QUENCHING AND NON-QUENCHING FOR NONLINEAR WAVE EQUATIONS WITH DAMPING

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ABSTRACT. In this article we consider an initial-boundary value problem for a wave equation in one space dimension with nonlinear damping and singular source terms. We establish the existence of local weak solutions. Moreover, the behavior of solutions is investigated. Under mild conditions, we prove several results concerning quenching and non-quenching of solutions.

1. Introduction. In this paper interest is focused on the initial-boundary value problem

\begin{equation}
\begin{aligned}
&u_{tt} - u_{xx} + g(u)u_t = f(u), \quad \text{in } (0, L) \times (0, T), \\
&u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \text{in } (0, L), \\
&u(0, t) = 0, \quad u_x(L, t) = h(t), \quad \text{in } (0, T);
\end{aligned}
\end{equation}

where throughout the paper \(f, g\) and \(h\) satisfy the following structural conditions:

\begin{equation}

(1.2) \quad f : (-\infty, 1) \to (0, \infty), \quad f \in C^1(-\infty, 1) \text{ is monotone increasing, convex, and } \lim_{u \to 1^-} f(u) = \infty,

(1.3) \quad g \in C^1(\mathbb{R}), \quad g \geq 0, \quad \text{and} \quad h \in C^1[0, \infty).
\end{equation}

A typical example of the function \(f\) is \(f(u) = a(1-u)^{-\beta} + b\), where \(a, \beta > 0\), and \(b \geq 0\).

In the last twenty years, there has been an extensive body of work on quenching of solutions to various partial differential equations, particularly for parabolic equations (see the definition of quenching in Section 4). However, very few results have been established for hyperbolic equations. In [4] Levine and Chang considered a version
of problem (1.1) but without the presence of damping. They have established the existence of weak solutions and obtained some results on quenching of solutions. Later Smith [21] generalized their results to higher space dimensions. The effect of nonlinear boundary conditions on the homogeneous wave equation was investigated by Levine [12] in one space dimension and by Rammaha [18] in two and three space dimensions. For other related work we refer the reader to [3, 13, 19] and the references therein.

Motivated by the work of Chang and Levine [4] and others, (see, e.g., [8, 12, 21]), and the fact that there has not been much work done on quenching for hyperbolic equations with damping, we prove in this article that the initial-boundary value problem (1.1) has a unique local weak solution. Moreover, we show that this solution must quench in finite time under mild conditions. In this case there are no global solutions to (1.1). The quenching results in theorems 5 and 6 are the outcome of large initial data. However, the results of theorem 7 are for small initial data, and quenching here occurs due to some structural conditions imposed on the nonlinearities $f, g$ and $L$. Theorem 8 deals with the special case $g(u) = g_0$ where $g_0$ is a positive constant. The result in Theorem 8 shows that whenever $g_0$ is sufficiently large and the initial data is sufficiently small, then the solution cannot quench in finite time. In Theorem 9 we show that weak solutions cannot quench even in infinite time, provided $L$ is sufficiently small.

The plan for this paper is as follows. In Section 2 we present the technical assumptions, definitions and notation that are necessary for the remaining sections of the paper. Section 3 is devoted to establishing the existence and uniqueness of local weak solutions. In Section 4 we investigate quenching and non-quenching of solutions.

2. Preliminaries. In this section we introduce some notation, definitions and the technical assumptions that are necessary for the remaining sections of the paper. Let $L^2(0, L)$ denote the standard Lebesgue space and $H^s(0, L)$ denote the standard Sobolev space. Also, for $s > 1/2$, we set

$$H^s_{0,0}(0, L) = \{u \in H^s(0, L) : u(0) = 0\}.$$ 

We also introduce the operator

$$A : L^2(0, L) \rightarrow L^2(0, L).$$
where \( A = -\left( \frac{\partial^2}{\partial x^2} \right) \) with its domain

\[
\mathcal{D}(A) = \{ u \in H^2(0, L) : u(0) = 0, u'(L) = 0 \}.
\]

It is well known that \( A \) is positive, self-adjoint, and \( A \) is the inverse of a compact operator. Moreover, \( A \) has the infinite sequence of positive eigenvalues \( \{ \lambda_n = ((2n - 1)\pi/2L)^2 : n = 1, 2, \ldots \} \) and a corresponding sequence of eigenfunctions \( \{ e_n = \sqrt{(2/L)} \sin(\lambda_n^{1/2} x) : n = 1, 2, \ldots \} \) that forms an orthonormal basis for \( L^2(0, L) \). Moreover, the powers of \( A \) are defined as follows:

\[
A^s : \mathcal{D}(A^s) \subseteq L^2(0, L) \rightarrow L^2(0, L),
\]

\[
A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n e_n, \quad \text{with the domain of } A^s \text{ given by: } \mathcal{D}(A^s) = \{ u \in L^2(0, L) : u = \sum_{n=1}^{\infty} u_n e_n, \sum_{n=1}^{\infty} \lambda_n^{2s} |u_n|^2 < \infty \}. \]

We remark here that the results of Grisvard [6] and Seeley [20] give the following characterization:

\[
\mathcal{D}(A^s) = \begin{cases} 
H^{2s}(0, L); & 0 \leq s < \frac{1}{4} \\
H^{2s}_{0,0}(0, L); & \frac{1}{4} < s < \frac{3}{4} \\
\{ u \in H^{2s}(0, L) : u(0) = 0, u'(L) = 0 \}; & \frac{3}{4} < s \leq 1.
\end{cases}
\]

Moreover, \( \mathcal{D}(A^{1/4}) \hookrightarrow H^{1/2}(0, L), \mathcal{D}(A^{3/4}) \hookrightarrow H^{3/2}_{0,0}(0, L) \), and the norm \( \| u \|_{H^s} \) is equivalent to \( (\sum_{n=1}^{\infty} \lambda_n^s |u_n|^2)^{1/2} \). Therefore, we set

\[
\| u \|_{H^s}^2 = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2.
\]

Also, throughout the paper, \( S(t) \) and \( C(t) \) denote the sine and cosine operators associated with \( A \). More specifically, \( S(t), C(t) : L^2(0, L) \rightarrow L^2(0, L) \) are given by \( S(t) = A^{-1/2} \sin(A^{1/2}t) \) and \( C(t) = \cos(A^{1/2}t) \).

