THE CANONICAL PRODUCT OF
THE SOLUTION OF THE STURM-LIOUVILLE
EQUATION IN ONE TURNING POINT CASE

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ABSTRACT. The present paper is concerned with the function theoretic property of solutions of the equation

\[ y'' + (\lambda t - q(t))y = 0, \quad -1 \leq t \leq 1. \]

Using the asymptotic solution as well as the distribution of positive and negative eigenvalues, we derive the canonical product of a particular solution of the Sturm-Liouville in one turning-point case.

1. Introduction. Basic existence theory contains the facts, based on the uniform convergence of successive approximations, that every solution of the Sturm-Liouville equation

\[ y'' + (\lambda t - q(t))y = 0, \quad -1 \leq t \leq 1 \]

is an entire function of the complex parameter \( \lambda \) of order \( 1/2 \) for any fixed \( t \in (a, b) \), \( y, y' \) having fixed values, independent of \( \lambda \), at some fixed \( c \in (-1, 1) \) and that \( q(t) \) is a real function locally Lebesgue integrable on real open interval \((-1, 1)\). See [5].

Now let \( U(t, \lambda) \) solve the initial value problem (1) with initial condition

\[ U(-1, \lambda) = 0, \quad \frac{\partial U}{\partial t}(-1, \lambda) = 1. \]

By Halvorsen’s result, \( U(x, \lambda) \) is an entire function of order \( 1/2 \) for each fixed \( x \) in \((-1, 1)\); therefore, by using Hadamard’s theorem, see [7, page 24], \( U(x, \lambda) \) can be represented in the form

\[ U(x, \lambda) = c(x) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\omega_n(x)}\right) \]
where \( c(x) \) is a function independent of \( \lambda \) but may depend on \( x \). The sequence of \( \{ \omega_n(x) \} \) is a zero set of \( U(x, \lambda) \) for each \( x \), so that \( U(x, \omega_n(x)) = 0 \), which corresponds to eigenvalues of the Dirichlet problem for equation (1) on the closed interval \([-1, x]\). We see that each \( \omega_n(x) \), \( n = 1, 2, \ldots \) for each fixed \( x \) appears in the denominator and must be nonzero. By adding the extra condition \( q(t) \geq 0 \), we will have \( \omega_n(x) \neq 0 \) for any \( x \) by Sturm’s comparison theorem.

It is known that, for a nonnegative continuous function \( q(x) \), the eigenvalues of the Dirichlet problem for (1) on \([-1, x]\) are real and simple, see [6]. Hence,

\[
\frac{\partial U}{\partial \lambda}(x, \lambda_n(x)) \neq 0
\]

for each \( x \in (-1, 0) \). It follows from the implicit function theorem that \( \omega_n(x, q) \) is \( C^2 \) in \( x \) and

\[
\omega_n'(x) = -\left\{ \frac{\partial U}{\partial x}(x, \lambda) \right\}_{\lambda = \omega_n(x)} \frac{\partial U}{\partial \lambda}(x, \lambda).
\]

For \( x \in [-1, 0] \) fixed, the Dirichlet problem corresponding to equation (1) on \([-1, x]\) has an infinite number of negative eigenvalues, say \( \{\lambda_n(x)\} \) (note that in this case \( \omega_n(x) = \lambda_n(x) \)). The asymptotic distribution of each function \( \lambda_n(x) \) is of the form

\[
\sqrt{-\lambda_n(x)} = \frac{n\pi}{\int_{-1}^{x} \sqrt{-t} \, dt} + O\left(\frac{1}{n}\right), \quad x < 0
\]

and

\[
\lim_{x \to -1} \lambda_n(x) = -\infty, \quad \lambda_1(x) > \lambda_2(x) > \cdots.
\]

