STABILITY ANALYSIS OF A VOLterra
PREDATOR-PREY SYSTEM WITH TWO DELAYS

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ABSTRACT. The objective of this paper is to systematically study the qualitative properties of a predator-prey system with two delays. First, we investigate the effect of two distinct delays on stability of the unique positive equilibrium by analyzing the corresponding characteristic equation. The stability criteria involving the delays and the parameters are given. Second, we prove a permanence result for the system. Third, we describe the global stability of the positive equilibrium while the sum of two delays is small, via constructing a proper Lyapunov functional in a restricted region. Finally, we give an example to show the application of our results.

1. Introduction. We consider the following Lotka-Volterra predator-prey system with two distinct discrete delays $\tau, \sigma \in \mathbb{R}_{+0} := [0, +\infty)$

\[
\begin{align*}
\dot{x}(t) &= x(t)[e_1 - a_1 x(t) - a_2 y(t - \sigma)], \\
\dot{y}(t) &= y(t)[-e_2 + a_3 x(t - \tau) - a_4 y(t)],
\end{align*}
\]

where $e_1, e_2, a_2, a_3$ are positive constants and $a_1, a_4$ are nonnegative. In biological terms, $x(t)$ and $y(t)$ may be interpreted as the densities of prey and predator populations, respectively, and $a_1, a_4$ self-limitation constants. In the absence of predator, the prey species follows the logistic equation $\dot{x}(t) = x(t)[e_1 - a_1 x(t)]$. In the presence of predator there is a hunting term, $a_2 y(t - \sigma)$, $a_2 > 0$, with a certain delay $\sigma$, called the hunting delay. In the absence of prey species, the predator decreases. The positive feedback $a_3 x(t - \tau)$ has a positive delay $\tau$ which is the delay in the predator maturation.

System (1.1) with some constants vanishing (most of the literature considers $\sigma = 0, a_4 = 0$) as with some more constraints on the constants, and also predator-prey models with distributed delays, have
been widely studied [1, 2, 5, 7]. Here we assume that

\begin{equation}
\frac{e_1}{e_2} > \frac{a_1}{a_3}.
\end{equation}

The condition (1.2) guarantees the existence of a positive equilibrium \( E_\ast = (x_\ast, y_\ast) \) for system (1.1) where

\begin{equation}
x_\ast = \frac{e_1a_4 + e_2a_2}{a_1a_4 + a_2a_3}, \quad y_\ast = \frac{e_1a_3 - e_2a_1}{a_1a_4 + a_2a_3}.
\end{equation}

In this paper we study the effect of two distinct discrete delays on the stability of the unique positive equilibrium \( E_\ast \), and we prove a nonlocal convergence result to \( E_\ast \) when \( \sigma + \tau \) is less than a given threshold.

We assume that the following initial conditions hold for (1.1):

\begin{equation}
\begin{aligned}
(x(\theta), y(\theta)) &= (\phi_1(\theta), \phi_2(\theta)) \geq 0, \\
\theta &\in [-\max(\sigma, \tau), 0], \\
\phi_1(0) > 0, \phi_2(0) > 0,
\end{aligned}
\end{equation}

where \( \phi = (\phi_1, \phi_2) \in \mathcal{C}([-\max(\sigma, \tau), 0], \mathbb{R}^2_{+0}) \).

We finish this introduction by noting that the paper contains three subsequent sections. In Section 2 we investigate the effect regions of two distinct delays on the local asymptotical stability of the positive equilibrium by analyzing the corresponding characteristic equation. In Section 3 we present the global stability of the boundary equilibrium and the permanence result of the system. Then, in Section 4, we discuss in detail the global stability of the positive equilibrium. Finally, we briefly discuss an example.

2. Local stability analysis. It is easy to check that for all parameter values system (1.1) admits both the trivial equilibrium \( E_0 = (0, 0) \) and the boundary equilibrium \( E_{+0} = [(e_1/a_1), 0] \). By the characteristic equation we see that \( E_0 \) is a saddle point for all values of delays \( \sigma, \tau \), whereas \( E_{+0} \) is an asymptotically stable node if the positive equilibrium \( E_\ast \) is not feasible, i.e., when \( e_2a_1 - e_1a_3 > 0 \). \( E_{+0} \) becomes critically stable when \( e_2a_1 - e_1a_3 = 0 \) and an unstable saddle point when \( E_\ast \) is feasible, (i.e. \( e_2a_1 - e_1a_3 < 0 \)). These results are independent of the delay values \( \sigma, \tau \).
After this short review, our local stability analysis will be concerned with the positive equilibrium $E_\ast = (x_\ast, y_\ast)$. At the positive equilibrium $E_\ast$, the characteristic equation is

$$\lambda^2 + a\lambda + b + ce^{-\lambda(\sigma+\tau)} = 0,$$

where

$$a = a_1 x_\ast + a_4 y_\ast, \quad b = a_1 a_4 x_\ast y_\ast, \quad c = a_2 a_3 x_\ast y_\ast.$$ 

If in the following we set

$$\mu = \sigma + \tau, \quad \mu \in \mathbb{R}_+,$$

the characteristic equation (2.1) can be written as

$$D(\lambda, \mu) = 0$$

with

$$D(\lambda, \mu) = \lambda^2 + g(\lambda, \mu)$$

where

$$g(\lambda, \mu) = a\lambda + b + ce^{-\lambda\mu}.$$ 

We remark that both the particular cases $\sigma = 0$, $\tau > 0$ and $\sigma > 0$, $\tau = 0$ are included in the general structure (2.4)–(2.6). For the characteristic equation (2.4)–(2.6), the following theorem holds true (see Freedman and Kuang [4]):

**Theorem 2.1.** Let

$$\alpha = \limsup_{\substack{\operatorname{Re} \lambda \geq 0 \\ |\lambda| \to \infty}} |\lambda^{-2} g(\lambda, \mu)| < 1.$$ 

Then, as $\mu$ varies in $\mathbb{R}_+$, the sum of multiplicities of roots of $D(\lambda, \mu) = 0$ in the open right complex plane can change only if a root appears on or crosses the imaginary axis.
Furthermore, observe that $\lambda = 0$ cannot be a characteristic root of (2.4)–(2.6) since $b > 0$. In the following, by stability switch we mean a value $\mu^* = \sigma^* + \tau^*$ crossing which, say for increasing $\mu$, the stability changes from (local) asymptotic stability to instability (or vice versa). Hence, stability switches in (2.4)–(2.6) can only occur with a pair of imaginary roots, say $\lambda = \pm i\omega, \omega \in \mathbb{R} - \{0\}$.