Let \( X \) and \( Y \) be Banach spaces. We write \( X \hookrightarrow Y \) if \( X \) is continuously imbedded in \( Y \). More precisely, the natural injection \( i : X \rightarrow Y \) is continuous and we identify \( X \) with the subspace \( i(X) \) of \( Y \). We denote by \( \text{L}(X, Y) \) the space of all continuous linear operators from \( X \) to \( Y \).

**Definition 2.1.** Let \( u^0 \in H^{1,1}_{0,0}(0, L), u^1 \in L^2(0, L) \). We say that \( u \) is a weak solution to the initial-boundary value problem (1.1) on \([0, T]\)
if $u \in L^2(0, T, H^1_{0,0}(0, L))$, $u' \in L^2(0, T, L^2(0, L))$ and $u$ satisfies:

\[(2.2) \quad \langle u'(t), \phi \rangle_{L^2} - \langle u^1, \phi \rangle_{L^2} + \int_0^t \langle A^{1/2}u(s), A^{1/2}\phi \rangle_{L^2} ds \]
\[\phantom{(2.2)} - \int_0^t h(s)\phi(L) ds + \langle G(u(t)), \phi \rangle_{L^2} - \langle G(u^0), \phi \rangle_{L^2} \]
\[= \int_0^t \langle f(u(s)), \phi \rangle_{L^2} ds,\]

for all $\phi \in H^1_{0,0}(0, L)$ and almost every $t \in [0, T]$; where $G(u) := \int_0^u g(w) dw$ and $u' = (d/dt)u(t)$.

Let us now consider the regularized version of (1.1). For $0 < \delta < 1/2$, define

\[f_\delta(u) = \begin{cases} f(u), & u \leq 1 - 2\delta, \\ f(1-\delta), & u \geq 1 - \delta, \end{cases}\]

where $f_\delta(u)$ is defined on the interval $[1 - 2\delta, 1 - \delta]$ so that $f_\delta \in C^1(\mathbb{R})$. Thus, the regularized version of (1.1) is given by

\[(2.3) \quad u_{ttt} - u_{xxx} + G'(u)u_t = f_\delta(u), \quad \text{in } (0, L) \times (0, T),
\]
\[u(x, 0) = u^0, \quad u_t(x, 0) = u^1, \quad \text{in } (0, L),
\]
\[u(0, t) = 0, \quad u_x(L, t) = h(t), \quad \text{in } (0, T).\]

In order for us to easily obtain certain estimates, we now derive the integral equations associated with the initial-boundary value problem (2.4). Let $v = u - w$, where $w(x, t) = h(t)x$. Then $v$ formally satisfies the initial-value problem:

\[(2.4) \quad v'' + Av = -w'' - G'(v + w)(v + w)' + f_\delta(v + w), \quad \text{on } (0, T),
\]
\[v(0) = u^0 - w(0), \quad v'(0) = u^1 - w'(0).\]

Thus, by the variation of constants formula, we have

\[v(t) = C(t)(u^0 - w(0)) + S(t)(u^1 - w'(0))
\]
\[- \int_0^t S(t - \tau)[w''(\tau) + G'(u(\tau))u'(\tau) - f_\delta(u(\tau))] d\tau.\]
Formal integration by parts yields

\[
    u(t) = C(t)(u^0 - w(0)) + S(t)(u^1 + G(u^0)) + w(t) - \int_0^t C(t - \tau)w'(\tau) \, d\tau
\]

\[
    - \int_0^t C(t - \tau)G(u(\tau)) \, d\tau + \int_0^t S(t - \tau)f_\delta(u(\tau)) \, d\tau.
\]

(2.5)

By differentiating (2.5), one has

\[
    u'(t) = C(t)(u^1 + G(u^0)) - AS(t)(u^0 - w(0)) + \int_0^t AS(t - \tau)w'(\tau) \, d\tau
\]

\[
    + \int_0^t AS(t - \tau)G(u(\tau)) \, d\tau - G(u(t)) + \int_0^t C(t - \tau)f_\delta(u(\tau)) \, d\tau.
\]

Now let

\[
    U_0(t) = C(t)(u^0 - w(0)) + S(t)(u^1 + G(u^0)) + w(t) - K w'(t),
\]

\[
    V_0(t) = C(t)(u^1 + G(u^0)) - AS(t)(u^0 - w(0)) + L w'(t),
\]

where the operators \( K \) and \( L \) are given by: \( K w(t) = \int_0^t C(t - \tau)w(\tau) \, d\tau \)
and \( L w(t) = \int_0^t AS(t - \tau)w(\tau) \, d\tau \).

The following regularity results are well-known, for example, (see [10, 11]), and thus their proofs are omitted.

**Lemma 2.2.** For \( s \geq 0 \), we have

(i) \( C(\cdot) \in L(D(A^s), C([0,T], D(A^s))) \),

(ii) \( S(\cdot) \in L(D(A^s), C([0,T], D(A^{s+1/2}))) \).
Remark 2.3. It can be shown that if $u \in L^2(0, T, H_{0,0}^1(0, L))$, $u' \in L^2(0, T, L^2(0, L))$ and $u$ satisfies the integral equations

$$u(t) = U_0(t) - \int_0^t C(t - \tau)G(u(\tau)) \, d\tau$$

$$+ \int_0^t S(t - \tau)f_\delta(u(\tau)) \, d\tau,$$

(2.6)

$$u'(t) = V_0(t) - G(u(t))$$

$$+ \int_0^t AS(t - \tau)G(u(\tau)) \, d\tau$$

$$+ \int_0^t C(t - \tau)f_\delta(u(\tau)) \, d\tau,$$

(2.7)

then $u$ is a weak solution to (2.4) in the sense of Definition 1. Moreover, the converse is also valid. The proof of this remark is very similar to the proof of Remark 2.1 in [2] and thus it is omitted.

3. Local existence. In this section we establish the existence of local weak solutions to (1.1) by using the integral equation (2.6) and the contraction mapping principle in an appropriate setting. To begin, we prove the following lemma.

