GLOBAL STABILITY OF CHEMOSTAT MODELS INVOLVING TIME DELAYS AND ZONES OF NO ACTIVATION

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ABSTRACT. A chemostat model with distributed time delays in both nutrient recycling and biotic species growth is considered. In this paper we introduce a new notion relating to the nonconsumption of nutrient by the consumer at the equilibrium value and this new notion is termed as ‘zone of no activation.’ This may also be viewed as a measure of restoring stability, especially when excess consumption of the nutrient drives the system to instability. The influence of zones of no activation on the global asymptotic stability of the positive equilibrium solution of the model equations is discussed in detail. It is found that the criteria obtained in this paper are easily verifiable and actually independent of each other. Further, the asymptotic nature of solutions, in the event of the model not possessing a positive equilibrium is also discussed.

1. Introduction. In this paper we consider a chemostat model with distributed time lags both in nutrient recycling and biotic species growth. A chemostat is a device used for growing microorganisms in a continuous cultured environment. It is an appropriate laboratory approximation of nature for phenomenon such as the growth of planktonic communities in a lake. A first attempt to model the chemostat problem has stemmed from the work of Hsu, Hubbell and Waltman [12], in which the extensive use of the chemostat in such a laboratory work has been demonstrated.

Thereafter, several authors have studied the model and proposed various improvements. For an excellent treatment of the chemostat problem, its historical account together with all developments, we refer the readers to a recent monograph of Smith and Waltman [15]. For subsequent developments, we refer the readers to [1–7, 9–12, 14–17].

In [4], Beretta and Takeuchi have considered a pair of integro-differential equations with distributed time lags both in material re-
cycling and biotic species growth and studied the global asymptotic stability of the positive equilibrium.

The nutrient uptake represented by the function \( U(x) \), in general, is assumed to be a monotonically increasing function. We disagree with this view and observe that, when the populations are at equilibrium or when the consumer population is completely fed, there cannot be any increase in the consumption at least for a while. Hence, we introduce a new notion called a ‘zone of no activation’ for the consumer by which we mean that the equilibrium renders the nonconsumption of the nutrient for a while. It can also be viewed as a measure of restoring the stability in a situation where excess consumption of the nutrient lead to abnormalities in the biomass growth which may in turn leads to instability of the system.

The following discussion provides a good understanding of the zones of no activation with regard to plants growing under the application of fertilizers. Consider a situation where a plant biomass feeds on a fertilizer. It may be true that a continuous use of fertilizer enriches the nutrient levels of the soil, thus leading to an enhanced plant growth. At the same time, excess use of the fertilizers lead to imbalances in the nutrient levels of the soil causing abnormalities in the plant growth, thus effecting the yield. Therefore, a situation arises when a constant supply of the fertilizer should be considered or even the supply of the fertilizer should be stopped until such a time that the soil has balanced nutrient levels. This we consider by \( U(x) = U(x^*) \) (maintenance of nutrient uptake at equilibrium level) in a certain neighborhood of \( x^* \) and this neighborhood may be regarded as a ‘zone of no activation.’

In our earlier work [16], we have considered the same model, in the absence of zones of no activation, with a more general class of uptake functions, which includes all continuous functions having a saturation effect. Besides several other results, we have presented several easily verifiable independent sets of sufficient conditions for the global asymptotic stability of the positive equilibrium. It turns out that the methods followed in this paper contribute to the improvement of some of the results of [16] in terms of yielding larger regions of asymptotic stability of the positive equilibrium in the absence of zones of no activation.
This paper is organized as follows. The model, its equilibria and properties of solutions are described briefly in Section 2. Section 3 deals with the global asymptotic stability of the equilibrium solution in the presence of a zone of no activation. Examples are given to establish that the results are actually independent of each other. A discussion follows in Section 4.

2. Preliminaries. We consider the following system of integro-differential equations proposed by Beretta and Takeuchi in [4].

\begin{align*}
  x'(t) &= D(x_0 - x(t)) - aU(x(t))y(t) \\
         &\quad + b\gamma \int_{-\infty}^{t} f(t-s)y(s)\,ds \\
  y'(t) &= -(\gamma + D)y(t) + cy(t) \\
         &\quad \times \int_{-\infty}^{t} g(t-s)U(x(s))\,ds - dy^2(t)
\end{align*}

with nonnegative initial conditions,

\begin{align*}
  x(s) &= \phi_1(s), \quad y(s) = \phi_2(s), \quad -\infty < s \leq 0
\end{align*}

which are continuous bounded functions.

In system (2.1), $x$ denotes the limiting nutrient and $y$ is the biotic species feeding on $x$. $x_0 > 0$ is the constant nutrient input concentration. The positive constants $D, a, c (< a), \gamma$ represent the washout rate, the consumption rate of the nutrient, the specific growth rate of the biotic species, the natural death rate coefficient of the biotic species, respectively. The constant $b \in (0, 1)$ represents the fraction of dead biomass that is recycled. The positive constant $d$ accounts for the finite carrying capacity of the environment. The delay kernels for the nutrient recycling and the growth of biomass, respectively $f$ and $g$, are nonnegative and satisfy

\begin{align*}
  &\text{(H}_1\text{)} \quad \int_{0}^{\infty} f(s)\,ds = 1 = \int_{0}^{\infty} g(s)\,ds \\
  \text{and}
  &\text{(H}_2\text{)} \quad \int_{0}^{\infty} sf(s)\,ds < \infty \quad \text{and} \quad \int_{0}^{\infty} sg(s)\,ds < \infty.
\end{align*}
$U(x)$ is called the nutrient uptake function and the following assumptions are made on $U(x)$.

(A$_1$) $U(x)$ is a continuous real valued function defined on $R_+ = [0, \infty)$ such that $U(x) \geq 0$ and $U(x) = 0$ if and only if $x = 0$ and $\lim_{x \to \infty} U(x) = L_1$.

It is easy to see that there exists a constant $L > 0$ such that $U(x) \leq L$ for all $x$.

(A$_2$) There exists a constant $K > 0$ such that

$$|U(x_1) - U(x_2)| \leq K|x_1 - x_2| \quad \forall x_1, x_2 \in R_+ = [0, \infty).$$

An equilibrium solution of the system (2.1) is a solution of the algebraic system,

$$Dx_0 - Dx - aU(x)y + b\gamma y = 0$$
$$[-(\gamma + D) + cU(x) - dy]y = 0,$$

for which $(x_0, 0)$ is a solution which is an axial equilibrium solution of system (2.1). Any positive equilibrium solution of (2.1), say $(x^*, y^*)$, should satisfy

(2.3)
$$Dx_0 - Dx^* - aU(x^*)y^* + b\gamma y^* = 0$$
$$-(\gamma + D) + cU(x^*) - dy^* = 0$$

and is given by

$$x^* = x_0 - \frac{(aU(x^*) - b\gamma)(cU(x^*) - \gamma - D)}{dD}$$
$$y^* = \frac{cU(x^*) - \gamma - D}{d}.$$

A set of necessary and sufficient conditions for the existence of a unique positive equilibrium for system (2.1) is given in [16, Theorem 3.1]. Also, it is proved that the solutions are nonnegative and bounded ([16, Theorem 3.5]).