It is easy to check that if $\lambda = i\omega, \omega > 0$, is a characteristic root then even $\lambda = -i\omega, \omega > 0$, is a characteristic root. We can therefore prove the following:

**Theorem 2.2.** (i) If $a_1a_4 \geq a_2a_3$, then $E_*$ is locally asymptotically stable for all $\sigma, \tau \in \mathbb{R}_+$;

(ii) if $a_1a_4 < a_2a_3$, then $E_*$ is locally asymptotically stable for all $\sigma, \tau \in \mathbb{R}_+$ satisfying

$$\sigma + \tau < \mu_0^* = \frac{\theta}{\omega_+}, \quad (2.8)$$

and unstable for all $\sigma, \tau \in \mathbb{R}_+$ satisfying $\sigma + \tau > \mu_0^*$, where $\theta \in [0, 2\pi]$ is a solution of

$$\cos \theta = \frac{\omega_+^2 - b}{c}, \quad \sin \theta = \frac{\omega_+a}{c} \quad (2.9)$$

and $\omega_+ > 0$ is given by

$$\omega_+^2 = \frac{1}{2} \left[ \sqrt{\Delta} - a_1^2x_*^2 - a_4^2y_*^2 \right], \quad \Delta = (a_1^2x_*^2 + a_4^2y_*^2)^2 + 4x_*^2y_*^2(a_2^2a_3^2 - a_1^2a_4^2) \quad (2.10)$$

**Proof.** First of all we remark that when $\sigma = \tau = 0$, the characteristic equation (2.5) can have only roots with negative real parts, i.e., $\text{Re} \lambda < 0$, and therefore $E_*$ is locally asymptotically stable.

Let us look now for the occurrence of stability switches when $\mu = \sigma + \tau > 0$. According to the previous remarks we must look for the occurrence of roots $\lambda = i\omega, \omega > 0$, of (2.5). Following Freedman and
Kuang [4], these roots can occur if and only if $\mu$ assumes one of the values

$$
\mu_n^* = \frac{\theta + n2\pi}{\omega}, \; n \in N_0
$$

where $\theta \in [0, 2\pi]$ is a solution of

$$
\cos \theta = \frac{\omega^2 - b}{c}, \; \sin \theta = \frac{\omega a}{c}
$$

and $\omega$ is the positive root of

$$
\omega_{\pm}^2 = \frac{1}{2} \left[ (2b - a^2) \pm \sqrt{\Delta} \right], \; \Delta = (2b - a^2)^2 - 4(b^2 - c^2).
$$

Since

$$
2b - a^2 = -(a_1^2x_*^2 + a_4^2y_*^2) < 0, \; b^2 - c^2 = a_1^2a_4^2 - a_2^2a_3^2,
$$

we have the following cases:

(i) $a_1a_4 \geq a_2a_3$. Since $2b - a^2 < 0$ and $b^2 - c^2 > 0$, neither $\omega_+ > 0$ nor $\omega_- > 0$ are feasible, that is, $\lambda = i\omega$, $\omega > 0$, cannot be a root of (2.5) for all $\sigma, \tau \in R_+^0$, i.e., we cannot have stability switches even when $\tau = 0$, $\sigma > 0$ or $\sigma = 0$, $\tau > 0$. At $\sigma = \tau = 0$, $E_*$ is locally asymptotically stable. Therefore, $E_*$ will remain locally asymptotically stable for all $(\tau, \sigma)$ in the positive cone $K^+ := \{(\sigma, \tau) \in R_+^{20}\}$. This proves (i).

(ii) $a_1a_4 < a_2a_3$. Since $2b - a^2 < 0$ and $b^2 - c^2 < 0$, $\omega_+ > 0$ is feasible and $\omega_- > 0$ is not feasible. Let $\theta \in [0, 2\pi]$ be a solution of

$$
\cos \theta = \frac{\omega_+^2 - b}{c}, \; \sin \theta = \frac{\omega_+ a}{c}.
$$

Then the imaginary roots $\lambda = \pm i\omega_+$, $\omega_+ > 0$, will appear at the $\mu$ value

$$
\mu_n^* = \frac{\theta + n2\pi}{\omega_+}, \; n \in N_0.
$$

From Kuang [5] and the characteristic equation (2.5), by straightforward computations, we have

$$
\text{sign} \left\{ \frac{d\Re(e^{\mu^*})}{d\mu} \right|_{\mu = \mu_n^*} \right\} = \text{sign} \left\{ \sqrt{\Delta(\mu_n^*)} \right\} = 1,
$$
therefore

\begin{equation}
\left. \frac{d\Re \lambda}{d\mu} \right|_{\mu=\mu^*_n} > 0, \quad n \in N_0.
\end{equation}

In fact, for each fixed value of \( \sigma \), the roots \( \lambda = \pm i\omega_+, \quad \omega_+ > 0 \), occur at the \( \tau \) values

\begin{equation}
\tau^*_n = \mu^*_n - \sigma = \frac{\theta + n2\pi}{\omega_+} - \sigma, \quad n \in N_0
\end{equation}

where, according to (2.17),

\begin{equation}
\text{sign} \left\{ \left. \frac{d\Re \lambda}{d\tau} \right|_{\tau=\tau^*_n} \right\} = \text{sign} \left\{ \left. \frac{d\Re \lambda}{d\mu} \right|_{\mu=\mu^*_n} \right\} > 0.
\end{equation}

Assume that \( \sigma > \sigma_0 = (\theta/\omega_+) \). Then at \( \tau = 0 \), \( E_* \) is unstable and, according to (2.19), it will remain unstable when \( \tau \) crossing any other \( \tau^*_n \), \( n \geq 1 \), value, that is, for all \( \tau > 0 \). If \( \sigma < \sigma_0 \), then at \( \tau = 0 \), \( E_* \) is locally asymptotically stable and it remains asymptotically stable for increasing \( \tau \) up to the value \( \tau^*_0 = \mu^*_0 - \sigma \), and then, according to (2.19), unstable for all \( \tau > \mu^*_0 - \sigma \).

The same kind of arguments can be applied when keeping \( \tau \) fixed and letting \( \sigma \) vary from \( 0 \) on \( \Re_+ \). The effect regions of the delays on local stability of \( E_* \) are depicted in Figure 1a and b. \( \blacksquare \)
We remark that in Theorem 2.2 we assume that $a_1^2 + a_4^2 > 0$. If $a_1 = a_4 = 0$, then for all $\mu = \sigma + \tau > 0$, $E_*$ is unstable.