For more details, see [1]. The eigenfunctions corresponding to the negative eigenvalues, \( \lambda_n(x) \), have asymptotic representation

\[
U(t, \lambda_n(x)) = \frac{p(x) \sin(n\pi p(t)/p(x))}{(-t)^{1/4} n\pi} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}
\]

and

\[
\frac{\partial U}{\partial t}(t, \lambda_n(x)) = (-t)^{1/4} \cos \frac{n\pi p(t)}{p(x)} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}.
\]
where

\[(3) \quad p(t) = \int_{-1}^{t} \sqrt{v} \, dv,\]

and the solution of the initial-value problem (1), \(U(t, \lambda)\), for fixed \(x = t\), \(-1 \leq x < 0\), can be represented in the form

\[U(t, \lambda) = \frac{p(x)}{(-x)^{1/4}} \prod_{n=1}^{\infty} \frac{\lambda - \lambda_k(x)}{Z_k^2(x)},\]

where \(p(x)\) is given in (3) and \(\lambda_k(x)\) is the sequence of eigenvalues for Dirichlet problem associated with (1) on \([-1, x]\), i.e.,

\[y(-1, \lambda) = 0 = y(x, \lambda),\]

and \(Z_m = (m\pi/p(x))\), \(m = 1, 2, \ldots\). For a proof see [9].

For \(x \in (0, 1]\) fixed, the Dirichlet problem for (1) on \([-1, x]\) has an infinite number of positive and negative eigenvalues, say, respectively, \(\{u_n(x)\}, \{r_n(x)\}\).

From [1], the asymptotic form of \(u_n(x)\) is of the form

\[\sqrt{u_n(x)} = \frac{n\pi - \pi/4}{\int_{x}^{0} \sqrt{t} \, dt} + O\left(\frac{1}{n}\right), \quad 0 < x;\]

and the asymptotic form of \(r_n(x)\) is of the form

\[\sqrt{-r_n(x)} = \frac{n\pi - \pi/4}{\int_{-1}^{x} \sqrt{-t} \, dt} + O\left(\frac{1}{n}\right), \quad 0 < x.\]

By Hadamard’s theorem, the solution on \([-1, x]\) for \(x > 0\) is of the form

\[U(x, \lambda) = c \prod \left(1 - \frac{\lambda}{r_n(x)}\right) \prod \left(1 - \frac{\lambda}{u_n(x)}\right).\]

Now let \(\tilde{j}_n, n = 1, 2, \ldots\), be the positive zeros of \(J_1'(z)\), derivative of the Bessel function of order one. The distribution of \(\tilde{j}_n\) is of the form

\[\frac{\tilde{j}_n^2}{j_n^2} = m^2 \pi^2 - \frac{m\pi^2}{2} + O(1),\]
see [8]; consequently, we have
\[
\frac{j_n^2}{f^2(x)u_n(x)} = 1 + O(1/n^2)
\]
and
\[
\frac{-j_n^2}{p^2(0)r_n(x)} = 1 + O(1/n^2),
\]
where
\[
f(x) = \int_{0}^{x} \sqrt{t} \, dt, \quad x > 0,
\]
and \(p(x)\) is defined in (3).

Consequently, the infinite products
\[
\prod (\frac{j_n^2}{f^2(x)u_n(x)})
\]
and
\[
\prod (\frac{-j_n^2}{p^2(0)r_n(x)})
\]
are absolutely convergent for each \(x > 0\), see [4]. Therefore we may write
\[
U(x, \lambda) = c_3 \prod \frac{(\lambda - r_n(x))p^2(0)}{j_n^2} \prod \frac{f^2(x)(u_n(x) - \lambda)}{j_n^2},
\]
where
\[
c_3 = c \prod \frac{j_n^2}{f^2(x)u_n(x)} \prod \frac{-j_n^2}{p^2(0)r_n(x)}.
\]

In this paper, we will first approximate the infinite products; then, by using the asymptotic form of \(U(x, \lambda)\), we will determine \(c_3\).

2. The asymptotic representation of the canonical product. The first and second term of the distribution of the positive eigenvalues \(u_m(x)\) is similar to the distribution of the positive zeros of \(J_1^2(z)\). We can use this fact to find the asymptotics of the infinite product.
By means of Hadamard’s theorem, we first find an infinite expansion of $J_{1/3}(z)$ and $J_1'(z)$.

**Lemma 1.** Let $J_{1/3}$ be the Bessel function of order $1/3$ and $b$ be a positive number. Then

\[
J_{1/3}(ib\sqrt{\lambda}) = \left\{\frac{i\sqrt{\lambda}b/2}{\Gamma(4/3)}\right\}^{1/3} \prod_{m=1}^{\infty} \left(1 + \frac{b^2\lambda}{j_m^2}\right)
\]

where the $j$s, $m = 1, 2, \ldots$, are the positive zeros of $J_{1/3}(z)$ and, for complex $\lambda$, the domain of the function $\sqrt{\lambda}$ is the complement of negative real axis $\lambda \leq 0$, while the range of $\sqrt{\lambda}$ is the right half of the $\lambda$ plane with the imaginary axis excluded.