Lemma 3.1. Let $f_\delta \in C^1(\mathbb{R})$ and $G \in C^2(\mathbb{R})$ be as previously defined. Then

(i) the mapping $u \mapsto f_\delta(u)$ from $H_{0,0}^1(0, L)$ to $L^2(0, L)$ is Lipschitz continuous on bounded sets.

(ii) The mapping $u \mapsto G(u)$ from $H_{0,0}^1(0, L)$ to $H_{0,0}^1(0, L)$ is Lipschitz continuous on bounded sets.

Proof. The proof of (i) is trivial and follows from the mean value theorem, the fact that $f_\delta'$ is bounded on $\mathbb{R}$, and the imbedding $H_{0,0}^1(0, L) \hookrightarrow C[0, L]$. To prove (ii), let $u, v \in B_R = \{w \in H_{0,0}^1(0, L) : \|w\|_{H^1} \leq R\}$. By the mean value theorem and the imbedding $H_{0,0}^1(0, L) \hookrightarrow C[0, L]$,
we have that
\[
\|G(u) - G(v)\|_{H^1}^2 = \|G(u) - G(v)\|_{L^2}^2 + \|g(u)u_x - g(v)v_x\|_{L^2}^2 \\
\leq C_0\|u - v\|_{L^2}^2 + (\|g(u)u_x - g(v)v_x\|_{L^2} + \|g(u)v_x - f(v)v_x\|_{L^2})^2 \\
\leq C_0\|u - v\|_{L^2}^2 + (C_0\|u_x - v_x\|_{L^2} + g(u) - g(v))\|_{C[0,1]}\|v_x\|_{L^2})^2 \\
\leq C_0\|u - v\|_{H^1}^2 + (C_0\|u - v\|_{H^1} + C_1\|u - v\|_{C[0,1]}\|v\|_{H^1})^2 \\
\leq C(g, R)\|u - v\|_{H^1}^2.
\]

The fact that \(G(u)|_{x=0} = 0\) is trivial, and the proof is complete. \(\square\)

Now let \(\delta, \varepsilon \in (0, 1/3)\) with \(2\delta < \varepsilon\). Fix \(R\) small enough so that \(B_R \subset \{u \in C[0, L] : \|u\|_{C[0, L]} \leq 1 - \varepsilon\}\) where \(B_R\) is the ball of radius \(R\) in \(H^1_{0,0}(0, L)\) as specified above. Let \(X = C([0, T], B_R)\) be endowed with the norm \(\|u\|_X = \sup_{0 \leq t \leq T} \|u(t)\|_{H^1}\). Next define the operator
\[
\Phi u(t) = U_0(t) - \int_0^t C(t - \tau)G(u(\tau)) d\tau \\
+ \int_0^t S(t - \tau)f_5(u(\tau)) d\tau.
\]

Notice that if \(u^0 \in H^1_{0,0}(0, L)\), \(u^1 \in L^2(0, L)\), and \(h \in C^1[0, \infty)\), then we have the following:

Since \(w \in C([0, T], D(A^{1/2}))\), then \(C(t)(u^0 - w(0)) \in C([0, T], D(A^{1/2}))\). The fact \(u^1 + G(u^0) \in L^2(0, L)\) yields \(S(t)(u^1 + G(u^0)) \in C([0, T], D(A^{1/2}))\). Similarly, \(w' \in C([0, T], D(A^{1/2}))\) implies \(Kw' \in C([0, T], D(A^{1/2}))\).

Thus if we assume that
\[
\|U_0(t)\|_{H^1} \leq R/2 \quad \text{and} \quad \|U_0(t)\|_{C[0, L]} \leq 1 - 2\varepsilon
\]
for all \(t \in [0, T]\), we have that \(U_0 \in X\). Now for \(\rho > 0\), we let
\[
X_\rho = \{u \in X : \|u - U_0\|_X < \rho\}.
\]

We will show that, for \(T > 0\) sufficiently small, \(\Phi\) is a contraction from \(X_\rho\) into itself.
Let $u, v \in X_\rho$. It follows from Lemma 2.2, (3.2) and the observations above that $\Phi u \in C([0, T], D(A^{1/2}))$. By using the mean value theorem, (3.2), and the fact that $f_\delta$ is bounded, one has

$$\|\Phi u(t)\|_{H^1} \leq R/2 + \int_0^t \|f_\delta(u(\tau))\|_{L^2} d\tau + \int_0^t \|G(u(\tau))\|_{H^1} d\tau$$

$$\leq R/2 + C_0(f_\delta) t + C_1(g, R) \int_0^t \|u(\tau)\|_{H^1} d\tau$$

$$\leq R/2 + C(f_\delta, g, R)t.$$ 

Thus, $\|\Phi u\|_X \leq R/2 + C(f_\delta, g, R)T$. Similarly, $\|\Phi u - U_0\|_X \leq C(f_\delta, g, R)T$. Therefore, for a sufficiently small $T > 0$, $\Phi u \in X_\rho$. In addition, Lemma 3.1 yields that

$$\|\Phi u(t) - \Phi v(t)\|_{H^1} \leq \int_0^t \|f_\delta(u(\tau)) - f_\delta(v(\tau))\|_{L^2} d\tau$$

$$+ \int_0^t \|G(u(\tau)) - G(v(\tau))\|_{H^1} d\tau$$

$$\leq C(f_\delta, g) \int_0^t \|u(\tau) - v(\tau)\|_{H^1} d\tau$$

$$\leq C(f_\delta, g)t\|u - v\|_X.$$ 

Hence $\|\Phi u - \Phi v\|_X \leq C(f_\delta, g)T\|u - v\|_X$ and $\Phi : X_\rho \to X_\rho$ is a contraction provided $T > 0$ is sufficiently small. Therefore, there exists a unique fixed point $u \in X_\rho$ such that $\Phi u = u$. Finally, the fact that $u' \in C([0, T], L^2(0, L))$ follows immediately from the integral equation (2.6). The fact that $u'' \in L^2(0, T, (H^1_{0, 0}(0, L))')$ follows easily from (2.2). Consequently, the following theorem has been proven.