In fact, the characteristic equation (2.5) becomes

\[(2.20) \quad \lambda^2 + e_1 e_2 e^{-\lambda \mu} = 0.\]

Let $\lambda = i\omega$, $\omega > 0$, be a solution of (2.20), then from (2.11)-(2.13),

\[\omega_+ = \sqrt{e_1 e_2}, \quad \mu_n^* = \frac{n2\pi}{\omega_+}, \quad n \in \mathbb{N},\]

and

\[\frac{d\Re \lambda}{d\mu} \bigg|_{\mu=\mu_n^*} = \frac{2\omega_+^2}{4 + (\mu_n^* \omega_+)^2} > 0, \quad n \in \mathbb{N}.\]

Now we choose the first value $\mu_1^* = 2\pi/\sqrt{e_1 e_2}$ such that (2.20) has a pair of pure imaginary roots $\lambda(\mu_1^*) = \pm i\omega_+$ and no stability switch may occur in $0 < \mu < \mu_1^*$, then we consider the stability of (2.20) for a special case $\mu_1^* > \mu > 3\mu_1^*/4$. According to the method of Taboas [6], we rewrite (2.20) as

\[(2.21) \quad z^2 e^{2z} = -\frac{\mu^2}{4} e_1 e_2.\]

Let $z = x + iy$, $0 < y < \pi/2$, be a solution of (2.21), then we have

\[(2.22) \quad \frac{\mu^2}{4} e_1 e_2 = e^{2x} (x^2 + y^2) < e^{2x} \left(x^2 + \frac{\pi^2}{4}\right).\]
Define a function

\[ f(t) = e^t - \mu^2 e_1 e_2 (t^2 + \pi^2)^{-1}, \quad t \in (0, +\infty). \]

Obviously, \( f(t) \) is strictly increasing for \( t \in (0, +\infty) \). From (2.22) we have

\[ f(2x) > 0, \quad f(0) = 1 - \mu^2 e_1 e_2 \pi^{-2} < -\frac{5}{4} < 0. \]

Hence, there exists at least one positive zero \( x_0 \) of \( f(t) \) on \( (0, +\infty) \). This implies that (2.21) has at least one root \( z = x_0 + iy, \quad 0 < y < \pi/2 \) with positive real part. Then, for all \( \mu = \sigma + \tau > 0 \), \( E^* \) is unstable.

3. A permanence result. Before considering the convergence result for the positive equilibrium \( E^* = (x^*, y^*) \), we prove a global convergence result for the boundary equilibrium \( E_{+0} = [(e_1/a_1), 0] \) when the positive equilibrium \( E^* \) is not feasible, i.e.,

\[ a_3 e_1 < a_1 e_2 \]

holds true.

**Theorem 3.1.** If inequality (3.1) holds true then for all nonnegative delay values \( \tau, \sigma \) the boundary equilibrium \( E_{+0} = [(e_1/a_1), 0] \) is asymptotically stable for all initial conditions \( \phi \) such that \( \phi(0) \in \mathbb{R}_{+}^2 \) (we say that \( E_{+0} \) is globally asymptotically stable with respect to \( \mathbb{R}_{+}^2 \)).

**Proof.** If \( \phi(0) \in \mathbb{R}_{+}^2 \), the corresponding solution \( (x(t), y(t)) \) of (1.1) remains positive whenever it exists. Then the first equation of (1.1) gives

\[ \dot{x}(t) = x(t)[e_1 - a_1x(t) - a_2y(t - \sigma)] \leq x(t)[e_1 - a_1x(t)] \]

from which

\[ \limsup_{t \to +\infty} x(t) \leq \frac{e_1}{a_1}. \]

Define

\[ u_1(t) := x(t) - \frac{e_1}{a_1}. \]
Hence (3.2) implies that for all $\varepsilon > 0$, there exists $t_1 > 0 : u_1(t) < \varepsilon$ for all $t > t_1$. From the second part of (1.1), we get

\begin{equation}
\dot{y}(t) = y(t) \left[ -e_2 + \frac{a_3 e_1}{a_1} + a_3 u_1(t - \tau) - a_4 y(t) \right],
\end{equation}

where we set

\[ e_2 - \frac{a_3 e_1}{a_1} = \frac{a_1 e_2 - a_3 e_1}{a_1} = A. \]

Due to (3.1), $A > 0$. Furthermore, for all $\varepsilon > 0$, there exists $T_1 = t_1 + \tau > 0$ such that $u_1(t - \tau) < \varepsilon$ for all $t > T_1$. From (3.3) we get

\begin{equation}
\dot{y}(t) \leq y(t) \left[ - (A - a_3 \varepsilon) - a_4 y(t) \right] \leq -(A - a_3 \varepsilon) y(t)
\end{equation}

for all $t > T_1$ where $A - a_3 \varepsilon > 0$. Hence $y(t) \to 0$ as $t \to +\infty$. This implies that for all $\varepsilon^1 > 0$, there exists $t^* > T_1$ such that $0 < y(t) < \varepsilon^1$. Therefore, from the first part of (1.1), we can say that for all $\varepsilon^1 > 0$, there exists $T^* = t^* + \sigma$ such that

\begin{equation}
x(t)[e_1 - a_1 x(t) - a_2 \varepsilon^1] \leq \dot{x}(t) \leq x(t)[e_1 - a_1 x(t)]
\end{equation}

for all $t > T^*$. Hence

\begin{equation}
\frac{e_1 - a_2 \varepsilon^1}{a_1} \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq \frac{e_1}{a_1},
\end{equation}

and letting $\varepsilon^1 \to 0$ we get $\lim_{t \to +\infty} x(t) = (e_1/a_1)$. In conclusion, we have proved that for all initial conditions such that $\phi(0) \in \mathbb{R}_+^2$ the solution $(x(t), y(t)) \to [(e_1/a_1), 0]$ as $t \to +\infty$ provided that (3.1) holds true. The proof remains true even if $a_4 = 0$. \hfill \Box

Assume now that the positive equilibrium $E^*_*$ is feasible, i.e., the following inequality

\begin{equation}
a_3 e_1 - a_1 e_2 > 0
\end{equation}

holds true. Then we can prove ([2, Theorem 2.1]) that all solutions of (1.1) with initial conditions (1.4) satisfy:

\begin{equation}
\limsup_{t \to +\infty} x(t) \leq \frac{e_1}{a_1} \Delta M_1,
\end{equation}

and

\begin{equation}
\limsup_{t \to +\infty} y(t) \leq \frac{a_3 e_1 - a_1 e_2}{a_1 a_4} \Delta M_2.
\end{equation}
Furthermore, when (3.7) holds true, we can prove