**Proof.** From [8] we have

\[
J_\nu(z) = \frac{[(1/2)(z)^\nu}{\Gamma(\nu + 1)} \prod \left(1 - \frac{z^2}{j_m^2}\right)
\]

where

\[
j_m \sim \beta - \frac{\alpha - 1}{8\beta} - \frac{4(\alpha - 1)(7\alpha - 31)}{3(8\beta)^3} - \ldots
\]

and

\[
\beta = (m + \nu/2 - 1/4)\pi, \quad \alpha = 4\nu^2.
\]

By inserting $z = ib\sqrt{\lambda}$ and $\nu = 1/3$, we get

\[
J_{1/3}(ib\sqrt{\lambda}) = \left\{\frac{i\sqrt{\lambda}b/2}{\Gamma(4/3)}\right\}^{1/3} \prod \left(1 + \frac{b^2\lambda}{j_m^2}\right)
\]

where

\[
j_m^2 = m^2\pi^2 - \frac{m\pi^2}{6} + O(1).
\]

**Lemma 2.** Let $J_1'(z)$ be the derivative of the Bessel function of order 1 and let $c$ be a positive constant. Then

\[
J_1'(c/\sqrt{\lambda}) = \frac{1}{2} \prod \left(1 - \frac{\lambda c^2}{j_m^2}\right)
\]

\[
J_1'(ic\sqrt{\lambda}) = \frac{1}{2} \prod \left(1 + \frac{\lambda c^2}{j_m^2}\right),
\]
where the $\tilde{j}_m$s, $m = 1, 2, \ldots$, are the positive zeros of $J'_1(z)$ and, for $\lambda$ complex, $\sqrt{\lambda}$ is defined as in Theorem 1.

Proof. From [8], we have

$$J'_\nu(z) = \frac{(z/2)^{\nu-1}}{2\Gamma(\nu)} \prod \left(1 - \frac{z^2}{\tilde{j}_m^2}\right), \quad \nu > 0$$

where

$$\tilde{j}_m \sim \beta' - \frac{\alpha + 3}{8\beta'} - \frac{4(7\alpha^2 + 82\alpha - 9)}{3(8\beta')^3} - \ldots$$

$$\beta' = (m + \nu/2 - 3/4)\pi, \quad \alpha = 4\nu^2;$$

by putting $z = c\sqrt{\lambda}$, $\Gamma(1) = 1$, we have

$$J'_1(c\sqrt{\lambda}) = \frac{1}{2} \prod \left(1 - \frac{\lambda c^2}{\tilde{j}_m^2}\right)$$

and similarly we can get

$$J'_1(\imath c\sqrt{\lambda}) = \frac{1}{2} \prod \left(1 + \frac{\lambda c^2}{\tilde{j}_m^2}\right)$$

where

$$(7) \quad \tilde{j}_m^2 = m^2\pi^2 - \frac{m\pi^2}{2} + O(1).$$

The following theorems play an important role in estimating the infinite product:

**Theorem 1.** $\prod_0^\infty (1 + p_n)$ converges absolutely if and only if $\sum_0^\infty p_n$ converges absolutely, where the $p_n$ are arbitrary complex constants.

**Theorem 2.** $\prod_0^\infty (1 + p_n)$ converges absolutely if and only if $\sum_0^\infty p_n$ converges absolutely, where the $p_n$ are arbitrary complex constants.

Proof. See [4].
**Theorem 3.** If \( p_n(z) \) is analytic in a simply connected domain \( D \), and if \( \sum_0^\infty |p_n(z)| \) converges uniformly in every closed region \( R \) of \( D \), then

\[
\prod_0^\infty (1 + p_n(z))
\]

converges uniformly to \( f(z) \) in every such \( R \) and \( f(z) \) is analytic in \( D \).