**Theorem 3.2.** Let $\varepsilon \in (0, (1/3))$. If $u^0 \in H^1_{0, 0}(0, L)$, $u^1 \in L^2(0, L)$ and $h \in C^1[0, \infty)$ such that (3.2) holds, then there exists a constant $T > 0$ and a unique solution $u$ to (1.1) with

$$u \in C([0, T], H^1_{0, 0}(0, L)), u' \in C([0, T], L^2(0, L))$$

$$u'' \in L^2(0, T, (H^1_{0, 0}(0, L))')$$ and $\sup_{0 \leq t \leq T} \|u(t)\|_{C[0, L]} < 1 - \varepsilon$.

Moreover, if $\varepsilon' < \varepsilon$, this solution may be extended to the interval $[0, T + T']$, for some $T' > 0$ with $\sup_{0 \leq t \leq T + T'} \|u(t)\|_{C[0, L]} \leq 1 - \varepsilon'$. 
4. **Quenching and non-quenching.** Let \( u \) be a solution to (1.1) on the interval \([0, T]\) in the sense of Theorem 3.2. In particular, \( u \in C([0, T], C[0, L]) \). We say that the solution \( u \) quenches at time \( T_0 \), \( 0 < T_0 \leq \infty \) if there exists \( x_0 \in [0, L] \) such that \( \lim_{t \to T_0^-} u(x_0, t) = 1 \). In this section we derive several results on quenching and non-quenching of weak solutions to the initial-boundary value problem (1.1). For simplicity, we assume throughout this section that \( h \equiv 0 \).

Since \( u_t \) is not sufficiently regular, obtaining the energy identity (4.1) below is not straightforward. However, by modifying the proof of Lemma 8.3 of Lions and Magenes [16], we have

\[
E(t) + \int_0^t \int_0^L g(u(x, \tau))(u_{\tau}(x, \tau))^2 \, dx \, d\tau = E(0),
\]

(4.1)

\[
E(t) = \int_0^L \left( \frac{1}{2}(u_t(x, t))^2 + \frac{1}{2}(u_x(x, t))^2 - \int_0^{u(x,t)} f(\xi) \, d\xi \right) \, dx.
\]

(4.2)

If \( E(0) \leq 0 \), (4.1) shows that \( E(t) \leq 0 \) for all \( t \in [0, T) \). Let \( \Omega^+(t) = \{ x \in [0, L] : u(x, t) \geq 0 \} \) and \( \Omega^-(t) = \{ x \in [0, L] : u(x, t) < 0 \} \). Then for all \( t \in [0, T) \), one has

\[
\frac{1}{2} \int_0^L ((u_t(x, t))^2 + (u_x(x, t))^2) \, dx + \int_{\Omega^-(t)} \int_0^{u(x,t)} f(\xi) \, d\xi \, dx
\]

\[
\leq \int_{\Omega^+(t)} \int_0^{u(x,t)} f(\xi) \, d\xi \, dx.
\]

(4.3)

Now set \( \lambda_1 = (\pi/2L)^2 \), \( \phi(x) = (\pi/2L) \sin((\pi/2L)x) \), and define

\[
F(t) = \int_0^L u(x, t)\phi(x) \, dx.
\]

(4.4)

Then \( F \in C^1[0, T] \) and, by (2.3)

\[
F'(t) = \langle u^1, \phi \rangle_{L^2} - \int_0^t \langle u_x(s), \phi' \rangle_{L^2} \, ds
\]

\[
- \langle G(u(t)), \phi \rangle_{L^2} + \langle G(u_0), \phi \rangle_{L^2} + \int_0^t \langle f(u(s)), \phi \rangle_{L^2} \, ds.
\]
Hence, $F''(t)$ exists on $[0, T)$ and

\[
F''(t) = -\langle u_x(t), \phi' \rangle_{L^2} - \langle g(u(t))u_t(t), \phi \rangle_{L^2} \\
+ \langle f(u(t)), \phi \rangle_{L^2} \\
= -\lambda_1 \langle u(t), \phi \rangle_{L^2} - \langle g(u(t))u_t(t), \phi \rangle_{L^2} \\
+ \langle f(u(t)), \phi \rangle_{L^2}.
\]

(4.5)

Thus, it follows from Jensen’s inequality that

\[
F''(t) \geq -\lambda_1 F(t) + f(F(t)) - \langle g(u(t))u_t(t), \phi \rangle_{L^2},
\]

(4.6)

for all $t \in [0, T)$.

Before stating our main results in this section, we make the following definitions. If \( \int_0^1 f(\xi) \, d\xi < \infty \), we set

\[
a_L = \sqrt{2L} \left( \int_0^1 f(\xi) \, d\xi \right)^{1/2}
\]

and

\[
K_L = \frac{\pi}{\sqrt{2}} \sup_{-a_L \leq u \leq 1} g(u) \left( \int_0^1 f(\xi) \, d\xi \right)^{1/2}.
\]

In addition, let

\[
\alpha = F(0), \quad \beta = F'(0), \quad \Psi(z) = -\lambda_1 z + f(z) - K_L, \\
\sigma(z) = -\lambda_1 z + f(z), \quad \tilde{\Psi}(z) = \int_{\alpha}^{z} \Psi(\xi) \, d\xi.
\]

The following lemma will be needed.

**Lemma 4.1.** For any $\beta \in \mathbb{R}$ and $\alpha < 1$ with $\Psi(\alpha) > 0$ and $\Psi'(\alpha) > 0$, we have

\[
\int_{\alpha}^{1} \frac{1}{\sqrt{\beta^2 + 2\tilde{\Psi}(z)}} \, dz < \infty.
\]
Proof. Since \( f \) is convex, then \( \Psi'(z) \geq \Psi'(\alpha) > 0 \), for \( z \in [\alpha, 1) \). Thus \( \Psi \) is increasing on \([\alpha, 1)\). By the mean value theorem we have \( \tilde{\Psi}(z) \geq \Psi(\alpha)(z - \alpha) \). Hence the lemma follows.

The following theorem asserts that weak solutions must quench in finite time, provided the initial data is sufficiently large and \( \int_0^1 f(\xi) \, d\xi < \infty \).