**Theorem 3.2.** Assume that

\[(H_1) \quad m_1 := \frac{1}{a_1 a_4} \left\{ \frac{e_1}{a_1} (a_1 a_4 - a_2 a_3) + a_2 e_2 \right\} > 0,\]

then

\[\liminf_{t \to +\infty} x(t) \geq m_1.\]  

Furthermore, assume that

\[(H_2) \quad m_2 := \frac{(a_3 e_1 - a_1 e_2)(a_1 a_4 - a_2 a_3)}{a_1^2 a_4^2} > 0,\]

then

\[\liminf_{t \to +\infty} y(t) \geq m_2.\]

**Proof.** Equation (3.7) implies that \(\limsup_{t \to +\infty} y(t) \leq M_2.\) Hence, from the first equation of (1.1) and for sufficient large \(t\),

\[\dot{x}(t) \geq a_1 x(t) \left[ \frac{e_1 - a_2 M_2}{a_1} - x(t) \right],\]

where it is easy to check that \((e_1 - a_2 M_2)/a_1 = m_1.\) Therefore, if \((H_1)\) holds true, then

\[\dot{x}(t) \geq a_1 x(t) [m_1 - x(t)]\]  

and (3.9) follows from inequality (3.11).

Similarly, from the second of (1.1) and (3.9) we have that for sufficient large \(t\)

\[\dot{y}(t) \geq a_4 y(t) \left[ \frac{-e_2 + a_3 m_1}{a_4} - y(t) \right],\]
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where \((-e_2 + a_3m_1)/a_4 = m_2\). Therefore, if (H_2) holds true (3.12) implies (3.10). \(\square\)

We remark that if \(a_3e_1 - a_1e_2 > 0\), i.e., \(E_*\) is locally asymptotically stable for all nonnegative \(\sigma, \tau\), then (1.1) is uniformly persistent in the sense that all solutions of (1.1) with initial conditions (1.4) satisfy that

\[
\begin{align*}
    m_1 &\leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq M_1, \\
    m_2 &\leq \liminf_{t \to +\infty} y(t) \leq \limsup_{t \to +\infty} y(t) \leq M_2.
\end{align*}
\]

If \(a_3e_1 - a_1e_2 \leq 0\), we may still have uniform persistence for the prey species \(x(t)\) (provided that (H_1) holds true), but we lose the persistence for the predator species \(y(t)\) since \(m_2 \leq 0\). Furthermore, we remark that thanks to Theorem 3.1, (or to (3.8)), the solution of (1.1) are always bounded from above.

4. A convergence result. Let \(E_* = (x_*, y_*)\) be positive equilibrium of system (1.1). We now define

\[
\begin{align*}
    u_1 &= \frac{x - x_*}{x_*}, \\
    u_2 &= \frac{y - y_*}{y_*},
\end{align*}
\]

then (1.1) becomes

\[
\begin{align*}
    u'_1(t) &= -(1 + u_1)[Au_1 + Bu_2(t - \sigma)], \\
    u'_2(t) &= (1 + u_2)[Cu_1(t - \tau) - Du_2(t)],
\end{align*}
\]

where

\[
A = a_1x_*, \quad B = a_2y_*, \quad C = a_3x_*, \quad D = a_4y_*.
\]

It is easy to verify that solutions of (1.1) with initial conditions (1.4) exist and stay positive for all \(t > 0\). The following two lemmas will be useful in the proof of our main results in this section.

Consider an autonomous system of delay differential equation

\[
\dot{x}(t) = F(x_t)
\]

such that \(F(0) = 0\) and \(F: C([-\tau^*, 0], \mathbb{R}^2) \to \mathbb{R}^2, \tau^* > 0,\) is Lipschitzian, where \(C = C([-\tau^*, 0], \mathbb{R}^2)\) is the set of continuous functions defined on \([-\tau^*, 0]\) with the norm

\[
\|\phi\| = \max_{\theta \in [-\tau^*, 0]} |\phi(\theta)|,
\]
where $|\cdot|$ is any norm in $\mathbb{R}^2$.

The following lemma needs no proof being a direct consequence of a result in Kuang ([5, Corollary 5.2]).

**Lemma 4.1.** Let $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$ be nonnegative continuous scalar functions such that $\varpi_i(0) = 0$, $i = 1, 2$; $\lim_{r \to +\infty} \varpi_1(r) = +\infty$, $\varpi_2(r) > 0$ for $r > 0$. Let $V : C \to \mathbb{R}$ be a continuously differentiable scalar functional and $S$ a nonempty subset of $C$ for which the following are satisfied

\begin{equation}
V(\phi) \geq \varpi_1(|\phi(0)|), \quad \dot{V}(\phi)|_{(4.4)} \leq -\varpi_1(|\phi(0)|).
\end{equation}

Then $x = 0$ is asymptotically stable with respect to the set $S$. That is, solutions that stay in $S$ converge to $x = 0$.

Our strategy in the proof of the convergence result of the positive equilibrium is to construct a suitable Liapunov functional, and for that, we need the following lemma.

**Lemma 4.2.** Let $m_i$, $i = 0, 1, \ldots, 9$, and $\gamma$ be positive constants and $w, \beta$ be two positive variables satisfying $\beta > \gamma$ and $w = (\gamma + m_0)/(\beta - \gamma)$. Then the optimization problem of finding

\begin{equation}
\eta = \max_{\beta} \left\{ \min \left\{ \frac{m_1 w}{m_2 + m_3 w + m_4 \beta w}, \frac{m_5 + m_6 w \beta}{m_7 + m_8 w + m_9 \beta w} \right\} \right\}
\end{equation}

has a unique solution $\beta = \beta^* > \gamma$ if and only if

\begin{equation}
\frac{m_1}{m_3 + m_4 \gamma} > \frac{m_6 \gamma}{m_8 + m_9 \gamma}.
\end{equation}

The proof of Lemma 4.2 is similar to Lemma 2.2 in Beretta and Kuang [1]. If $m_6 = 0$, condition (4.7) holds forever, hence the Lemma 2.2 in Beretta and Kuang [1] is the special case of Lemma 4.2. We shall use system (4.2) in our analysis below. Note that, for $t \geq 0$, $1 + u_i(t) > 0$, $i = 1, 2$. Consider first the following scalar function $V_0(t)$ which is defined as

\begin{equation}
V_0(t) = \log(1 + u(t)) + \alpha \log(1 + u_2(t));
\end{equation}
here $\alpha$ is a positive constant whose value is to be determined later. For convenience, we let

$$z_i \equiv z_i(u_i(t)) = \log(1 + u_i(t)), \quad i = 1, 2.$$  

Then along the solution of (4.2) we have

$$\left(\frac{1}{2}V_0^2\right)' = V_0V_0'$$

$$= (z_1 + \alpha z_2)[-Au_1 - Bu_2(t - \sigma) + \alpha Cu_1(t - \tau) - \alpha Du_2].$$