Throughout the rest of the paper we assume that there exists a unique positive equilibrium $(x^*, y^*)$ for the system (2.1), the solutions of which are nonnegative and bounded.
3. Zones of no activation. In this section we discuss the global asymptotic stability of the positive equilibrium solution of the system (2.1), when the uptake functions have zones of no activation as described in Section 1. We present several examples for illustration.

We make the following change of variables

\[ u(t) = x(t) - x^*, \quad v(t) = y(t) - y^* \]

and

\[ G(u) = U(x) - U(x^*) \]

which transforms system (2.1) into

\[
\begin{align*}
    u'(t) &= -Du(t) - a(G(u(t)) + U(x^*))v(t) \\
          &\quad \quad - ay^* G(u(t)) + b\gamma \int_{-\infty}^{t} f(t-s)v(s) \, ds \\
    v'(t) &= (v(t) + y^*) \left[ c \int_{-\infty}^{t} g(t-s)G(u(s)) \, ds - dv(t) \right].
\end{align*}
\]

In the zone of no activation we define the uptake function defined by \( G(u) \) as

\[
G(u) = \begin{cases} 
    G_1(u), & \text{for } u > \delta_1, \\
    G_2(u), & \text{for } u < -\delta_1, \\
    0, & \text{for } |u| \leq \delta_1.
\end{cases}
\]

We observe that \((0,0)\) is an equilibrium solution of (3.1).

Before proving our first result we observe that, for any \( a, b \) and \( \eta > 0 \), the following inequality holds

\[
ab \leq \frac{1}{4\eta} a^2 + \eta b^2.
\]

We now state and prove our first result in this section.

**Theorem 3.1.** Assume that the delay kernels satisfy the conditions (H1) and (H2) and the uptake function satisfies (A1). The equilibrium
solution \((0,0)\) of \((3.1)\) is globally asymptotically stable provided there exist constants \(\eta_1 > 0\) and \(\eta_2 > 0\) such that
\[
\min \left\{ D - \frac{b\gamma}{4\eta_1} + A_1, D - \frac{b\gamma}{4\eta_1} + A_2, D - \frac{b\gamma}{4\eta_1} \right\} > 0,
\]
\[
d - \frac{c}{4\eta_2} - b\gamma\eta_1 > 0
\]
and
\[
a^2L^2 < 4B \left[ d - \frac{c}{4\eta_2} - b\gamma\eta_1 \right]
\]
where
\[
B = \min \left\{ D - \frac{b\gamma}{4\eta_1} + A_1, D - \frac{b\gamma}{4\eta_1} + A_2, D - \frac{b\gamma}{4\eta_1} \right\} > 0,
\]
\[
A_1 = \min_{u > \delta_1} \left\{ ay^* \left( \frac{G_1(u)}{u} \right) - c\eta_2 \left( \frac{G_1(u)}{u} \right)^2 \right\}
\]
and
\[
A_2 = \min_{u < -\delta_1} \left\{ ay^* \left( \frac{G_2(u)}{u} \right) - c\eta_2 \left( \frac{G_2(u)}{u} \right)^2 \right\}.
\]

**Proof.** Consider the functional
\[
V(t) \equiv V(u(t), v(t)) = \frac{u^2(t)}{2} + \int_0^{v(t)} \frac{z}{z + y^*} \, dz
\]
\[
+ c\eta_2 \int_0^\infty g(s) \int_{t-s}^t \frac{G^2(u(t_1))}{u} \, dt_1 \, ds
\]
\[
+ b\gamma\eta_1 \int_0^\infty f(s) \int_{t-s}^t \frac{v^2(t_1)}{u} \, dt_1 \, ds.
\]
Clearly, \(V(0,0) = 0\) and
\[
V(u(t), v(t)) \geq \frac{u^2(t)}{2} + \int_0^{v(t)} \frac{z}{z + y^*} \, dz > 0.
\]
The time derivative of $V$ along the solutions of system (3.1) is given by
\[
\frac{dV}{dt} = u(t)u'(t) + \frac{v(t)}{v(t) + y^*}v'(t) + c\eta_2 G^2(u(t))
\]
\[- c\eta_2 \int_0^\infty g(s)G^2(u(t - s)) \, ds
\]
\[+ b\gamma \eta_1 v^2(t) - b\gamma \eta_1 \int_0^\infty f(s)v^2(t - s) \, ds
\]
\[= -Du^2(t) - a(G(u(t)) + U(x^*))u(t)v(t)
\]
\[- ay^* G(u(t))u(t) + b\gamma \int_0^\infty f(s)u(t)v(t - s) \, ds
\]
\[+ c \int_0^\infty g(s)v(t)G(u(t - s)) \, ds - dv^2(t)
\]
\[+ c\eta_2 G^2(u(t)) - c\eta_2 \int_0^\infty g(s)G^2(u(t - s)) \, ds
\]
\[+ b\gamma \eta_1 v^2(t) - b\gamma \eta_1 \int_0^\infty f(s)v^2(t - s) \, ds.
\]

Utilizing the inequality (3.3) with $\eta = \eta_1$ and $\eta = \eta_2$ in the fourth and fifth terms of the right hand side of the above equation, invoking $(H_1)$ on the delay kernels and simplifying, we get
\[
\frac{dV}{dt} \leq -\left(D - \frac{b\gamma}{4\eta_1}\right)u^2(t) - a(G(u) + U(x^*))u(t)v(t)
\]
\[- \left[d - \frac{c}{4\eta_2} - b\gamma \eta_1\right]v^2(t) - ay^* G(u(t))u(t) + c\eta_2 G^2(u(t)).
\]

Now, from (3.2), we have
\[
\frac{dV}{dt} \leq -\left\{
\begin{array}{ll}
[D - (b\gamma/4\eta_1) + A_1]u^2(t) + a(G_1(u) + U(x^*))u(t)v(t) \\
+ [d - (c/4\eta_2) - b\gamma \eta_1]v^2(t) \quad \text{for } u > \delta_1
\end{array}
\right.
\]
\[
\frac{dV}{dt} \leq -\left\{
\begin{array}{ll}
[D - (b\gamma/4\eta_1) + A_2]u^2(t) + a(G_2(u) + U(x^*))u(t)v(t) \\
+ [d - (c/4\eta_2) - b\gamma \eta_1]v^2(t) \quad \text{for } u < -\delta_1
\end{array}
\right.
\]
\[
\frac{dV}{dt} \leq -\left\{
\begin{array}{ll}
[D - (b\gamma/4\eta_1)]u^2(t) + aU(x^*)u(t)v(t) \\
+ [d - (c/4\eta_2) - b\gamma \eta_1]v^2(t) \quad \text{for } |u| \leq \delta_1.
\end{array}
\right.
\]
Thus, we have

\[
\frac{dV}{dt} \leq - \left[ Bu^2(t) + a(G(u) + U(x^*))u(t)v(t) + \left( d - \frac{c}{4\eta_2} - b\gamma \eta_1 \right) v^2(t) \right].
\]