**Theorem 4.2.** Let \( u \) be a solution to (1.1) on \([0, T)\) in the sense of Theorem 3.2. Assume that \( u^0, u^1, f \) and \( g \) satisfy the following conditions:

\[
E(0) \leq 0, \quad 0 \leq \alpha < 1, \quad \beta \geq 0, \tag{4.7}
\]
\[
\int_0^1 f(\xi) \, d\xi < \infty, \quad \Psi(\alpha) > 0, \quad \text{and} \quad \Psi'(\alpha) > 0. \tag{4.8}
\]

Then \( u \) quenches in finite time, i.e., \( T < \infty \). Moreover, we have the following bounds on \( T \)

\[
T \leq -\beta + \sqrt{\beta^2 + 2\Psi(\alpha)(1 - \alpha)} \over \Psi(\alpha), \tag{4.9}
\]
\[
T \leq \int_\alpha^1 (2\tilde{\Psi}(z) + \beta^2)^{-1/2} \, dz. \tag{4.10}
\]

Proof. First (4.3) yields that

\[
\int_0^L \left((u_t(x,t))^2 + (u_x(x,t))^2\right) \, dx \leq 2L \int_0^1 f(\xi) \, d\xi. \tag{4.11}
\]

Therefore, \( \sup_{(x,t) \in [0, L] \times [0, T]} |u(x,t)| \leq \sup_{0 \leq t < T} \int_0^L |u_x(x,t)| \, dx \leq a_L \).

Thanks to (4.11) and the fact that \( g \) is continuous, one has

\[
\left| \int_0^L g(u(x,t))u_t(x,t)\phi(x) \, dx \right| \leq K_L.
\]

Thus it follows from (4.6) that, for all \( t \in [0, T) \),

\[
F''(t) \geq \Psi(F(t)). \tag{4.12}
\]
We will show that $F$ is increasing on $[0,T)$. First notice that $F''(0) \geq \Psi(\alpha) > 0$. Hence there exists some $t_0 > 0$ such that $F''(t) > 0$ on $[0,t_0]$. If $F$ is not increasing on $[0,T)$, then let $t_1 = \inf\{t \in [t_0,T) : F'(t) = 0\}$. For $z \in [\alpha,1)$, $\Psi'(z) \geq \Psi'(\alpha) > 0$, and consequently $\Psi$ is increasing on $[\alpha,1)$. Since $F$ is increasing on $[0,t_1]$, then $F''(t) \geq \Psi(\alpha) > 0$ on $[0,t_1]$. Integrating from 0 to $t_1$, we have that $\beta < 0$, which contradicts (4.8). Thus, $F$ is increasing on $[0,T)$.

Hence, for all $t \in [0,T)$, $F''(t) \geq \Psi(\alpha) > 0$. Therefore, we have

$$F(t) \geq \frac{1}{2}\Psi(\alpha)t^2 + \beta t + \alpha. \quad (4.13)$$

On the other hand, for $t \in [0,T)$, $F(t) \leq (\pi/2L) \int_0^L \sin((\pi/2L)x) \, dx = 1$. Therefore, $(1/2)\Psi(\alpha)t^2 + \beta t + \alpha \leq 1$ for $t \in [0,T)$. Thus $u$ must quench in finite time and (4.9) follows immediately.

To show (4.10), we multiply (4.12) by $F'(t)$ and integrate from 0 to $t$ to obtain $F'(t)/\sqrt{2\Psi(F(t)} + \beta^2 \geq 1$. By integrating from 0 to $T$, inequality (4.10) follows and the proof is complete.

The following theorem is also an outcome of large initial data, but without the restriction $\int_0^1 f(\xi) \, d\xi < \infty$.

**Theorem 4.3.** Let $u$ be a solution to (1.1) in the sense of Theorem 3.2. Assume that $u^0, u^1, f$ and $g$ satisfy the following conditions

$$0 \leq \alpha < 1, \quad \beta \geq 0, \quad \sigma(\alpha) > 0, \quad \sigma'(\alpha) > 0, \quad \text{and} \quad g(u) \equiv g_0, \quad (4.14)$$

for some $g_0 > 0$. Then $u$ quenches in finite time $T$. Moreover, we have the bound

$$T \leq -\gamma + \sqrt{\gamma^2 + 2\sigma(\alpha)(1 - \alpha)} \quad (4.15)$$

where $\gamma = g_0\alpha + \beta - g_0$.

**Proof.** From (4.6) we see that

$$F''(t) + g_0F'(t) \geq \sigma(F(t)). \quad (4.16)$$
As in the proof of Theorem 4.2, we show now that $F(t)$ is increasing for $t > 0$. It follows from (4.16) that $F''(0) + g_0 F'(0) \geq \sigma(F(0))$. Since $\beta = F'(0) \geq 0$, then there exists a $t_0 > 0$ such that $F'(t) > 0$ for $t \in (0, t_0]$. If $F$ is not increasing on $(0, T)$, let $t_1 = \inf \{ t \in (t_0, T) : F'(t) = 0 \}$. By the assumptions on $\sigma$, namely $\sigma(\alpha) > 0$ and $\sigma'(\alpha) > 0$, then $\sigma$ is increasing on $[0, 1)$. Since $F$ is increasing on $(0, t_1)$, then (4.16) yields $F''(t) + g_0 F'(t) \geq \sigma(\alpha)$ for $t \in (0, t_1)$. Therefore one has $(d/dt)(e^{g_0 t} F'(t)) \geq \sigma(\alpha) e^{g_0 t}$ on $(0, t_1)$. By integrating the last inequality from 0 to $t_1$, we find $\beta = F'(0) < 0$, contradicting our assumption. Hence $F$ is increasing on $(0, T)$. Thus, for $t \in (0, T)$,

\begin{equation}
F''(t) + g_0 F'(t) \geq \sigma(\alpha) > 0.
\end{equation}

By integrating (4.23) twice and using the fact $F(t) \leq 1$, one has

\begin{equation}
\frac{1}{2} \sigma(\alpha) t^2 + (g_0 \alpha + \beta - g_0) t + \alpha \leq 1.
\end{equation}

Thus $u$ quenches in finite time, and (4.15) follows.

Theorem 4.4 below shows that, regardless of the size of the initial data, weak solutions to (1.1) must quench in finite time provided the nonlinearities $f$ and $g$ and $L$ satisfy some structural conditions. Let

\begin{equation}
K_1 = \sup_{0 \leq u \leq 1} \frac{u}{f(u)}, \quad K_2 = \inf_{-\infty < u \leq 0} \left( f(u) - \left( \frac{\pi}{2L} \right)^2 u \right).
\end{equation}

Note that $K_1$ and $K_2 > 0$.