Notice that

$$u_1(t - \tau) = u_1(t) - \int_{t-\tau}^t u'_1(s) \, ds,$$

$$u_2(t - \sigma) = u_2(t) - \int_{t-\sigma}^t u'_2(s) \, ds,$$

we obtain

$$\left(\frac{1}{2}V_0^2\right)' = (z_1 + \alpha z_2)[(\alpha C - A)u_1 - (B + \alpha D)u_2(t)]

+ (z_1 + \alpha z_2) \left[ B \int_{t-\sigma}^t u'_2(s) \, ds - \alpha C \int_{t-\tau}^t u'_1(s) \, ds \right].$$

By choosing $\alpha = A/C$, we have

$$\left(\frac{1}{2}V_0^2\right)' = -(z_1 + \alpha z_2)(B + \alpha D)u_2

+ (z_1 + \alpha z_2) \left[ B \int_{t-\sigma}^t u'_2(s) \, ds - \alpha C \int_{t-\tau}^t u'_1(s) \, ds \right].$$

Now we define the following scalar function

$$V_1(t) = u_1 - z_1 + \beta(u_2 - z_2),$$

where $\beta$ is a positive constant to be determine later. We have the derivative of $V_1(t)$ along the solution of (4.2)

$$V'_1(t) = -Au_1^2 - Bu_1 u_2(t - \sigma) + \beta Cu_2 u_1(t - \tau) - D\beta u_2^2

= -Au_1^2 - Bu_1 u_2 + \beta Cu_2 u_1

+ Bu_1 \int_{t-\sigma}^t u'_2(s) \, ds - \beta Cu_2 \int_{t-\tau}^t u'_1(s) \, ds - D\beta u_2^2.$$
Hence we have

$$\left(\frac{1}{2}V_0^2 + wV_1\right)' = -(B + \alpha D)(z_1 + \alpha z_2)u_2 - A wu_1^2 - Bw u_1 u_2$$

$$+ \beta C wu_2 u_1 + B(z_1 + \alpha z_2 + wu_1) \int_{t-\sigma}^{t} u_2'(s) \, ds$$

$$- D\beta wu_2^2 - C(\alpha z_1 + \alpha^2 z_2 + \beta wu_2) \int_{t-\tau}^{t} u_1'(s) \, ds.$$  

Here \( w \) is a positive constant to be determined later. Using the inequality

$$2 \int_{t-\sigma}^{t} z_1(t)(1 + u_2(s))u_1(s - \tau) \, ds$$

$$\leq z_1^2 \sigma + \int_{t-\sigma}^{t} (1 + u_2(s))^2 u_1^2(s - \tau) \, ds,$$

and by similar manipulations for other integral terms, we obtain that

$$B(z_1 + \alpha z_2 + wu_1) \int_{t-\sigma}^{t} u_2'(s) \, ds$$

$$= B(z_1 + \alpha z_2 + wu_1) \int_{t-\sigma}^{t} (1 + u_2)[Cu_1(s - \tau) - Du_2(s)] \, ds$$

$$\leq \frac{1}{2}BC\sigma(z_1^2 + \alpha z_2^2 + wu_1^2) + \frac{1}{2}BD\sigma(z_1^2 + wu_1^2)$$

$$+ \frac{1}{2}BC(1 + \alpha + w) \int_{t-\sigma}^{t} [1 + u_2(s)]^2 u_1^2(s - \tau) \, ds$$

$$+ \frac{1}{2}BD(1 + w) \int_{t-\sigma}^{t} [1 + u_2(s)]^2 u_2^2(s) \, ds,$$

$$-C(\alpha z_1 + \alpha^2 z_2 + \beta wu_2) \int_{t-\tau}^{t} u_1'(s) \, ds$$

$$= C(\alpha z_1 + \alpha^2 z_2 + \beta wu_2)$$

$$\cdot \int_{t-\tau}^{t} (1 + u_1)[Au_1(s) + Bu_2(s - \sigma)] \, ds$$

$$\leq \frac{1}{2}AC\tau(\alpha z_1^2 + \alpha^2 z_2^2 + wu_1^2) + \frac{1}{2}BC\tau(\alpha z_1^2 + \alpha^2 z_2^2 + \beta wu_2^2)$$
\[ + \frac{1}{2} AC(\alpha + \alpha^2 + \beta w) \int_{t-\tau}^{t} [1 + u_1(s)]^2 u_1^2(s) \, ds \]
\[ + \frac{1}{2} BC(\alpha + \alpha^2 + \beta w) \int_{t-\tau}^{t} [1 + u_1(s)]^2 u_2^2(s - \sigma) \, ds. \]

Denote

\[ (4.11) \quad p = \frac{1}{2} B(1 + \alpha + w), \quad q = \frac{1}{2} C(\alpha + \alpha^2 + \beta w), \]

therefore,

\[
\left( \frac{1}{2} V_0^2 + wV_1 \right)'
\leq -\alpha(B + \alpha D)z_2 u_2 - Awu_2^2 - D\beta wu_2^2 - \frac{1}{2} BD\alpha\sigma z_2^2
\]
\[- (B+\alpha D)z_1 u_2 + (\beta C - B)wu_1 u_2 + \frac{1}{2} B(C + D)\sigma (z_1^2 + \alpha z_2^2 + wu_1^2)
\]
\[ + \frac{1}{2} (A + B)C\tau (\alpha z_1^2 + \alpha^2 z_2^2 + \beta wu_2^2) + Aq \int_{t-\tau}^{t} [1 + u_1(s)]^2 u_1^2(s) \, ds
\]
\[ + \left( Dp - \frac{1}{2} BD\alpha \right) \int_{t-\tau}^{t} [1 + u_2(s)]^2 u_2^2(s) \, ds
\]
\[ + Cp \int_{t-\sigma}^{t} [1 + u_2(s)]^2 u_1^2(s - \tau) \, ds + Bq \int_{t-\tau}^{t} [1 + u_1(s)]^2 u_2^2(s - \sigma) \, ds. \]