Then \( dV/dt \) is negative definite, provided

\[ [a(G(u) + U(x^*))]^2 < 4B \left[ d - \frac{c}{4\eta_2} - b\gamma \eta_1 \right]. \]

Since \( G(u) + U(x^*) \leq L \), for all \( u \), it follows that \( dV/dt \) is negative definite if

\[ a^2L^2 < 4B \left[ d - \frac{c}{4\eta_2} - b\gamma \eta_1 \right]. \]

Now let \( Q(t) = (u(t), v(t))^T \),

\[ P = \begin{pmatrix} B & (aL/2) \\ (aL/2) & d - (c/4\eta_2) - b\gamma \eta_1 \end{pmatrix}. \]

Then from (3.4) we have

\[ \frac{dV}{dt} \leq -Q(t)^T PQ(t) \leq -\bar{\lambda}(u^2(t) + v^2(t)), \]

where \( \bar{\lambda} = \min\{\lambda(P)\} \), implying the negative definiteness of \( dV/dt \).

Now the conclusion follows from [8]. The proof is complete. \( \square \)

The following examples illustrate Theorem 3.1.

**Example 3.2.** Consider the system (3.1) with

\[
G(u) = \begin{cases} 
(u - \delta_1)/(p + u - \delta_1) & \text{for } u > \delta_1 \\
0 & \text{for } |u| \leq \delta_1 \\
(u + \delta_1)/(p + u + \delta_1) & \text{for } u < -\delta_1,
\end{cases}
\]

where \( p = 1 + x^* \).
This uptake function corresponds to Michaelis-Menten kinetics. Clearly, \( uG_1(u) > 0 \) and \( uG_2(u) > 0 \) and
\[
\lim_{u \to \infty} \frac{G_1(u)}{u} = 0 = \lim_{u \to \infty} \frac{G_2(u)}{u},
\]
where
\[
G_1(u) = \frac{u - \delta_1}{p + u - \delta_1} \quad \text{and} \quad G_2(u) = \frac{u + \delta_1}{p + u + \delta_1}.
\]
Therefore, \( A_1 = A_2 = 0. \)

Thus, the equilibrium solution \((0,0)\) of (3.1) is globally asymptotically stable provided
\[
D - \frac{b\gamma}{4\eta_1} > 0, \quad d - b\gamma\eta_1 - \frac{c}{4\eta_2} > 0
\]
and
\[
a^2L^2 < 4 \left[ D - \frac{b\gamma}{4\eta_1} \right] \left[ d - b\gamma\eta_1 - \frac{c}{4\eta_2} \right].
\]

**Example 3.3.** Consider the system (3.1) with
\[
G(u) = \begin{cases} 
  z(\delta_1)(u - \delta_1) & \text{for } u > \delta_1 \\
  0 & \text{for } |u| \leq \delta_1 \\
  z(\delta_1)(u + \delta_1) & \text{for } u < -\delta_1,
\end{cases}
\]
where \( z(\delta_1) \) satisfies \( \lim_{\delta_1 \to 0} z(\delta_1) = (ay^*/cn_2) \), with \( G_1(u) = z(\delta_1)(u - \delta_1), \ G_2(u) = z(\delta_1)(u + \delta_1). \)

We have \( A_1 = A_2 = z(\delta_1)ay^* - z^2(\delta_1)cn_2 = A, \) (say).

The condition \( D - (b\gamma/4\eta_1) + A > 0 \) gives an estimate for \( z(\delta_1) \) and is given by
\[
z(\delta_1) < \frac{ay^* + \sqrt{a^2y^{*2} + 4cn_2[D - (b\gamma/4\eta_1)]}}{2cn_2}.
\]

Now the equilibrium solution \((0,0)\) of (3.1) is globally asymptotically stable if
\[
D - \frac{b\gamma}{4\eta_1} > 0, \quad d - b\gamma\eta_1 - \frac{c}{4\eta_2} > 0
\]
and
\[ a^2L^2 < 4 \left[ D - \frac{b\gamma}{4\eta_1} + A \right] \left[ d - b\gamma \eta_1 - \frac{c}{4\eta_2} \right]. \]

From the definitions of \( A_1 \) and \( A_2 \) in Theorem 3.1 and as seen in the above examples, it follows that the functions \( G_1(u) \) and \( G_2(u) \) cannot be superlinear in ‘\( u \).’ For, if \( G_1(u) \) and \( G_2(u) \) are superlinear, the values of \( A_1 \) and \( A_2 \) may be in the extended real number system. The following theorem accommodates the superlinear uptake functions which have saturation effects and accordingly we define \( G(u) \) as follows

\[
G(u) = \begin{cases} 
G_1^*(u) & \text{for } u(t) > \delta_1 \\
0 & \text{for } |u(t)| \leq \delta_1 \\
G_2^*(u) & \text{for } u(t) < -\delta_1 
\end{cases}
\]

where

\[
G_1^*(u) = \begin{cases} 
P_1(u) & \text{for } \delta_1 < u \leq \alpha_1 \\
P_1(\alpha_1) & \text{for } u > \alpha_1 
\end{cases}
\]

and

\[
G_2^*(u) = \begin{cases} 
P_2(u) & \text{for } -\alpha_2 \leq u < -\delta_1 \\
P_2(-\alpha_2) & \text{for } u < -\alpha_2, 
\end{cases}
\]

in which \( \alpha_1, \alpha_2 \) are positive real numbers and \( P_1, P_2 \) are continuous and can be superlinear in their arguments.