**Theorem 4.4.** Let $u$ be a solution to (1.1) in the sense of Theorem 3.2. Then $u$ quenches in finite time, provided one of the following conditions is satisfied:

(i) $E(0) \leq 0$, $\int_0^1 f(\xi) \, d\xi < \infty$, $\lambda_1 K_1 < 1$ and $\min\{f(0)(1 - \lambda_1 K_1), K_2\} > K_L$.

(ii) $E(0) \leq 0$, $\int_0^1 f(\xi) \, d\xi < \infty$, $\lambda_1 K_1 < 1$, $\min\{f(0)(1 - \lambda_1 K_1), K_2\} = K_L$ and $\beta > 0$.

(iii) $g(u) \equiv g_0$, $g_0$ is a positive constant and $\lambda_1 K_1 < 1$. 
Proof. We first show that \( u \) quenches in finite time if (i) is valid. From (4.5) one has

\[
F''(t) = \int_{\Omega^+(t)} (f(u(x,t)) - \lambda_1 u(x,t))\phi(x) \, dx \\
- \int_0^L g(u(x,t))u_t(x,t)\phi(x) \, dx \\
+ \int_{\Omega^-(t)} (f(u(x,t)) - \lambda_1 u(x,t))\phi(x) \, dx \\
\geq \int_{\Omega^+(t)} f(u(x,t))(1 - \lambda_1 K_1)\phi(x) \, dx \\
+ \int_{\Omega^-(t)} K_2\phi(x) \, dx - K_L \\
\geq \min\{f(0)(1 - \lambda_1 K_1), K_2\} - K_L := K_0 > 0.
\]

Integrating twice from 0 to \( t \), we have that \((1/2)K_0 t^2 + \beta t + \alpha \leq 1\). Thus \( u \) quenches in finite time. If (ii) is valid, then we still have the same conclusion. Now assume that (iii) is valid. Then, from (4.5) we have

\[
F''(t) + g_0 F'(t) = \int_{\Omega^+(t)} (f(u(x,t)) - \lambda_1 u(x,t))\phi(x) \, dx \\
+ \int_{\Omega^-(t)} (f(u(x,t)) - \lambda_1 u(x,t))\phi(x) \, dx \\
\geq \min\{f(0)(1 - \lambda_1 K_1), K_2\} \int_0^L \phi(x) \, dx := J_0 > 0.
\]

By integrating twice, one has \((1/2)J_0 t^2 + (\beta - g_0(1 - \alpha))t + \alpha \leq 1\). Therefore, \( u \) quenches in finite time and the proof is complete. \( \square \)

Remark 4.5. Condition (i) in Theorem 4.4 above is a structural condition on \( f, g \) and \( L \). It can be easily verified for many examples. Interestingly, the computations carried out in Example 4.6 below shows that, if \( g \) is bounded on \((-\infty, 1)\), then condition (i) above is satisfied provided either \( L \) or \( f(0) \) is sufficiently large; but \( \sup_{-\infty < u \leq 1} g(u) \) cannot be too large. Therefore, it can be viewed as a restriction on the size of \( g \) relative to the size of \( f(0) \) and \( L \).
Example 4.6. We show that the structural conditions in (i) of Theorem 4.4 can be met in the case when \( f(u) = C_0/\sqrt{T-u} \), \( C_0 > 0 \) and \( g \) is bounded on \((-\infty,1)\). Let \( M := \sup_{-\infty < u \leq 1} g(u) > 0 \), \( \mu_0 := (2C_0L^2/\pi^2) \geq 1 \). A straightforward computation shows that \( K_1 \) is attained at \( u = 2/3 \) and \( K_1 = 2/(3\sqrt{3}C_0) \). Similarly, since \( \mu_0 := (2C_0L^2/\pi^2) \geq 1 \), then \( K_2 \) is attained at \( 1 - \mu_0^{2/3} \leq 0 \) and \( K_2 = (C_0/2)(3\mu_0^{-1/3} - \mu_0^{-1}) \). Since \( \int_0^1 f(\xi) d\xi = 2C_0 \), then \( K_L = \pi M\sqrt{C_0} \). In order for condition (i) of Theorem 4.4 to hold, the inequalities \( C_0(1-\lambda K_1) \geq K_L \) and \( K_2 \geq K_L \) must hold. However, these inequalities are equivalent to \( \sqrt{C_0}(1-(1/3\sqrt{3}\mu_0)) \geq \pi M \), and \( \sqrt{C_0}/2(3\mu_0^{-1/3} - \mu_0^{-1}) \geq \pi M \), respectively. Since \( \mu_0 \geq 1 \), then both inequalities can be made valid for suitable choices of the parameters \( C_0, L \) and \( M \).

The following theorem deals with the special case when \( g(u) = g_0 \), where \( g_0 \) is a positive constant. More specifically, Theorem 4.7 shows that if \( g_0 \) is sufficiently large and the initial data is sufficiently small, then \( u \) cannot quench for any time \( t \in [0,T] \) for any given \( T > 0 \). However, before stating Theorem 4.7, we introduce the following fixed parameters

\[
T > 0, \quad 0 < \delta < 1/2, \quad \varepsilon = \min \left\{ \frac{LT}{\sqrt{2}}, \frac{(1-2\delta)^2}{16L^2(f(1-\delta))^2} \right\} \tag{4.22}
\]

\[
\varepsilon_0 = \frac{(1-2\delta)^2}{16L^2(f(0))^2}, \quad \gamma = \frac{1}{\sqrt{2LT}} \sqrt{\varepsilon}, \quad \eta = \frac{1}{4\gamma}.
\]

Also we set \( e(t) = (1/2) \int_0^L ((u_t(x,t))^2 + (u_x(x,t))^2) dx \) and

\[
P(t) = e(t) - \int_0^L \int_0^L u(x,t) f(\xi) d\xi
+ \gamma \int_0^L \left( u(x,t) u_t(x,t) + \frac{1}{2} g_0(u(x,t))^2 \right) dx.
\]

Theorem 4.7. Assume \( g(u) = g_0 \), where \( g_0 \) is a positive constant. Let \( u \) be the solution to (1.1) in the sense of Theorem 3.2. Let \( T, \delta, \varepsilon, \varepsilon_0 \) and \( \gamma \) be as specified in (4.22). If \( g_0 \geq (1/\varepsilon) \) and the initial data \( u^0 \) and \( u^1 \) satisfy the condition \( P(0) \leq (1/16L)(1-2\delta)^2 \), then \( u \) does not at
any time quench $t \in [0,T]$. More specifically, we have $\|u(t)\|_{\infty} \leq 1 - 2\delta$, for all $t \in [0,T]$.