**Proof.** See [4].

**Theorem 4.** (a) Suppose \( a_{mn}, \ m, n > 1, \) are complex numbers satisfying

\[
|a_{mn}| = O\left(\frac{1}{|m^2 - n^2|}\right), \quad m \neq n
\]

then, for each \( 1 \leq n \),

\[
\prod_{m=1}^{\infty} (1 + a_{mn}) = 1 + O\left(\frac{\log n}{n}\right).
\]

(b) In addition, if \( b_n, \ 1 \leq n, \) is a square summable sequence of complex numbers, then

\[
\prod_{m,n>1}^{\infty} (1 + a_{mn}b_n) < \infty.
\]

**Proof.** See [9, page 165].

**Lemma 3.** Let \( \tilde{\gamma}_m \) be the positive zeros of \( J_1'(z) \) and \( u_m(x), \ 1 \leq m, \) a sequence of continuous functions defined on any compact subinterval of \( (0,1) \) such that

\[
u_m(x) = \frac{m^2 \pi^2}{f^2(x)} - \frac{m \pi^2}{2f^2(x)} + O(1), \quad 1 \leq m,
\]
where \( f(x) = \int_0^x \sqrt{t} \, dt \). Then, the infinite product

\[
\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2}
\]

is an entire function of \( \lambda \) for fixed \( x \), whose roots are precisely \( u_m(x) \), \( 1 \leq m \). Moreover,

\[
\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2} = 2J'_1(\sqrt{\lambda f(x)}) \left(1 + O\left(\frac{\log n}{n}\right)\right)
\]

uniformly on the circles \(|\lambda| = (n^2\pi^2/f^2(1)) = (9n^2\pi^2/4)\).

**Proof.** Let \( x \) be fixed. Since, from (6),

\[
\tilde{j}_m^2 = m^2\pi^2 - \frac{m\pi^2}{2} + O(1),
\]

therefore

\[
\sum_{1 \leq m} \left| \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2} - 1 \right| = \sum_{1 \leq m} \left| \frac{\lambda + O(1)}{\tilde{j}_m^2} \right|
\]

converges uniformly on bounded subsets of complex plane. Therefore, by Theorem 3, the infinite product converges to an entire function of \( \lambda \) whose roots are precisely \( u_m(x) \), \( 1 \leq m \).

Now, by Lemma 2,

\[
J'_1(\sqrt{\lambda f(x)}) = \frac{1}{2} \prod \left(1 - \frac{\lambda f^2(x)}{\tilde{j}_m^2}\right);
\]

thus, the quotient of the products is

\[
\prod_{1 \leq m} \left( \frac{(u_m(x) - \lambda)f^2(x)/\tilde{j}_m^2}{(1/2) \prod[1 - (\lambda f^2(x)/\tilde{j}_m^2)]} \right) = 2 \prod_{1 \leq m} \frac{u_m - \lambda}{[(\tilde{j}_m^2/f^2(x)) - \lambda]}.
\]

Furthermore,

\[
\left| \frac{u_m(x) - \lambda}{(\tilde{j}_m^2/f^2(x)) - \lambda} - 1 \right| = \left| \frac{u_m(x) - (\tilde{j}_m^2/f^2(x))}{(\tilde{j}_m^2/f^2(x)) - \lambda} \right| \leq \frac{|O(1)|}{|(\tilde{j}_m^2/f^2(x)) - \lambda|}.
\]
Therefore, on the circles $|\lambda| = (n^2\pi^2/f^2(1))$, the uniform estimates

$$
\frac{u_m - \lambda}{(j_n^2/f^2(x)) - \lambda} = \begin{cases} 
1 + O(1/n) & \text{if } m = n \\
1 + O(1/|m^2 - n^2|) & \text{if } m \neq n
\end{cases}
$$

hold. By Theorem 4,

$$\prod_{1 \leq m} \frac{u_m - \lambda}{(j^2/f^2(x)) - \lambda} = \left(1 + O\left(\frac{\log n}{n}\right)\right)(1 + O(1/n)) = 1 + O\left(\frac{\log n}{n}\right)
$$

whence

$$\prod_{1 \leq m} \frac{(u_m(x) - \lambda)f^2(x)}{\tilde{j}_m^2} = 2J'_1(\sqrt{\lambda}f(x))\left(1 + O\left(\frac{\log n}{n}\right)\right)
$$

uniformly on these circles.