**Theorem 3.4.** Assume that the delay kernels satisfy the conditions \( (H_1), (H_2), \) and that the uptake function satisfies \( (A_1) \). The equilibrium solution \((0,0)\) of (3.1) is globally asymptotically stable provided

\[
\min \left\{ D - \frac{b\gamma}{4\eta_1} + A_1^*, D - \frac{b\gamma}{4\eta_1} + A_2^*, D - \frac{b\gamma}{4\eta_1} \right\} > 0,
\]

\[
d - b\gamma \eta_1 - \frac{c}{4\eta_2} > 0 \quad \text{and} \quad a^2L^2 < 4B^* \left[ d - b\gamma \eta_1 - \frac{c}{4\eta_2} \right]
\]

where

\[
B^* = \min \left\{ D - \frac{b\gamma}{4\eta_1} + A_1^*, D - \frac{b\gamma}{4\eta_1} + A_2^*, D - \frac{b\gamma}{4\eta_1} \right\} > 0,
\]

\[
A_1^* = \min_{u > \delta_1} \left\{ ay^* \left[ \frac{P_1(u)}{u} \right] - c\eta_2 \left[ \frac{P_1(u)}{u} \right]^2 \right\}
\]
and

\[ A_2^* = \min_{u < -\delta_1} \left\{ ay^* \left[ \frac{P_2(u)}{u} \right] - c\eta_2 \left[ \frac{P_2(u)}{u} \right]^2 \right\}. \]

Proof. The proof of this theorem is similar to that of Theorem 3.1 and hence is omitted. \(\square\)

**Theorem 3.5.** Assume that the delay kernels satisfy \((H_1)\) and \((H_2)\) and the uptake function satisfies \((A_1)\) and \((A_2)\). The equilibrium solution \((0,0)\) of \((3.1)\) is globally asymptotically stable provided

\[ \mu = \min\{D - (c - ay^*)K_1, D - (c - ay^*)K_2\} > 0, \]

\[ \nu_1 = \min_{u > \delta_1} \{ d + a(G_1(u) + U(x^*)) - b\gamma \} \]

and

\[ \nu_2 = \min_{u < -\delta_1} \{ d + a(G_2(u) + U(x^*)) - b\gamma \} > 0 \]

in which

\[ K_1 > 0 \quad \text{and} \quad K_2 > 0 \]

such that

\[ |G_1(u)| \leq K_1|u| \quad \text{for} \ u > \delta_1 \quad \text{and} \quad |G_2(u)| \leq K_2|u| \quad \text{for} \ u < -\delta_1. \]

Remark 3.6. The existence of the positive constants \(K_1\) and \(K_2\) is assured by the assumption \((A_2)\) on \(G(u)\). In case \(\delta_1 = 0\) we have \(K_1 = K_2 = K\) (the Lipschitz constant defined in \((A_2)\)).

Proof of Theorem 3.5. We consider the following functional

\[
V(t) = |u(t)| + \left| \log \frac{v(t) + y^*}{y^*} \right| + b\gamma \int_0^\infty f(s) \int_{t-s}^t |v(z)| \, dz \, ds + c \int_0^\infty g(s) \int_{t-s}^t |G(u(z))| \, dz \, ds.
\]
The upper Dini derivative of $V$ along the solutions of system (3.1), after some simplifications, is given by

$$D^+ V \leq -D|u(t)| + (c - ay^*)|G(u(t))|$$

$$- [d + a(G(u) + U(x^*)) - b\gamma]|v(t)|$$

and from (3.2) we have

$$D^+ V \leq - \begin{cases} [D - (c - ay^*)K_1]|u(t)| \\ + [d + a(G_1(u) + U(x^*)) - b\gamma]|v(t)| & \text{for } u(t) > \delta_1 \\ [D - (c - ay^*)K_2]|u(t)| \\ + [d + a(G_2(u) + U(x^*)) - b\gamma]|v(t)| & \text{for } u(t) < -\delta_1 \\ D|u(t)| + (d + aU(x^*) - b\gamma)|v(t)| & \text{for } |u(t)| \leq \delta_1. \end{cases}$$

Let $\nu = \min\{\nu_1, \nu_2\}$. Then it follows that

$$D^+ V \leq -[\mu|u(t)| + \nu|v(t)||.$$

Integrating this from 0 to $t$, we get

$$V(t) + \mu \int_0^t |u(s)| \, ds + \nu \int_0^t |v(s)| \, ds \leq V(0),$$

which implies that

$$V(t) + \mu \int_0^t |u(s)| \, ds + \nu \int_0^t \log \left(\frac{v(s) + y^*}{y^*}\right) \, ds \leq V(0).$$

Therefore, $V(t)$ is bounded on $[0, \infty)$ and, since the solutions of system (3.1) are bounded, it follows that $u(t)$ and $\log((v(t) + y^*)/y^*)$ are bounded, implying that $D^+ V$ is bounded. Now the conclusion follows from Lemma 1 in [13, pp. 601–602].

Remark 3.7. In the hypotheses of the above theorem, it is taken that $c - ay^* > 0$. In case $c - ay^* < 0$, we have the upper Dini derivative after some simplifications given by

$$D^+ V \leq -D|u(t)| + (c - ay^*)|G(u(t))|$$

$$- (d + a(G(u(t) + U(x^*)) - b\gamma)|y(t) - y^*| < 0,$$
since $\nu_1 > 0, \nu_2 > 0$. Thus we observe that hypotheses $\nu_1 > 0, \nu_2 > 0$ are alone sufficient to ensure the global asymptotic stability of $(0,0)$ in this case.

In the next result we relax the condition $(H_2)$ on the delay kernels at the expense of placing more restrictions on the parameters of the system (3.1).

**Theorem 3.8.** Assume that the delay kernels satisfy $(H_1)$, the uptake function satisfies $(A_1)$ and $(A_2)$. The equilibrium solution $(0,0)$ of system (3.1) is globally asymptotically stable provided

\[ b? + cK_1 < \min \{ D - aK_1 y^*, \min_{u(t) > \delta_1} \{ d + a(G_1(u) + U(x^*)) \} \} = \beta_1 \text{ (say)}, \]
\[ b? < \min \{ D, d + aU(x^*) \} = \beta \text{ (say)}, \]

and

\[ b? + cK_2 < \min \{ D - aK_2 y^*, \min_{u(t) < -\delta_1} \{ d + a(G_2(u) + U(x^*)) \} \} = \beta_2 \text{ (say)}, \]

where $K_1$ and $K_2$ are as in Theorem 3.5.

**Proof.** Consider the functional

\[ V(t) = V(u(t), v(t)) = |u(t)| + \log \left( \frac{V(t) + y^*}{y^*} \right). \]

Clearly $V(0,0) = 0$, $V(t) > 0$.

The upper Dini derivative of $V$ along the solutions of (3.1) is given
by

\[ D^+ V \leq -D|u(t)| - a(G(u) + U(x^*))|v(t)| - a\gamma|G(u(t))| \\
+ b\gamma \int_0^\infty f(s)|v(t-s)| \, ds + c \int_0^\infty g(s)|G(u(t-s))| \, ds - d|v(t)| \\
\begin{cases} 
-(D - aK_1)u(t)| - [d + a(G_1(u) + U(x^*))]|v(t)| \\
+ b\gamma \int_0^\infty f(s)|v(t-s)| \, ds + c \int_0^\infty g(s)|G_1(u(t-s))| \, ds & \text{for } u(t) > \delta_1 \\
-(D - aK_2)u(t)| - [d + a(G_2(u) + U(x^*))]|v(t)| \\
+ b\gamma \int_0^\infty f(s)|v(t-s)| \, ds + c \int_0^\infty g(s)|G_2(u(t-s))| \, ds & \text{for } u(t) < -\delta_1 \\
-D|u(t)| - (d + aU(x^*))|v(t)| + b\gamma \int_0^\infty f(s)|v(t-s)| \, ds & \text{for } |u(t)| \leq \delta_1,
\end{cases} \]