Proof. Let $0 < \delta < 1/2$ be fixed, and recall the definition of $f_\delta$, the regularization of the nonlinearity $f$. Note that the solution to the regularized problem (2.4) coincides with the solution to (1.1), provided $u(x,t) \leq 1 - 2\delta$ for all $x \in [0,L]$ and $t > 0$. For simplicity we shall use $f$ in place of $f_\delta$ in the entire proof. First it follows from (4.1) and (4.2)

$$\frac{d}{dt} \left(e(t) - \int_0^L \int_0^{u(x,t)} f(\xi) d\xi dx\right) = -g_0 \int_0^L u_t^2 dx.$$  

Let $\langle \cdot, \cdot \rangle$ denote the standard pairing of $(H^1_{0,0}(0,L))'$ with $H^1_{0,0}(0,L)$. Since $u \in H^1_{0,0}(0,L)$, then we have $\langle u'' , u \rangle = (d/dt) \langle u' , u \rangle_{L^2(0,L)} - \langle u' , u' \rangle_{L^2(0,L)}$, and $\langle Au , u \rangle = \langle A^{1/2} u , A^{1/2} u \rangle_{L^2(0,L)}$. Therefore, it follows from the PDE that

$$\frac{d}{dt} \int_0^L \left(uu_t + \frac{1}{2}g_0 u^2\right) dx = \int_0^L \left(u f(u) + u_t^2 - u_x^2\right) dx.$$  

Therefore, (4.23) and (4.24) yield that

$$\frac{d}{dt} P(t) = \int_0^L \left[\gamma uf(u) + (\gamma - g_0)u_t^2 - \gamma u_x^2\right] dx.$$  

Since $g_0 \geq 1/(\gamma \varepsilon)$, then by the definitions of $\varepsilon$ and $\gamma$ we must have $g_0 \geq 4\gamma$. By using the fact that $f(u) \leq 1 - \delta$, we have

$$\frac{d}{dt} P(t) \leq -\gamma e(t) + \gamma L f(1 - \delta)\|u(t)\|_{\infty}.$$  

We now show that $P(t)$ satisfies the following inequality:

$$\frac{1}{2} e(t) - L \varepsilon(f(1 - \delta))^2 \leq P(t) \leq c_0 e(t) + L \varepsilon_0(f(0))^2,$$

where $c_0 > 0$ will be specified below.
Let $\varepsilon_0$ and $\varepsilon_1$ be as specified in (4.22). In the estimates below we shall use the Young and the Poincaré inequalities repeatedly. By the mean value theorem, we have

$$-\int_0^L \int_0^u f(\xi) \, d\xi \, dx \leq \int_0^L |u| f(0) \, dx$$

(4.28)

$$\leq \int_0^L \left( \frac{1}{4\varepsilon_0} u^2 + \varepsilon_0 f(0)^2 \right) \, dx$$

$$\leq \frac{1}{4\varepsilon_0} \frac{4L^2}{\pi^2} \int_0^L u^2 \, dx + L\varepsilon_0 f(0)^2.$$  

Since $|uu_t| \leq \eta u^2 + \frac{1}{4\eta} u^2$, then

$$P(t) \leq e(t) + \frac{1}{4} \int_0^L u_t^2 \, dx$$

(4.29)

$$+ \frac{4L^2}{\pi^2} \left( \frac{\gamma}{2} g_0 + \frac{\gamma}{4\eta} + \frac{1}{4\varepsilon_0} \right) \int_0^L u^2 \, dx + L\varepsilon_0 f(0)^2$$

$$\leq c_0 e(t) + L\varepsilon_0 f(0)^2,$$

where $c_0 = \max \{ (3/2), 1 + (8L^2/\pi^2)[(\gamma/2)g_0 + \gamma^2 + (1/4\varepsilon_0)] \}$.

Now, to obtain the other half of inequality (4.27), we first note that

$$\int_0^L \int_0^u f(\xi) \, d\xi \, dx \leq \int_0^L |u| f(1-\delta) \, dx$$

(4.30)

$$\leq \int_0^L \frac{1}{4\varepsilon} u^2 \, dx + L\varepsilon (f(1-\delta))^2.$$  

Note that, since $g_0 \geq 1/(\gamma\varepsilon)$, and thus $(1/4)g_0 \geq \gamma$, then $(\gamma/2)g_0 - \gamma^2 - (1/4\varepsilon) \geq 0$. Therefore, (4.30) and the fact $|uu_t| \leq \eta u_t^2 + (1/4\eta)u^2$ yield the lower bound

$$P(t) \geq e(t) - \frac{1}{4} \int_0^L u_t^2 \, dx + \left( \frac{\gamma}{2} g_0 - \gamma^2 - \frac{1}{4\varepsilon} \right) \int_0^L u^2 \, dx$$

(4.31)

$$- L\varepsilon (f(1-\delta))^2$$

$$\geq \frac{1}{2} e(t) - L\varepsilon (f(1-\delta))^2.$$
Thus, (4.27) is established. Now it follows from (4.26) and (4.27) that

$$\frac{d}{dt} P(t) \leq -\frac{\gamma}{c_0} P(t) + \frac{\gamma}{c_0} L\varepsilon_0 (f(0))^2 + \gamma Lf(1-\delta)\|u(t)\|_\infty. $$

By multiplying (4.32) by $e^{(\gamma/c_0)t}$ and integrating, one has

$$P(t) \leq P(0) e^{-(\gamma/c_0)t} + L\varepsilon_0 (f(0))^2 (1 - e^{-(\gamma/c_0)t})$$

$$+ \gamma Lf(1-\delta) e^{-(\gamma/c_0)t} \int_0^t e^{(\gamma/c_0)s} \|u(s)\|_\infty ds$$

$$\leq P(0) + L\varepsilon_0 (f(0))^2 + \gamma Lf(1-\delta) \int_0^t \|u(s)\|_\infty ds. $$

By using the fact $\|u(s)\|_\infty^2 \leq 2 Le(t)$ and inequality (4.27), we have

$$e(t) \leq 2P(0) + 2L\varepsilon_0 (f(0))^2 + 2L\varepsilon (f(1-\delta))^2$$

$$+ \gamma \sqrt{8L^3} f(1-\delta) \int_0^t \sqrt{e(s)} ds$$

$$= m_0 + \gamma \sqrt{8L^3} f(1-\delta) \int_0^t \sqrt{e(s)} ds,$$

where $m_0 = 2P(0) + 2L\varepsilon_0 (f(0))^2 + 2L\varepsilon (f(1-\delta))^2$.