Similarly, we can prove the following lemma

**Lemma 4.** Let $\tilde{j}_m$, $m = 1, 2, \ldots$, be the positive zeros of $J'_1(z)$, and, for fixed $x$ in $(0,1),

$$r_m(x) = -\frac{m^2\pi^2}{p^2(0)} + \frac{m\pi^2}{2p^2(0)} + O(1), \quad 1 \leq m,
$$

be a negative sequence of continuous functions where $p(x) = \int_{-1}^x \sqrt{-t} \, dt$. Then, the infinite product

$$\prod_{1 \leq m} (\lambda - r_m(x))p^2(0) \quad \tilde{j}_m^2
$$

is an entire function of $\lambda$ for fixed $x$, whose roots are precisely $r_m(x)$, $1 \leq m$. Moreover,

$$\prod_{1 \leq m} \frac{(\lambda - r_m(x))p^2(0)}{\tilde{j}_m^2} = 2J'_1(\sqrt{\lambda}p(0))\left(1 + O\left(\frac{\log n}{n}\right)\right)
$$

uniformly on the circles $|\lambda| = (n^2\pi^2/p^2(0)) = (9n^2\pi^2/4)$. 
Proof. This follows from Theorem 4 and use of the method of the proof of the preceding lemma. Similarly, we have

**Lemma 5.** Let $j_m, m = 1, 2, \ldots,$ be the positive zeros of $J_{1/3}(z)$ and

$$
\lambda_m(0) = -\frac{m^2\pi^2}{p^2(0)} + \frac{m\pi^2}{6p^2(0)} + O(1), \quad 1 \leq m,
$$

be a negative sequence where $p(0) = \int_{-1}^{0} \sqrt{-t} \, dt$. Then the infinite product

$$
\prod_{1 \leq m} \frac{(\lambda - \lambda_m(0))p^2(0)}{j_m^2}
$$

is an entire function of $\lambda$ whose roots are precisely $\lambda_m(0), 1 \leq m$. Moreover,

$$
\prod_{1 \leq m} \frac{(\lambda - \lambda_m(0))p^2(0)}{j_m^2} = \frac{\Gamma(4/3)}{\{i\sqrt{\lambda b}/2\}^{1/3}} J_{1/3}(ib\sqrt{\lambda}) \left( 1 + O\left(\frac{\log n}{n}\right) \right)
$$

uniformly on the circles $|\lambda| = (n^2\pi^2/p^2(0)) = (9n^2\pi^2/4)$, where $b = p(0) = 2/3$.

Proof. This follows from Lemma 1 and use of the method of Lemma 3.

In [2] it was shown that

(8) \quad U(t, \lambda) =

$$
\begin{cases}
(1/(-t)^{1/4}\sqrt{\lambda})(1+O(1/\sqrt{\lambda})) \sinh(p(t)\sqrt{\lambda}), & \text{if } -1 \leq t < 0, \\
(\pi^{1/2}Ai(0)/\lambda^{5/12})\{e^{2/3\sqrt{\lambda}} - (\sqrt{3}/2)e^{-2/3\sqrt{\lambda}}\} \\
\times\{1 + O(1/\sqrt{\lambda})\}, & \text{if } t = 0, \\
(1/t^{1/4}\sqrt{\lambda})\{e^{2/3\sqrt{\lambda}}\cos(2/3t^{3/2}\sqrt{\lambda} - \pi/4) + e^{-2/3\sqrt{\lambda}} \\
\times(1/2)\sin((2/3)t^{3/2}\sqrt{\lambda} - \pi/4)\} \{1 + O(1/\sqrt{\lambda})\}, & \text{if } 0 < t;
\end{cases}
$$
and
(9)

\[
\frac{\partial U}{\partial t}(t, \lambda) = \begin{cases} 
(-t)^{1/4} \left\{ 1 + O(\lambda^{-1/2}) \right\} \cosh(p(t)\sqrt{\lambda}) 
& \text{if } -1 \leq t < 0 \\
\pi^{1/2} \lambda^{-1/12} Bi'(0) \left\{ (1/2)e^{-2/3\sqrt{\lambda}} + 1/\sqrt{3}e^{2/3\sqrt{\lambda}} \right\} 
& \text{if } 0 \leq t \leq 1 \\
\pi^{1/2} \lambda^{-1/12} Bi'(0) \left\{ (1/2)e^{-2/3\sqrt{\lambda}} \cos((2/3)t^{3/2}/\sqrt{\lambda} - \pi/4) - e^{2/3\sqrt{\lambda}} 
& \times \sin((2/3)t^{3/2}/\sqrt{\lambda} - \pi/4) \right\} \left\{ 1 + O(1/\sqrt{\lambda}) \right\} 
& \text{if } 0 < t \leq 1,
\end{cases}
\]

where

\[
p(t) = \int_{-1}^{t} \sqrt{v} dv
\]