using (3.2), and

\[ \begin{cases} 
-\beta_1 V(t) + b\gamma \int_0^\infty f(s)|v(t-s)| \, ds \\
+ cK_1 \int_0^\infty g(s)|u(t-s)| \, ds & \text{for } u > \delta_1 \\
-\beta_2 V(t) + b\gamma \int_0^\infty f(s)|v(t-s)| \, ds \\
+ cK_2 \int_0^\infty g(s)|u(t-s)| \, ds & \text{for } u < -\delta_1 \\
-\beta V(t) + b\gamma \int_0^\infty f(s)|v(t-s)| \, ds & \text{for } |u| \leq \delta_1
\end{cases} \]

\[ \begin{cases} 
-\beta_1 V(t) + b\gamma \int_0^\infty f(s)V(t-s) \, ds \\
+ cK_1 \int_0^\infty g(s)V(t-s) \, ds & \text{for } u > \delta_1 \\
-\beta_2 V(t) + b\gamma \int_0^\infty f(s)V(t-s) \, ds \\
+ cK_2 \int_0^\infty g(s)V(t-s) \, ds & \text{for } u < -\delta_1 \\
-\beta V(t) + b\gamma \int_0^\infty f(s)V(t-s) \, ds & \text{for } |u| \leq \delta_1
\end{cases} \]

\[ < 0, \text{ using the hypotheses.} \]

Since the rest of the proof is similar to that of Theorem 5.6 of [16], we omit the details here.
The following examples illustrate that Theorems 3.1 and 3.5 are independent of each other.

Example 3.9. Consider the model

\begin{align*}
  x'(t) &= 2.75(x_0 - x(t)) - 18U(x(t))y(t) + (0.0625) \int_{-\infty}^{t} f(t - s)y(s) \, ds \\
  y'(t) &= -3y(t) + 16y(t) \int_{-\infty}^{t} g(t - s)U(x(s)) \, ds - 44y^2(t),
\end{align*}

in which \( U(x) = x/(4 + x) \), \( b = 0.25 \), \( \gamma = 0.25 \) and \( D = 2.75 \).

The equilibrium solutions are \((x^*, y^*) = ((8/3), 0.0773)\) with \( U(x^*) = (2/5) \).

For this system, define

\[ G_1(u) = G_2(u) = \frac{9u}{5(3u + 20)}. \]

Clearly, \( K_1 = K_2 = (1/4) \).

We observe that \( D - cK_1 + aK_1y^* < 0 \), \( D - cK_2 + aK_2y^* < 0 \) and, hence, Theorem 3.5 cannot be applied here.

Let us verify the hypotheses of Theorem 3.1.

Clearly, for the choice of \( \eta_1 = 1 \), \( \eta_2 = 2 \) and \( \delta_1 = 1 \), we see that \( A_1 = -0.086 \), \( A_2 = -0.51 \) and, accordingly, \( B = 2.1775 \). It is easy to see that all conditions of Theorem 3.1 are satisfied and \((x^*, y^*) = ((8/3), 0.0773)\) is globally asymptotically stable in view of Theorem 3.1.

Example 3.10. Consider the following system

\begin{align*}
  x'(t) &= 2(x_0 - x(t)) - 18U(x(t))y(t) + (0.25) \int_{-\infty}^{t} f(t - s)y(s) \, ds \\
  y'(t) &= -3y(t) + 16y(t) \int_{-\infty}^{t} g(t - s)U(x(s)) \, ds - 3y^2(t),
\end{align*}

in which \( U(x) = x/(4 + x) \), \( b = 0.25 \), \( \gamma = 1 \) and \( D = 2 \).
The equilibrium solutions are \((x^*, y^*) = ((8/3), 1.133)\) with 

\[U(x^*) = \frac{2}{5}\] and \(x_0 \approx 29.716\) approximately.

Define \(G_1(u)\) and \(G_2(u)\) as in Example 3.8. We notice that Theorem 3.1 cannot be applied as the following inequality

\[a^2L^2 < 4B \left[ d - \frac{c}{4\eta_2} - b\gamma\eta_1 \right]\]

is not satisfied for any choice of \(\eta_1\) and \(\eta_2\). However, we observe that for \(K_1 = K_2 = (1/4)\), all the hypotheses of Theorem 3.5 are satisfied, and hence \((x^*, y^*)\) is globally asymptotically stable.

We shall now study the asymptotic nature of solutions when the given system of equations does not possess a saturated equilibrium solution. The following theorem gives sufficient conditions for the asymptotic equivalence of solutions of system (2.1) in the presence of a zone of no activation.

We now define

\[U(x) = \begin{cases} 
U_1(x), & 0 \leq x < \alpha, \\
U^*, & \alpha \leq x \leq \beta, \\
U_2(x), & x > \beta,
\end{cases}\]

where \(\alpha, \beta\) are positive real numbers such that \(\alpha < \beta\) and \(U_1(x)\) and \(U_2(x)\) are continuous, bounded in their intervals of definition and \(\lim_{x \to \alpha} U_1(x) = U^* = \lim_{x \to \beta} U_2(x)\).

Further assume that there exist constants \(K_1, K_2 > 0\) such that

\[(A_3) \quad |U_i(w) - U_i(z)| \leq K_i|w - z|, \quad i = 1, 2,\]

for all \(w, z \in [0, \infty)\).

Also, since the solutions of system (2.1) are bounded, we can find a constant \(M > 0\) such that \(|y(t)| \leq M\) for all \(t\).

We now state and prove our next theorem.

**Theorem 3.11.** Assume that the delay kernels satisfy the conditions \((H_1)\) and \((H_2)\) and \(U_1(x)\) and \(U_2(x)\) satisfy \((A_3)\). Further, assume that
there exists a continuously differentiable function $P(t) = (p_1(t), p_2(t))$, $p_i(t) > 0$, $i = 1, 2$, and bounded on $0 \leq t < \infty$ such that

$$p_1'(t) = Dx_0 - Dp_1(t) - aU(p_1(t))p_2(t) + b\gamma \int_0^\infty f(s)p_2(t-s) \, ds + q_1(t)$$

$$p_2'(t) = p_2(t)[-(\gamma + D) + c \int_0^\infty g(s)U(p_1(t-s)) \, ds - dp_2(t) + q_2(t)],$$

where $q_i(t), i = 1, 2$, are bounded on $[0, \infty)$ and $\int_0^\infty |q_i(t)| \, dt < \infty$.