By using a Gronwall-type inequality, one has

$$e(t) \leq \frac{1}{4} (2\sqrt{m_0} + \gamma \sqrt{8L^3} f(1-\delta)t)^2$$

$$\leq m_0 + 4\gamma^2 L^3 (f(1-\delta))^2 T^2,$$

for $t \in [0, T]$. By recalling (4.22), we have $e(t) \leq (1/2L)(1-2\delta)^2$ for $t \in [0, T]$. Therefore, $\|u(t)\|_\infty^2 \leq (1-2\delta)^2$ for $t \in [0, T]$, which completes the proof.

The following result provides a condition on $L$ under which $u$ cannot quench even in infinite time.

**Theorem 4.8.** Let $u$ be a solution to (1.1) in the sense of Theorem 3.2. Assume that $E(0) \leq 0$ and that

$$L^2 < \sup_{0 \leq u \leq 1} \frac{1}{2} u^2 \left( \int_0^u f(\xi) \, d\xi \right)^{-1}.$$
Then $u$ is a global solution of (1.1), and there exists $\eta_0 \in (0, 1)$ such that $u(x, t) \leq \eta_0$ for all $(x, t) \in [0, L] \times [0, \infty)$.

Proof. Let $\Phi(u) = (1/2)u^2(\int_0^u f(\xi) d\xi)^{-1}$ for $0 \leq u \leq 1$ and $K_3 = \sup_{0 \leq u \leq 1} \Phi(u)$. Since $\Phi(0^+) = 0$ and $0 \leq \Phi(1^-) < \infty$, then $0 < K_3 < \infty$ and there exists some $\eta_0 \in [0, 1]$ with $\Phi(\eta_0) = K_3$. Next we show that, in fact, $\eta_0 \in (0, 1)$. Clearly, $\eta_0 \neq 0$ since $\Phi(0) = 0$. Suppose that $\eta_0 = 1$. Then $\phi(1) = K_3$, and there exists some $\varepsilon > 0$ such that $\Phi$ is increasing on $(1 - \varepsilon, 1)$. Hence, on $(1 - \varepsilon, 1)$, we have

$$
\Phi'(u) = \frac{u \int_0^u f(\xi) d\xi - (1/2)u^2 f(u)}{(\int_0^u f(\xi) d\xi)^2} \geq 0.
$$

Consequently, $\int_0^u f(\xi) d\xi \geq (1/2)u f(u)$ for all $u \in (1 - \varepsilon, 1)$. However, as $u \to 1^-$, $f(u) \to \infty$, and so it must be that $\int_0^1 f(\xi) d\xi = \infty$. Thus, $\Phi(1) = 0$, which contradicts that $K_3 > 0$. Therefore, $\eta_0 \in (0, 1)$, $\Phi(\eta_0) = K_3$ and $\Phi'(\eta_0) = 0$. Thus, $\int_0^{\eta_0} f(\xi) d\xi = (1/2)\eta_0 f(\eta_0)$ and so $K_3 = (\eta_0/f(\eta_0)) \leq K_1$.

Next notice that, since $E(0) \leq 0$, then it follows from (4.1) that $|\Omega^+(t)|$, the Lebesgue measure of $\Omega^+(t)$, is positive for all $t \in [0, T)$. Let

$$
(4.36) \quad T_0 = \sup\{t \geq 0 : 0 \leq u(x, t) \leq \eta_0 \text{ for all } x \in \Omega^+(t)\}.
$$

Suppose that $T_0 < \infty$. Since $u$ is a continuous function, then we must have $0 \leq u(x, T_0) \leq \eta_0$ for all $x \in \Omega^+(T_0)$.

We shall show that there exists some $x_0 \in \Omega^+(T_0)$ with $u(x_0, T_0) = \eta_0$. Let $N \in \mathbb{N}$ be sufficiently large so that $T_0 + (1/N) \leq T$. Then for each $n \in \mathbb{N}$ with $n \geq N$, there exists $x_n \in \Omega^+(T_0 + (1/n))$ such that $u(x_n, T_0 + (1/n)) > \eta_0$. Since $[0, L]$ is compact, $\{x_n\}_{n=N}^{\infty}$ has a convergent subsequence, still denoted by $\{x_n\}$, with $x_n \to x_0$ for some $x_0 \in [0, L]$. By continuity, $u(x_n, T_0 + (1/n)) \to u(x_0, T_0)$. Thus, $u(x_0, T_0) \geq \eta_0$ and so $x_0 \in \Omega^+(T_0)$. Hence, $u(x_0, T_0) = \eta_0$. 


Finally, from (4.1) we have that, for all \( x \in [0, L] \),

\[
(u(x, T_0))^2 \leq 2L \int_{\Omega^+(T_0)} \int_0^{u(x, T_0)} f(\xi) \, d\xi \, dx
\]

(4.37)

\[
\leq 2L \int_{\Omega^+(T_0)} \int_0^{\eta_0} f(\xi) \, d\xi \, dx
\]

\[
\leq 2L^2 \int_0^{\eta_0} f(\xi) \, d\xi.
\]

By letting \( x \to x_0 \), (4.37) yields that \( L^2 \geq \Phi(\eta_0) = K_3 \), which contradicts the assumption. Thus, \( T_0 = \infty \) and so \( u(x, t) \leq \eta_0 \) for all \( (x, t) \in [0, L] \times [0, \infty) \).

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