Thus, if we define \( V_2(t) \) as

\[ V_2(t) = Cp \int_{t-\sigma}^{t} ds \int_{s}^{t} [1 + u_2(\gamma)]^2 u_1^2(\gamma - \tau) \, d\gamma
\]
\[ + Aq \int_{t-\tau}^{t} [1 + u_1(s)]^2 u_1^2(s) \, ds
\]
\[ + Bq \int_{t-\tau}^{t} ds \int_{s}^{t} [1 + u_1(\gamma)]^2 u_2^2(\gamma - \sigma) \, d\gamma
\]
\[ + Cp\sigma \int_{t-\sigma}^{t} \left( u_1^2 + \frac{1}{2} u_1^4 \right) \, ds
\]
\[ + \left( Dp - \frac{1}{2} BD\alpha \right) \int_{t-\tau}^{t} [1 + u_2(s)]^2 u_2^2(s) \, ds
\]
\[ + Bq\tau \int_{t-\tau}^{t} \left( u_2^2 + \frac{1}{2} u_2^4 \right) \, ds, \]
we obtain that

\[
\left( \frac{1}{2}V_0^2 + wV_1 + V_2 \right)'
\leq -\alpha(B + \alpha D)z_2 u_2 - Awu_1^2 - D\beta wu_2^2
- (B + \alpha D)z_1 u_2 + (\beta C - B)wu_1 u_2
+ \frac{1}{2} B(C + D)\sigma(z_1^2 + \alpha z_2^2 + wu_1^2) - \frac{1}{2} BD\alpha \sigma z_2^2
+ \frac{1}{2} (A + B) C \tau (\alpha z_1^2 + \alpha^2 z_2^2 + \beta wu_2^2)
+ \left( Dp - \frac{1}{2} BD\alpha \right) \sigma(1 + u_2)^2 u_2^2
+ Aq \tau (1 + u_1)^2 u_1^2 + C p \sigma \left( u_1^2 + \frac{1}{2} u_1^4 + \frac{1}{2} (u_2^2 + 2u_2)^2 \right)
+ Bq \tau \left( u_2^2 + \frac{1}{2} u_2^4 + \frac{1}{2} (u_1^2 + 2u_1)^2 \right).
\]

Recall that \( z_i = \log(1 + u_i(t)), \ i = 1, 2 \), we define for \( i = 1, 2 \),

\[
\varepsilon_i u_i \equiv \varepsilon_i(u_i)u_i \equiv z_i - u_i \leq 0;
\]

hence,

\[
z_i = u_i + \varepsilon_i u_i, \quad i = 1, 2.
\]

Then we can define our Liapunov functional \( V \equiv V(\phi_1, \phi_2) \) on

\[
\{(\phi_1, \phi_2) \in C([-\mu, 0], \mathbb{R}^2), \quad \phi_i(\theta) = 0, \ \theta \in [-\mu, -\max(\sigma, \tau)], \ i = 1, 2\}
\]

as

\[
V \equiv V(\phi_1, \phi_2)
= \frac{1}{2} V_0^2 (\phi_1(0), \phi_2(0)) + wV_1 (\phi_1(0), \phi_2(0)) + V_2 (\phi_1, \phi_2),
\]

where \( \phi_1, \phi_2 \) are the boundary values of \( \phi_i \).
therefore we obtain
\[
V' \leq \left[ (\beta C - B)w - (B + \alpha D) \right] u_1 u_2 - (B + \alpha D) \varepsilon_1 u_1 u_2 \\
- \alpha(B + \alpha D)u_2^2 - \alpha(B + \alpha D)\varepsilon_2 u_2^2 - Awu_1^2 - D\beta wu_2^2 \\
+ \frac{1}{2} B(C + D)\sigma(u_1^2 + \alpha u_2^2 + wu_1^2) - \frac{1}{2} BD\sigma u_2^2 \\
+ Aq\tau(u_1^2 + 2u_1^3 + u_1^4) + \frac{1}{2}(A + B)C\tau(\alpha u_1^2 + \alpha^2 u_2^2 + \beta wu_2^2) \\
+ \left( Dp - \frac{1}{2} BD\alpha \right)\sigma(u_2^2 + 2u_3^2 + u_4^2) \\
+ Bq\tau\left( u_2^2 + \frac{1}{2} u_4^2 + 2u_1^3 + 2u_1^4 + \frac{1}{2} u_1^4 \right) + \frac{1}{2} BD\sigma(2\varepsilon_1 + \varepsilon_1^2)u_1^2 \\
+ C\rho\sigma\left( u_1^2 + \frac{1}{2} u_4^2 + 2u_2^3 + 2u_3^2 + \frac{1}{2} u_2^4 \right) \\
+ \frac{1}{2}(A + B)C\tau(2\alpha\varepsilon_1 u_1^2 + \alpha\varepsilon_1^2 u_1^2 + 2\alpha^2\varepsilon_2 u_2^2 + \alpha^2\varepsilon_2^2 u_2^2) \\
+ \frac{1}{2} BC\sigma((2\varepsilon_1 + \varepsilon_1^2)u_1^2 + (2\varepsilon_2 + \varepsilon_2^2)\alpha u_2^2). 
\]

Denote
\[
f(\beta, w) = B(C + D)(1 + w) + (A + B)C\alpha + 2Aq + 2Cp + 4Bq, \\
g(\beta, w) = BC\alpha + (A + B)C(\alpha^2 + \beta w) + 2Dp - BD\alpha + 2Bq + 4Cp, 
\]
then
\[
V' \leq -u_1^2 \left[ Aw - \frac{1}{2} f(\beta, w) \right] \\
- u_2^2 \left[ \alpha(B + \alpha D) + D\beta w - \frac{1}{2} g(\beta, w) \right] \\
+ \left[ (\beta C - B)w - (B + \alpha D) \right] u_1 u_2 - (B + \alpha D)\varepsilon_1 u_1 u_2 \\
- \alpha(B + \alpha D)\varepsilon_2 u_2^2 + \left( Dp - \frac{1}{2} BD\alpha \right)\mu(2u_2^3 + u_4^2) \\
+ Aq\mu(2u_1^3 + u_4^4) + \frac{1}{2} C\rho\mu(u_1^4 + 4u_2^3 + u_4^4) + \frac{1}{2} Bq\mu(u_2^4 + 4u_1^3 + u_1^4) \\
+ \frac{1}{2} BC\mu(2\varepsilon_1 u_1^2 + \varepsilon_1^2 u_1^2 + 2\alpha\varepsilon_2 u_2^2 + \alpha\varepsilon_2^2 u_2^2) + \frac{1}{2} BD\mu(2\varepsilon_1 + \varepsilon_1^2)u_1^2 \\
+ \frac{1}{2}(A + B)C\mu(2\alpha\varepsilon_1 u_1^2 + \alpha\varepsilon_1^2 u_1^2 + 2\alpha^2\varepsilon_2 u_2^2 + \alpha^2\varepsilon_2^2 u_2^2). 
\]
Notice that $\beta$ and $w$ are yet to be determined. An obvious choice is to eliminate the third term by suitable values of $\beta$ and $w$. This can be done easily by choosing values $\beta$ and $w$ such that $\beta > \gamma$, $\gamma = B/C$, and $w = (\gamma + m_0)/(\beta - \gamma)(m_0 = \alpha D/C)$. In order to have negative coefficients for the first two terms, we must have $\mu$ smaller than a threshold value $\bar{\mu}$ which is defined as