Then for any solution $(x(t), y(t))$ of (2.1), one has

$$\lim_{t \to \infty} (x(t), y(t)) = (p_1(t), p_2(t))$$

provided

$$D > \max\{cK_1 + aK_1 M, cK_2 + aK_2 M\},$$

$$\min_{0 < p_1(t) < \alpha} \{d - b\gamma + aU_1(p_1(t))\} > 0,$$

$$\min_{p_1(t) > \beta} \{d - b\gamma + aU_2(p_1(t))\} > 0 \quad \text{and} \quad d - b\gamma + aU^* > 0.$$

**Proof.** We consider the functional

$$V(t) \equiv V(x(t), y(t))$$

$$= |x(t) - p_1(t)| + |\log y(t) - \log p_2(t)|$$

$$+ c \int_0^\infty g(s) \int_{t-s}^t |U(x(z)) - U(p_1(z))| \, dz \, ds$$

$$+ b\gamma \int_0^\infty f(s) \int_{t-s}^t |y(z) - p_2(z)| \, dz \, ds.$$

Clearly, $V(p_1(t), p_2(t)) = 0$ and $V(t) \geq |x(t) - p_1(t)| + |\log y(t) - \log p_2(t)| > 0$.

As before, it may be shown that the upper Dini derivative of $V$ along the solutions of (2.1) is negative definite, invoking the hypotheses. By employing standard arguments, the rest of the proof may be completed (see Theorem 3.5).
Now, for any solutions \((x_1(t), y_1(t))\) and \((x_2(t), y_2(t))\) of (2.1), Theorem 3.11 implies that
\[
|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| \\
= |x_1(t) - p_1(t) + p_1(t) - x_2(t)| + |y_1(t) - p_2(t) + p_2(t) - y_2(t)| \\
\leq |x_1(t) - p_1(t) + p_1(t) - x_2(t)| + |y_1(t) - p_2(t) + p_2(t) - y_2(t)| \\
\to 0 \quad \text{as } t \to \infty.
\]
Thus, all the solutions of (2.1) are asymptotic to each other.

In order to compare the present work with the results of [16], we consider the system (2.1) in the absence of zones of no activation. We establish that the following results improve the global stability results of [16] substantially in a number of ways.

**Theorem 3.12.** Assume that the delay kernels satisfy the conditions (H\(_1\)) and (H\(_2\)) and the uptake function satisfies (A\(_1\)). The equilibrium solution \((x^*, y^*)\) of (2.1) is globally asymptotically stable provided there exist constants \(\eta_1 > 0\) and \(\eta_2 > 0\) such that
\[
D - \frac{b\gamma}{4\eta_1} + A > 0, \quad d - \frac{c}{4\eta_2} - b\gamma\eta_1 > 0
\]
and
\[
a^2L^2 < 4 \left\{ D - \frac{b\gamma}{4\eta_1} + A \right\} \left[ d - \frac{c}{4\eta_2} - b\gamma\eta_1 \right]
\]
in which
\[
A = \min_{x \geq 0} \left\{ ay^* \left( \frac{U(x) - U(x^*)}{x - x^*} \right) - c\eta_2 \left( \frac{U(x) - U(x^*)}{x - x^*} \right)^2 \right\}.
\]

**Proof.** We consider the same functional used in Theorem 3.1 and, following along the lines of the proof of Theorem 3.1, the proof of Theorem 3.12 may be completed.

**Theorem 3.13.** Assume that the delay kernels satisfy the conditions (H\(_1\)) and (H\(_2\)) and the uptake function satisfies (A\(_1\)) and (A\(_2\)).
The equilibrium point \((x^*, y^*)\) of (2.1) is globally asymptotically stable provided
\[(i) \quad D - (c - ay^*)K > 0\]
\[(ii) \quad \min_{x \geq x^*} \{d + aU(x) - b\gamma\} > 0.\]

Proof. Following the proof of Theorem 3.5 and utilizing the same functional, we see that the proof of Theorem 3.13 may be completed.

Also it is easy to see that the conditions placed on the parameters of the system (3.1) in Theorem 3.13 are less restrictive than the corresponding conditions in Theorem 5.2 of [16]. This fact may be seen in the following example.

Example 3.14. Consider the system
\[
\begin{align*}
x'(t) &= 3(x_0 - x(t)) - 18U(x(t))y(t) + \int_{-\infty}^{t} f(t-s)y(s) \, ds \\
y'(t) &= -5y(t) + 16y(t) \int_{-\infty}^{t} g(t-s)U(x(s)) \, ds - 9y^2(t)
\end{align*}
\]
in which \(U(x) = x/(4 + x), b = 0.5, \gamma = 2\) and \(D = 3\).

We consider the equilibrium point \((x^*, y^*) = (16, 0.866)\) (approximately), with \(U(x^*) = 0.8, K = (1/4)\) and \(x_0 = 19.868\) (approximately). Clearly, all the hypotheses of Theorem 3.13 are satisfied, and hence \((x^*, y^*) = (16, 0.866)\) is globally asymptotically stable.

We observe that, since \(D - cK - aKy^* < 0\), Theorem 5.2 of [16] cannot be applied here.

Theorem 3.15. Assume that the delay kernels satisfy \((H_1)\) and \((H_2)\) and the uptake function \(U(x)\) satisfies \((A_1)\) and \((A_2)\). Further, assume that \((x - x^*)[U(x) - U(x^*)] > 0\) and \(U(x) = U(x^*)\) if and only if \(x = x^*\). The positive equilibrium point \((x^*, y^*)\) is globally asymptotically stable provided there exist positive constants \(\eta_1\) and \(\eta_2\) such that
\[
4D\eta_1 - b\gamma > 0, \quad 4d\eta_2 - c > 0
\]
and
\[
(4d\eta_2 - c)\sqrt{4D\eta_1 - b\gamma} \delta y^* \geq 8cL\eta_1 \eta_2^2 \sqrt{b\gamma}
\]
in which \( \delta = \min_{x \geq 0} \{(x - x^*)/(U(x) - U(x^*))\} > 0 \).

**Proof.** We consider the functional

\[
V(t) \equiv V(x(t), y(t)) = \frac{W_1}{2}(x(t) - x^*)^2 + W_2 \int_0^t \frac{z}{z + y^*} \, dz
\]

\[
+ W_3 \int_0^t g(s) \int_{t-s}^t [U(x(t_1)) - U(x^*)]^2 \, dt_1 \, ds
\]

\[
+ W_4 \int_0^t f(s) \int_{t-s}^t [y(t_1) - y^*]^2 \, dt_1 \, ds,
\]

where \( W_1, W_2, W_3 \) and \( W_4 \) are positive constants determined in the due course. Clearly,

\[
V(x^*, y^*) = 0
\]

and

\[
V(x(t), y(t)) \geq \frac{W_1}{2}(x(t) - x^*)^2 + W_2 \int_0^t \frac{z}{z + y^*} \, dz > 0.
\]