(10)

\[Ai(0) = \frac{1}{3^{2/3}\Gamma(2/3)}, \quad Bi'(0) = -\frac{\sqrt{3}}{3^{1/3}\Gamma(1/3)}\]

the error terms are uniform, \( \lambda \to \infty \). Now, by using (8) and (9), we find \( c_3 \) in (5).

**Theorem 5.** Let \( U(t, \lambda) \) be the solution of the initial value problem (1)–(2). Then, for \( 0 < x \),

\[U(x, \lambda) = \frac{\pi \sqrt{x}}{6} \prod \frac{(\lambda - r_k(x))p^2(0)}{j_k^2} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j_k^2}},\]

where \( f(x) = \int_{-1}^{x} \sqrt{t} dt, \ p(v) = \int_{-1}^{v} \sqrt{-t} dt, \) the sequence \( \{u_k(x)\} \) represents the positive eigenvalues and \( \{r_k(x)\} \) the negative eigenvalues of the Dirichlet problem associated with (1) on \([-1, x]\).

**Proof.** From [1], the asymptotic form \( \{u_m(x)\} \) and \( \{r_m(x)\} \) are of the form

\[r_m(x) = -\frac{m^2 \pi^2}{b^2} + \frac{m \pi^2}{2b^2} + O(1), \quad 1 \leq m,
\]

and

\[u_m(x) = \frac{m^2 \pi^2}{f^2(x)} - \frac{m \pi^2}{2f^2(x)} + O(1), \quad 1 \leq m
\]
where \( f(x) = \int_0^x \sqrt{t} \, dt \), \( b = p(0) = 2/3 \). Therefore, by (5) and (8), we have

\[
U(x, \lambda) = c_3 \prod \frac{(\lambda - r_k(x))p^2(0)}{\tilde{j}^2_k} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j}^2_k},
\]

\[
= \frac{1}{x^{1/4} \sqrt[4]{\lambda}} \left\{ e^{2/3 \sqrt{\lambda}} \cos \left( \frac{2}{3} x^{3/2} \sqrt{\lambda} - \pi/4 \right) \right\} \left( 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right), \lambda \to \infty.
\]

By Lemmas 3 and 4, on the circles \(|\lambda| = \left( n^2 \pi^2 / b^2 \right)\), we have

\[
\prod \frac{(\lambda - r_k(x))p^2(0)}{\tilde{j}^2_k} \prod \frac{f^2(x)(u_k(x) - \lambda)}{\tilde{j}^2_k} = 4J'_1(f(x)\sqrt{\lambda})J'_1(ib\sqrt{\lambda}) \left\{ 1 + O\left( \frac{\log n}{n} \right) \right\}.
\]

It is known that [8],

\[
J'_\nu(z) = \sqrt{\frac{2}{z\pi}} \{ - R\nu, z \} \sin X - S(\nu, z) \cos X \quad (| \arg z | < \pi)
\]

where \( \nu \) is fixed and

\[
X = z - \left( \frac{\nu}{2} + 1/4 \right) \pi,
\]

\[
R(\nu, z) \sim \sum_{k=0}^\infty ( -1 )^k \frac{4\nu^2 + 16k^2 - 1}{4\nu^2 - (4k - 1)^2} \left\{ \frac{(\nu, 2k)}{(2\zeta)^{2k}} \right\} = 1 - \frac{(\alpha - 1)(\alpha + 15)}{2(8z)^2} + \ldots,
\]

\[
S(\nu, z) \sim \sum_{k=0}^\infty ( -1 )^k \frac{4\nu^2 + 4(2k + 1)^2 - 1}{4\nu^2 - (4k + 1)^2} \left\{ \frac{(\nu, 2k + 1)}{(2\zeta)^{2k+1}} \right\} = \frac{(\alpha + 3)}{8z} - \frac{(\alpha - 1)(\alpha - 9)(\alpha + 35)}{3!(8z)^3} + \ldots,
\]