$$\bar{\mu} = \min \left\{ \frac{2Aw}{f(\beta, w)}, \frac{2\alpha(B + \alpha D) + 2D\beta w}{g(\beta, w)} \right\}. \quad (4.12)$$

Substituting the expressions $p$ and $q$ in (4.11) into (4.12), we have

$$\bar{\mu} = \min \left\{ \frac{m_1w}{m_2 + m_3w + m_4\beta w}, \frac{m_5 + m_6w\beta}{m_7 + m_8w + m_9\beta w} \right\}, \quad (4.13)$$

where

- $m_1 = 2A$,
- $m_2 = B(2C + D) + (A + 2B)C(2\alpha + \alpha^2)$,
- $m_3 = B(2C + D)$,
- $m_4 = (A + 2B)C$,
- $m_5 = 2\alpha(B + \alpha D)$,
- $m_6 = 2D$,
- $m_7 = B(2C + D) + (A + 2B)C\alpha^2 + 4BC\alpha$,
- $m_8 = B(2C + D)$,
- $m_9 = (A + 2B)C$.

It is easy to check that

$$\frac{m_1}{m_3 + m_4\gamma} = \frac{2A}{B(A + 2B + 2C + D)},$$

$$\frac{m_1}{m_6\gamma} = \frac{2D}{C(A + 2B + 2C + D)}.$$

Assume that

$$(H_3) \quad a_1a_3(e_1a_4 + e_2a_2)^2 > a_2a_4(a_3e_1 - a_1e_2)^2$$

holds true, this implies that $AC > BD$; hence, we have

$$\frac{m_1}{m_3 + m_4\gamma} > \frac{m_6\gamma}{m_8 + m_9\gamma}.$$
By taking advantage of the value $\mu^*$, we have
\[
V' \leq -w^* A \left( 1 - \frac{\mu}{\mu^*} \right) u_1^2 - (\alpha(B + \alpha D) + D\beta^* w^*) \left( 1 - \frac{\mu}{\mu^*} \right) u_2^2 \\
- (B + \alpha D)\varepsilon_1 u_1 u_2 - \alpha(B + \alpha D)\varepsilon_2 u_2^2 \\
+ \frac{1}{2} Bq\mu(u_2^4 + 4u_3^3 + u_4^4) + \frac{1}{2} Cp\mu(u_1^4 + 4u_2^3 + u_2^4) \\
+ \frac{1}{2} BD(1 + w^*)\mu(2u_2^3 + u_2^4) + Aq\mu(2u_1^3 + u_1^4) \\
+ \frac{1}{2} B(C + D)\mu(2\varepsilon_1 + \varepsilon_1^2)u_1^2 + \frac{1}{2} BC\mu\alpha(2\varepsilon_2 + \varepsilon_2^2)u_2^2 \\
+ \frac{1}{2} (A + B)C\mu((2\varepsilon_1 + \varepsilon_1^2)\alpha u_1^2 + \alpha^2(2\varepsilon_2 + \varepsilon_2^2)u_2^2).
\]

We are now ready to state and prove our main result similar to the proof of Theorem 3.1 in Beretta and Kuang [1]. As usual,
\[
\|u_0\| = \max\{|u_i(\theta)|, \quad i = 1, 2, \quad \theta \in [-\max(\sigma, \tau), 0]\}.
\]

**Theorem 4.3.** For system (1.1), assume that $\mu = \sigma + \tau < \mu^*$; then there is a positive constant $\delta$ such that if $\|u_0\| < \delta$, then the solution $(u_1(t), u_2(t))$ of (4.1) tends to $(0, 0)$. Equivalently, the solution $(x(t), y(t))$ of the original system (1.1) tends to $(x_*, y_*)$.

**Proof.** From the above computations we have
\[
V' \leq \left[ -2w^* A \left( 1 - \frac{\mu}{\mu^*} \right) + F \right] \frac{u_1^2}{2} \\
+ \left[ -2(\alpha(B + \alpha D) + D\beta^* w^*) \left( 1 - \frac{\mu}{\mu^*} \right) + G \right] \frac{u_2^2}{2}
\]

where
\[
F = (B + \alpha D)\varepsilon_1 + Bq\mu(4u_1 + u_1^2) + 2Aq\mu(2u_1 + u_1^2) + Cp\mu u_1^2 \\
+ B(C + D)\mu(2\varepsilon_1 + \varepsilon_1^2) + (A + B)C\mu(2\varepsilon_2 + \varepsilon_2^2)\alpha,
\]
\[
G = (B + \alpha D)\varepsilon_1 - 2\alpha(B + \alpha D)\varepsilon_2 \\
+ BD(1 + w^*)\mu(2u_2 + u_2^2) + Cp\mu(4u_2 + u_2^2) + Bq\mu u_2^2 \\
+ BC\mu\alpha(2\varepsilon_2 + \varepsilon_2^2) + (A + B)C\mu\alpha^2(2\varepsilon_2 + \varepsilon_2^2).
\]
It is easy to show that if $|u| < 1/4$, then

$$|\log(1 + u) - u| \leq \frac{8}{9}|u|^2 \leq |u|^2.$$  

We assume below that $|u_i| < 1/4$, $i = 1, 2$. Hence, we have

$$|\varepsilon_i| \leq |u_i|, \quad \varepsilon_i^2 < \frac{1}{4}|u_i| < |u_i|, \quad |u_i|^2 < |u_i|.$$  