Now the time derivative of \( V \) along the solutions of (2.1) is given by

\[
\frac{dV}{dt} = W_1(x(t) - x^*) \left\{ D x_0 - D x(t) - a U(x(t)) y(t) + b \gamma \int_{-\infty}^t f(t-s)y(s) \, ds \right\}
\]

\[
+ W_2(y(t) - y^*) \left\{ - (\gamma + D) + c \int_{-\infty}^t g(t-s) U(x(s)) \, ds - dy(t) \right\}
\]

\[
+ W_3[U(x(t)) - U(x^*)]^2 - W_3 \int_0^\infty g(s)[U(x(t-s)) - U(x^*)]^2 \, ds
\]

\[
+ W_4[y(t) - y^*]^2 - W_4 \int_0^\infty f(s)[y(t-s) - y^*]^2 \, ds.
\]

Using (2.3) in this, we obtain

\[
\frac{dV}{dt} = W_1(x(t) - x^*) \left\{ - D(x(t) - x^*) - a U(x(t))(y(t) - y^*)
\right.
\]

\[
- a y^*(U(x(t)) - U(x^*)) + b \gamma \int_0^\infty f(s)(y(t-s) - y^*) \, ds \right\}
\]
utilizing the inequality (3.3) for arbitrary \( \eta > 0 \) and \( \eta_2 > 0 \). Hence,

\[
\frac{dV}{dt} = W_1 ay^*(x(t) - x^*)(U(x(t) - U(x^*)) + W_3(U(x(t)) - U(x^*))^2 \\
- W_1 \left( D - \frac{b\gamma}{4\eta_1} \right) (x(t) - x^*)^2 + W_1 aU(x(t) - x^*)(y(t) - y^*) \\
- W_2 \left( d - \frac{c}{4\eta_2} \right) - W_4 (y(t) - y^*)^2 \\
- (W_3 - W_2 c\eta_2) \int_0^\infty g(s)[U(x(t - s)) - U(x^*)]^2 ds \\
- (W_4 - W_1 b\gamma\eta_1) \int_0^\infty f(s)[y(t - s) - y^*]^2 ds.
\]

Then \( dV/dt < 0 \) if
(i) \( W_4 \geq W_1 b \gamma \eta_1 \),
(ii) \( W_3 \geq W_2 c \eta_2 \),
(iii) \( W_1 ay^* \delta > W_3 \),
(iv) \( D - (b \gamma / 4 \eta_1) > 0, W_2[d - (c / 4 \eta_2)] > W_4 \)

and

\[
[ W_1 a U(x) ]^2 < 4 W_1 \left( D - \frac{b \gamma}{4 \eta_1} \right) \left[ W_2 \left( d - \frac{c}{4 \eta_2} \right) - W_4 \right].
\]

These four conditions imply that \( dV/dt \) is negative definite if

\[
(3.5) \quad a^2 L^2 < 4 \left( D - \frac{b \gamma}{4 \eta_1} \right) \left[ \frac{ay^* \delta}{cn_2} \left( d - \frac{c}{4 \eta_2} \right) - b \gamma \eta_1 \right].
\]

Now consider

\[
a^2 L^2 = 4 \left( D - \frac{b \gamma}{4 \eta_1} \right) \left[ \frac{ay^* \delta}{cn_2} \left( d - \frac{c}{4 \eta_2} \right) - b \gamma \eta_1 \right].
\]

This, after some simplification, becomes

\[
(3.6) \quad 4 c \eta_1 \eta_2^2 L^2 a^2 - (4D \eta_1 - b \gamma)(4d \eta_2 - c) \delta y^* a
+ 4b \gamma c(4D \eta_1 - b \gamma) \eta_1 \eta_2^2 = 0.
\]

Solving for \( a \), we get

\[
a = \frac{(4D \eta_1 - b \gamma)(4d \eta_2 - c) \delta y^*}{8cL^2 \eta_1 \eta_2^2}
\]

\[
\pm \sqrt{\frac{(4D \eta_1 - b \gamma)^2(4d \eta_2 - c)^2 \delta^2 y^*^2 - 64b \gamma c^2 L^2(4D \eta_1 - b \gamma) \eta_1^2 \eta_2^4}{8cL^2 \eta_1 \eta_2^2}}.
\]

By the hypotheses, \( 4D \eta_1 - b \gamma > 0, 4d \eta_2 - c > 0 \) and hence, \( a \) exists and is a positive root of (3.6) provided

\[
(4D \eta_1 - b \gamma)^2(4d \eta_2 - c)^2 \delta^2 y^*^2 \geq 64b \gamma c^2 L^2(4D \eta_1 - b \gamma) \eta_1^2 \eta_2^4.
\]

That is,

\[
(3.7) \quad (4d \eta_2 - c) \delta y^* \sqrt{4D \eta_1 - b \gamma} \geq 8cL \eta_1 \eta_2 \sqrt{b \gamma}.
\]
For this value of \( a \), the inequality (3.5) is clearly satisfied and, hence, the conclusion of the theorem follows.

**Remark 3.16.** We observe that Theorem 5.3 of [16] follows from Theorem 3.15 for the choice of \( \eta_1 = \eta_2 = (1/2) \). Moreover, Theorem 3.15 is quite an improvement over Theorem 5.3 of [16]. The flexibility in the choice of the constants \( \eta_1 \) and \( \eta_2 \) improves the scope of the application of Theorem 3.15 as may be seen in the following example.

**Example 3.17.** Consider the system

\[
\begin{align*}
x'(t) &= 5(x_0 - x(t)) - 22U(x(t))y(t) + \int_{-\infty}^{t} f(t-s)y(s)\,ds \\
y'(t) &= -7y(t) + 20y(t) \int_{-\infty}^{t} g(t-s)U(x(s))\,ds - 9y^2(t)
\end{align*}
\]

in which \( U(x) = x/(5 + x) \), \( b = 0.5 \), \( \gamma = 2 \) and \( D = 5 \).

Consider the equilibrium solution \( x^* = 20 \), \( y^* = 1 \) with \( U(x^*) = 0.8 \) and \( x_0 = 22.44 \) approximately. Clearly, \( L = 1 \) and \( \delta = 25 \). Since \( 2d - c < 0 \), Theorem 5.3 of [16] cannot be applied here.

However, if we choose \( \eta_2 = (3/5) \), then we have \( 4d\eta_2 - c > 0 \) and from the condition \( (4d\eta_2 - c) \sqrt{4D\eta_1 - b\gamma} \delta y^* \geq 8cL\eta_1\eta_2^2 \sqrt{b\gamma} \), we shall have \( \sqrt{20\eta_1 - 1} > 1.44\eta_1 \) (approximately).

It is easy to see that we can have a range of values for \( \eta_1 \) for which the above inequality holds, thus ensuring the global asymptotic stability of the positive equilibrium \( (x^*, y^*) = (20, 1) \).