as \(|z| \to \infty\), where \( \alpha = 4\nu^2 \). Now, after some lengthy but straightforward calculations, we find

\[
J'_1(f(x)\sqrt{\lambda})J'_1(ib\sqrt{\lambda}) = \frac{2 \sin(ib\sqrt{\lambda} - (3\pi/4))}{\nu^{1/2} p^{1/2}(x) b^{1/2} \sqrt{\lambda}} \{ \sin(f(x)\sqrt{\lambda} - (3\pi/4)) + O(1/\sqrt{\lambda}) \}.
\]
Since
\[
\sin \left( f(x) \sqrt{\lambda} - \frac{3\pi}{4} \right) \sin \left( b \sqrt{\lambda} - \frac{3\pi}{4} \right) = \frac{1}{2i} \cos \left( f(x) \sqrt{\lambda} - \pi/4 \right) \{ e^{b \sqrt{\lambda} + (3i\pi/4)} - e^{-b \sqrt{\lambda} - (3i\pi/4)} \},
\]
it follows that
\[
J'_1(f(x) \sqrt{\lambda}) J'_1(b \sqrt{\lambda}) = \frac{e^{b \sqrt{\lambda} + (3i\pi/4)}}{\pi i^{3/2} b^{1/2} f^{1/2}(x) \sqrt{\lambda}} \times \left\{ \cos \left( f(x) \sqrt{\lambda} - \pi/4 \right) + O \left( \frac{1}{\sqrt{\lambda}} \right) \right\}
\]
whence, for \(|\lambda| = (n^2 \pi^2 / b^2)\),
\[
U(x, \lambda) = 4c_3 \frac{e^{b \sqrt{\lambda} + (3i\pi/4)}}{\pi i^{3/2} \sqrt{bf(x)\lambda}} \left\{ \cos \left( f(x) \sqrt{\lambda} - \pi/4 \right) + O \left( \frac{1}{\sqrt{\lambda}} \right) \right\}
\]
\[
\times \left\{ 1 + O \left( \frac{\log n}{n} \right) \right\}
\]
\[
= \frac{1}{x^{1/4} \sqrt{\lambda}} \times \left\{ e^{2/3 \sqrt{\lambda}} \cos \left( (2/3)x^{3/2} \sqrt{\lambda} - \pi/4 \right) \left( 1 + O \left( \frac{1}{\sqrt{\lambda}} \right) \right) \right\},
\]
\[
\lambda \to \infty,
\]
by (8). Consequently,
\[
c_3 = \frac{\sqrt{2/3} \pi f^{1/2}(x) b^{1/2}}{4x^{1/4} e^{3i\pi/4}} \left( 1 + O \left( \frac{1}{\sqrt{\lambda}} \right) \right) = \frac{\pi \sqrt{x}}{6} \left( 1 + O \left( \frac{1}{\sqrt{\lambda}} \right) \right),
\]
where the \(O\)-term can be made uniform in \(x\) for \(x \in (0, 1)\). Since \(c_3\) depends only on \(x\), by Hadamard’s theorem, we may let \(\lambda \to \infty\) and find
\[
c_3 = \frac{\pi \sqrt{x}}{6}.
\]
Theorem 6. Let $U(t, \lambda)$ be the solution of the initial-value problem (1)–(2). Then, for $x = 0$, 

$$U(0, \lambda) = \prod_{n=1}^{\infty} \frac{4(\lambda - \lambda_k(0))}{9j_k^2},$$

where $\{\lambda_k(0)\}$ is the sequence of negative eigenvalues of the Dirichlet problem associated with (1) on $[-1, 0]$. As before $j_k$ represents the sequence of positive zeros of the Bessel function of order $1/3$.