Let $\|u\| = \max\{|u_1|, |u_2|\}$, then we obtain

(4.14) \quad $V' \leq -\Delta_1 u_1^2 - \Delta_2 u_2^2,$

where

$$\Delta_1 = w^* A \left(1 - \frac{\mu}{\mu^*}\right) - M\|u\|,$$

$$\Delta_2 = (\alpha (B + \alpha D) + D\beta^* w^*) \left(1 - \frac{\mu}{\mu^*}\right) - N\|u\|,$$

$$M = \frac{B + \alpha D}{2} + \frac{\mu}{16} (36Aq + Cp + 34Bq) + \frac{9\mu}{8} (B(C + D)(A + B)\alpha),$$

$$N = \frac{(B + \alpha D)(1 + 2\alpha)}{2} + \frac{9}{8} B\mu (D + Dw^* + C\alpha + C\alpha^2) + \frac{\mu}{8} (17Cp + Bq + 9CA\alpha^2).$$

Let

(4.15) \quad $\delta_0 = \min \left\{ \frac{w^* A}{M} \left(1 - \frac{\mu}{\mu^*}\right), \frac{\alpha (B + \alpha D) + D\beta^* w^*}{N} \left(1 - \frac{\mu}{\mu^*}\right) \right\}.$$

Then we see that $\|u(t)\| < \delta_0$ for $\mu \geq 0$ implies that $\Delta_1 > 0$, $\Delta_2 > 0$ and therefore $-\Delta_1 u_1^2 - \Delta_2 u_2^2$ is negative definite. Lemma 4.1 will then ensure that $\lim_{t \to \infty} u_i(t) = 0$, $i = 1, 2$, and hence $\lim_{t \to \infty} x(t) = x_*$, $\lim_{t \to \infty} y(t) = y_*$. Therefore, to complete the proof, we need to find $\delta$ such that if $\|u(0)\| < \delta$, this implying that $\|u(t)\| < \delta_0$ for all $t \geq 0$. To this end, we define

(4.16) \quad $L = \min \left\{ \frac{1}{2} V_0^2 + w^* V_1 : \|u\| = \delta_0 \right\},$
and the set
\[ S = \{ (\phi_1, \phi_2) \in C, \max\{\|\phi_1\|, \|\phi_2\|\} < \delta_0 \text{ and } V(\phi_1, \phi_2) < L \} , \]
where \( C = C([-\max(\sigma, \tau), 0], \mathbb{R}^2) \). We claim that for initial data chosen from \( S \), we must have \( \|u(t)\| < \delta_0 \) for all \( t \geq 0 \). Otherwise, we can deduce a contradiction (see Beretta and Kuang [1]). Since \( V \) is continuous, clearly there is a \( \delta : 0 < \delta < \delta_0 \) such that
\[ S_\delta = \{ (\phi_1, \phi_2) \in C, \max\{\|\phi_1\|, \|\phi_2\|\} < \delta \} \subset S. \]
This is a desired value for \( \delta \) in our theorem, hence the end of the proof.

**Example.** Consider a special case of (1.1) \((4.17)\)
\[
\begin{align*}
\dot{x}(t) &= x(t)[0.2 - 0.2x(t) - 0.2y(t - \sigma)], \\
\dot{y}(t) &= y(t)[-0.1 + 0.3x(t - \tau) - 0.1y(t)].
\end{align*}
\]
Then \( x_* = y_* = 1/2 \). Obviously, condition (1.2) and assumption (H3) hold true. We can compute the following
\[ a = 0.15, \quad b = 0.005, \quad c = 0.015, \quad \Delta = 9.5625 \times 10^{-4}, \]
\[ \omega_+^2 = 9.2 \times 10^{-3}, \quad \cos \theta = 0.28, \quad \sin \theta = 0.96, \]
\[ \theta = 1.29, \quad \mu^*_0 = 13.44. \]
Therefore, for \( 0 \leq \sigma + \tau < \mu^*_0 \), the positive equilibrium \( E_* \) is locally asymptotically stable.

After making the change of variables \( x = (1 + u_1)/2, \ y = (1 + u_2)/2, \) we have a special case of (4.2) with
\[
A = 0.1, \quad B = 0.1, \quad C = 0.15, \quad D = 0.05, \\
\alpha = 2/3, \quad \gamma = 2/3, \quad m_0 = 2/9, \quad w = 8/(9\beta - 6), \\
m_1 = 0.2, \quad m_2 = 0.115, \quad m_3 = 0.035, \quad m_4 = 0.045, \\
m_5 = 1.6/9, \quad m_6 = 0.1, \quad m_7 = 0.095, \\
m_8 = 0.035, \quad m_9 = 0.045, \\
2Aw/f(\beta, w) = 1.6/(1.395\beta - 0.41), \\
(2\alpha(B + \alpha D) + 2D\beta w)/g(\beta, w) = (21.6\beta - 9.6)/(9(1.215\beta - 0.29)).
\]
Hence
\[
\mu^* = \max_{\beta > \gamma} \left\{ \min \left\{ \frac{2Aw}{f(\beta, w)}, \frac{2\alpha(B + \alpha D) + 2D\beta w}{g(\beta, w)} \right\} \right\}
\]
\[
= \frac{2Aw}{f(\beta, w)} \bigg|_{\beta = \beta^*} = 1.48,
\]
where \(\beta^* = 1.07\) is the larger root of equation
\[
\frac{2Aw}{f(\beta, w)} = \frac{2\alpha(B + \alpha D) + 2D\beta w}{g(\beta, w)}
\]
and \(w^* = 2.2\). This in turn leads to
\[
p = 0.19, \quad q = 0.26, \quad M = 0.067 + 0.15\mu, \quad N = 0.16 + 0.11\mu.
\]
Hence the value \(\delta_0\) is defined by
\[
\delta_0 = \min \left\{ \frac{w^*A}{M} \left(1 - \frac{\mu}{\mu^*}\right), \frac{\alpha(B + \alpha D) + D\beta^* w^*}{N} \left(1 - \frac{\mu}{\mu^*}\right) \right\}
\]
\[
= \min \left\{ \frac{0.22(1 - \mu/\mu^*)}{0.067 + 0.15\mu}, \frac{0.21(1 - \mu/\mu^*)}{0.16 + 0.11\mu} \right\}.
\]
For example, we can choose \(\mu = \mu^*/2\), then
\[
\delta_0 = 0.43.
\]
Therefore, for system (4.17), assume that \(\mu = \sigma + \tau < \mu^*\), then there is a constant \(0 < \delta < \delta_0\) such that if \(\|u_0\| < \delta\), then the solution of (4.17) tends to \((x^*, y^*)\).

For the example system (4.17), the value of \(\mu_0^*\) is 13.44 while the value \(\mu^*\) is around 1.48. This is about nine times in difference.

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