**Example 3.18.** Consider the system

\[
\begin{align*}
x'(t) &= 3(x_0 - x(t)) - 18U(x(t))y(t) + \int_{-\infty}^{t} f(t-s)y(s)\,ds \\
y'(t) &= -5y(t) + 16y(t) \int_{-\infty}^{t} g(t-s)U(x(s))\,ds - 9y^2(t)
\end{align*}
\]

in which \( U(x) = x/(4 + x) \), \( x_0 = 19.406 \) (approximately), \( b = 0.5 \), \( \gamma = 2 \) and \( D = 3 \).
Then the equilibrium solutions are \( x^* = 16 \), \( y^* = 0.866 \) (approximately), with \( L = 1 \), \( U(x^*) = 0.8 \), \( \delta = 20 \) and \( K = 1/4 \).

It is easy to check that all the conditions of Theorem 5.3 of [16] are satisfied and, hence, \((x^*, y^*) = (16, 0.866)\) is globally asymptotically stable.

Thus, with \( \eta_1 = \eta_2 = (1/2) \), Theorem 3.15 ensures the global asymptotic stability of \((x^*, y^*)\).

Let us now fix \( \eta_1 = (1/2) \) and find out the range of values of \( \eta_2 \) that yields the global asymptotic stability of \((x^*, y^*)\).

We require

\[
4d\eta_2 - c > 0 \quad \text{and} \quad (4d\eta_2 - c)\sqrt{4D\eta_1 - b\gamma} \delta y^* \geq 8cL\eta_1\eta_2^2 \sqrt{b\gamma}.
\]

For the given set of values these conditions become \( \eta_2 > (4/9) \) and

\[
(9\eta_2 - 4) > (1.6)\eta_2^2 \quad \text{(approximately)}.
\]

That is, \( (1.6)\eta_2^2 - 9\eta_2 + 4 < 0 \) and \( \eta_2 > (4/9) \).

Clearly, for \( \eta_2 \in (0.484, 5.140) \) these conditions are satisfied, and hence, we have the global asymptotic stability of \((x^*, y^*)\).

**Example 3.19.** Consider the following model

\[
x'(t) = 2(x_0 - x(t)) - 18U(x(t))y(t) + (0.25) \int_{-\infty}^{t} f(t-s)y(s) \, ds
\]

\[
y'(t) = -3y(t) + 16y(t) \int_{-\infty}^{t} g(t-s)U(x(s)) \, ds - 10y^2(t)
\]

in which \( U(x) = x/(4 + x) \), \( b = 0.25 \), \( \gamma = 1 \), \( D = 2 \) and \( x_0 = 3.85 \), approximately.

The equilibrium solutions are \( x^* = (8/3) \) and \( y^* = 0.34 \) with \( U(x^*) = (2/5) \), \( \delta = (20/3) \).

Since \( D - (cK - aKy^*) < 0 \), Theorem 3.13 cannot be applied here.

Now with \( \eta_1 = \eta_2 = (1/2) \), it is easy to see that all the hypotheses of Theorem 3.15 are satisfied and the equilibrium \([8/3, 0.34]\) is globally asymptotically stable by virtue of Theorem 3.15.
Example 3.20. Consider the following model,

\[
x'(t) = 2(x_0 - x(t)) - 20U(x(t))y(t) + (0.5) \int_{-\infty}^{t} f(t - s)y(s) \, ds
\]

\[
y'(t) = -3y(t) + 19y(t) \int_{-\infty}^{t} g(t - s)U(x(s)) \, ds - 2y^2(t)
\]

in which

\[
U(x) = \begin{cases} 
    x/(10 + x^2), & 0 \leq x < 4 \\
    2/13, & \text{otherwise}.
\end{cases}
\]

Clearly \(U(x)\) is the generalized Michaelis-Menten uptake function introduced in [16] for the choice of \(\alpha = 1, \beta = 2\) and \(\omega = 10\).

Also in the above system it is chosen that \(b = 0.5, \gamma = 1\) and \(D = 2\). Then the equilibrium solutions are \(x^* = \sqrt{10}, y^* = 0.00195\) \(U(x^*) = 0.1581\) (approximately), \(K = (1/10), \delta = 20\) with \(x_0 = 3.16487\) approximately.

It is easy to check that all the hypotheses of Theorem 3.13 are satisfied here and, hence, \((x^*, y^*) = (\sqrt{10}, 0.00195)\) is globally asymptotically stable.

A straightforward computation yields that the inequality (3.5) in Theorem 3.15 is violated for any choice of \(\eta_1\) and \(\eta_2\).

Hence, Theorem 3.15 cannot establish the global asymptotic stability of \((x^*, y^*) = (\sqrt{10}, 0.00195)\).

From Examples 3.19 and 3.20 it follows that Theorems 3.13 and 3.15 are independent.

4. Discussion. In this paper we have considered a model dealing with the growth of microorganisms in a cultured environment and involving distributed time delays both in nutrient recycling and nutrient uptake, popularly known as a chemostat model. In this paper we have introduced a new notion called a ‘zone of no activation’ for the consumer. This means that near the equilibrium the microorganisms do not consume any further nutrient and the same consumption level is maintained near equilibrium. In terms of mathematics, this phenomenon may be expressed as the uptake function \(U(x)\) maintaining the same value \(U(x^*)\) not only at \(x^*\) but also in a certain neighborhood of \(x^*\). In this context one may say that the consumption of the
nutrient has a saturation near equilibrium. We notice that this saturation of the consumption of nutrient is altogether different from the saturation in the supply of the nutrient uptake as envisaged in the assumption \( (A_1) \) on \( U(x) \). In a zone of no activation since the consumers are completely fed, they show no tendency to consume any further nutrient while the saturation effect on \( U(x) \) implies that the consumption of nutrient does not increase even when the populations require more nutrient. Further, these zones of no activation play an important role in restoring the stability when excess consumption of the nutrient may lead to abnormalities in the growth of populations, which may adversely affect the stability of the system.

We have studied the influence of zones of no activation on the global asymptotic stability of the equilibrium solution to our model equations and presented three easily verifiable independent sets of sufficient conditions for the global asymptotic stability. We have presented various examples to illustrate these results. We have studied the asymptotic nature of the solutions in the event of the model equations not possessing an equilibrium solution.

Further, we have studied the global asymptotic stability of the positive equilibrium solution of the model equations in the absence of zones of no activation. We have presented three independent sets of sufficient conditions for the global stability. We have observed that these results improve some of our results in [16] in terms of yielding larger regions of asymptotic stability for the equilibrium solutions. We have presented several examples to establish this fact and to prove that the results are independent of each other. Taking into account Theorem 5.6 of [16] and Theorems 3.12, 3.13 and 3.15, we have four independent sets of sufficient conditions for the global asymptotic stability of the positive equilibrium in the absence of a zone of no activation and three sets of sufficient conditions, Theorems 3.1, 3.5 and 3.8, for the global asymptotic stability of the equilibrium solution in the presence of a zone of no activation for the model equations. It will be a matter of immense interest to know whether any further sets of sufficient conditions which will be independent of the above sets do exist for these types of chemostat models.
REFERENCES


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