Proof. For $x = 0$, the distribution of the eigenvalues of equation (1) on $[-1, 0]$ is of the form 

$$\sqrt{-\lambda_n(0)} = \frac{n\pi - \pi/12}{\int_{-1}^{0} \sqrt{-t} \, dt} + O\left(\frac{1}{n}\right),$$

see [1]. By Hadamard’s theorem we also have 

$$U(0, \lambda) = c \prod \left(1 - \frac{\lambda}{\lambda_n(0)}\right).$$

Now let $j_n$, $n = 1, 2, \ldots$, be the positive zeros of the Bessel function of order $1/3$. Then, 

$$\frac{-9j_n^2}{4\lambda_n(0)} = 1 + O(1/n^2),$$

and so the infinite product $\prod (-9j_n^2/4\lambda_n(0))$ is absolutely convergent. Consequently, we may write 

$$U(0, \lambda) = c_2 \prod_{n=1}^{\infty} \frac{4(\lambda - \lambda_n(0))}{9j_n^2},$$

where $c_2 = c \prod (-9j_n^2/4\lambda_n(0))$. From (8) we have 

$$U(0, \lambda) = c_2 \prod_{n=1}^{\infty} \frac{4(\lambda - \lambda_k(0))}{9j_k^2}$$

$$= \frac{\pi^{1/2} Ai(0)}{\lambda^{5/12}}$$

$$\times \left\{ e^{(2/3)\sqrt{\lambda}} - \frac{\sqrt{3}}{2} e^{(-2/3)\sqrt{\lambda}} \right\} \left\{ 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}, \lambda \to \infty, $$

$$= \frac{\pi^{1/2} Ai(0)}{\lambda^{5/12}} e^{(2/3)\sqrt{\lambda}} \left\{ 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}, \lambda \to \infty.$$
By Lemma 1 on the circles $|\lambda| = (9n^2\pi^2/4)$, we have
\[
\prod \frac{4(\lambda - \lambda_k(0))}{9j_k^2} = \frac{\Gamma(4/3)}{(i\sqrt{\lambda}b/2)^{1/3}} J_{1/3}(ib\sqrt{\lambda}) \left( 1 + O\left( \frac{\log n}{n} \right) \right).
\]

From [8], for fixed $\nu$ and $|z| \to \infty$, the asymptotic form of $J_\nu(z)$ is of the form
\[
J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos(z - \pi\nu/2 - \pi/4) + e^{i\text{Im}z}O\left( \frac{1}{|z|} \right) \right\}, \quad |\arg z| < \pi.
\]

Therefore,
\[
J_{1/3}(ib\sqrt{\lambda}) = \sqrt{\frac{2}{\nu\pi b\sqrt{\lambda}}} \left\{ \cos(ib\sqrt{\lambda} - \frac{5\pi}{12}) + e^{ib\sqrt{\lambda}}O\left( \frac{1}{\sqrt{\lambda}} \right) \right\}.
\]

Since
\[
\cos \left( ib\sqrt{\lambda} - \frac{5\pi}{12} \right) = \frac{1}{2} \left\{ e^{ib\sqrt{\lambda}+(5\pi/12)} + e^{-ib\sqrt{\lambda}-(5\pi/12)} \right\},
\]

\[
J_{1/3}(ib\sqrt{\lambda}) = e^{ib\sqrt{\lambda}+(5\pi/12)} \sqrt{\frac{1}{2\nu\pi b\sqrt{\lambda}}} \left( 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right).
\]

Since $Ai(0) = (1/3^{2/3}\Gamma(2/3))$,
\[
c_2 = \frac{U(0, \lambda)}{\prod(4(\lambda - \lambda_k(0))/9j_k^2)}
= \frac{\pi^{1/2}Ai(0)e^{(2/3)\sqrt{\lambda}}(b\pi/2)^{1/3}\lambda^{1/6}}{\lambda^{5/12}} \frac{\Gamma(4/3)}{(2\pi b)^{1/2}\lambda^{1/4}} \left( 1 + O\left( \frac{\log n}{n} \right) \right)
\times \frac{(2\pi b)^{1/2}\lambda^{1/4}}{e^{b\sqrt{\lambda}+(5\pi/12)}} \left( 1 + O\left( \frac{\log n}{n} \right) \right)
\]

whence
\[
c_2 = \frac{2\pi}{\sqrt{3}\Gamma(1/3)\Gamma(2/3)} \left( 1 + O\left( \frac{\log n}{n} \right) \right)
\]
or
\[
c_2 = 1 + O\left( \frac{\log n}{n} \right)
\]
on the circles $|\lambda| = (9n^2\pi^2/4)$, because $\Gamma(1/3)\Gamma(2/3) = (2\pi/\sqrt{3})$, see [2]. Using an argument similar to the one for $c_3$, we find that $c_2 = 1$. 
REFERENCES


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