps use the GN, and when approaching the solution one can benefit from faster the BQN iteration.

4.4. Image reconstruction results. Some test problems have been used to validate the image reconstruction approaches studied in this paper. Figure 11 shows the reconstruction of a plastic ring with permittivity 1.6 with one linear reconstruction step. Figure 12 shows a horizontal flow reconstructed by using six GN iteration and the norm of the mismatch between measured and simulated capacitance can be seen in Figure 13. Figure 14 shows the reconstructed images of a plastic object with a rather complicated cross shape. Iteration steps in nonlinear and semi-linear method improve the quality of the reconstructed image. Three GN and three BQN steps were used in this example.

Figure 15 (a) shows a test phantom, and this low contrast permittivity is reconstructed by using BQN scheme and is shown in Figure 15 (b). To avoid the so called inverse crime, 1 percent Gaussian noise was added to the simulated measurement data. Four BQN iterations, including the first iteration which involves calculation of the Jacobian matrix and its inverse, were enough to produce satisfactory result. Figure 16 shows an example with background permittivity 1.8, and three inclusions have permittivity 1, 2, 3. The BQN method was used to reconstruct the image. Again, 1 percent Gaussian noise was added to the simulation data. After six iterations the algorithm diverges, and the reconstructed image in iteration 5 shown in Figure 16 (b) are not satisfactory.
Figure 11. Reconstructed image from experimental ECT data of a plastic ring (of Figure 5) using one linear step.

Figure 12. Reconstructed image from experimental ECT data representing a horizontal flow using six non-linear steps.

Figure 13. Norm of the mismatch error between simulated and measured capacitances.
Figure 14. Reconstruction of a cross shape plastic object from the experimental data (a) and the reconstructed image (b).

Figure 15. True permittivity image (a), background has permittivity 1 and the inclusion has permittivity 1.6 and the reconstructed image (b).

Figure 17 shows an example with background permittivity 1, four inclusions with permittivity 6. A combined GN and BQN has been used to reconstruct the image. Here 1 percent Gaussian noise was added to the simulation data. The BQN method is used for the image reconstruction, and the algorithm diverges and it could not reconstruct the test phantom. Then a GN algorithm is used and after four iterations the reconstructed image was in a good agreement with the test phantom. A combined GN and BQN is used, after two GN iterations three more BQN iterations are used, and the reconstructed image are similar to four GN iterations.

Conclusion

This paper examined a finite element based image reconstruction for 2D ECT. Linear, semi-linear and nonlinear methods have been studied for the inverse problem
of ECT. A BQN method is applied to the ECT problem with low contrast permittivity. The algorithm is particularly faster than GN in each iteration. Updating the inverse of the Jacobian matrix with Broyden formula is faster than recalculating the Jacobian matrix and inverting it. Linear and semi-linear methods may not produce good results in all cases. A combined GN, BQN can be used for higher contrast ECT problem to produce accurate images with good speed.

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